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The geometric means of an entire function

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1. Introduction. Let $f(z)$ be an entire function of order ϱ ($0 \leqslant \varrho \leqslant \infty$). Let us introduce the following mean values:

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\right\}; \tag{1.1}$$

$$g_k(r) = \exp\left\{\frac{k+1}{2\pi r^{k+1}}\int_0^r\int_0^{2\pi} \log|f(xe^{i\theta})|\,x^k dx\,d\theta\right\}. \tag{1.2}$$

One of us (PKK) introduced these mean values and has obtained certain of their properties in an earlier paper [8]. It follows from a remark there that the orders of $G(r)$ and $g_k(r)$ do not exceed ϱ. However, the orders of $G(r)$ and $g_k(r)$ need not always be equal to ϱ (for counter examples, see [6]). It is not always possible to obtain all results similar to ordinary mean values (compare for instance Theorem 5 of this paper and Theorem 1 in [5]). Our purpose is therefore to explore as many results on the growth of $G(r)$ and $g_k(r)$ as possible.

2. Growths of $G(r)$ and $g_k(r)$. Let $0 < \varrho < \infty$ and let us then define

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log G(r)}{r^\varrho} = \begin{matrix}C\\ D\end{matrix}; \quad \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log g_k(r)}{r^\varrho} = \begin{matrix}A\\ B\end{matrix};$$

Then the limits A, B, C and D are related by the following

THEOREM 1. *One has*

$$A \geqslant (k+1)\,C^{1+\varrho/(k+1)}\varrho^{\varrho/(k+1)}/(\varrho+k+1)\{(\varrho+k+1)\,C-(k+1)\,D\}^{\varrho/(k+1)}, \tag{2.1}$$

$$B \geqslant (D+1)\,D/(\varrho+k+1); \tag{2.2}$$

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(2.3) $$B \leqslant \frac{D}{(\varrho+k+1)\, C^{(k+1)/\varrho}},$$

(2.4) $$A \leqslant \frac{(k+1)\, C}{\varrho+k+1}.$$

Proof. We have

$$\log g_k(r) = \frac{(k+1)}{r^{k+1}} \int\limits_0^r x^k \log G(x)\, dx .$$

Let $\eta > 1$. Then

$$\begin{aligned}
\log g_k(r\eta^{1/\varrho}) &= \frac{k+1}{(r\eta^{1/\varrho})^{k+1}} \int\limits_0^{r\eta^{1/\varrho}} x^k \log G(x)\, dx \\
&= \frac{k+1}{(r\eta^{1/\varrho})^{k+1}} \Big[\int\limits_0^{r_0} + \int\limits_{r_0}^{r} + \int\limits_{r}^{r\eta^{1/\varrho}}\Big] x^k \log G(x)\, dx \\
&> \frac{A_1}{r^{k+1}} + \frac{(k+1)(D-\varepsilon)}{(r\eta^{1/\varrho})^{k+1}} \int\limits_{r_0}^{r} x^{k+\varrho} dx + \frac{(k+1)\log G(r)}{(r\eta^{1/\varrho})^{k+1}} \int\limits_{r}^{r\eta^{1/\varrho}} x^k dx \\
&= \frac{A_1}{r^{k+1}} + \frac{(k+1)(D-\varepsilon)\big(1+o(1)\big)\, r^{\varrho}}{(\varrho+k+1)\, \eta^{(k+1)/\varrho}} + \log G(r)\, \frac{\eta^{(k+1)/\varrho}-1}{\eta^{(k+1)/\varrho}} .
\end{aligned}$$

Therefore

$$\frac{\log g_k(r\eta^{1/\varrho})}{(r\eta^{1/\varrho})^{\varrho}} > \frac{1}{\eta}\left\{\frac{A_1}{r^{\varrho+k+1}} + \frac{(k+1)(D-\varepsilon)\big(1+o(1)\big)}{(\varrho+k+1)\, \eta^{(k+1)/\varrho}} + \frac{\log G(r)}{r^{\varrho}}\, \frac{\eta^{(k+1)/\varrho}-1}{\eta^{(k+1)/\varrho}}\right\},$$

where A_1 denotes a constant not necessarily the same at each occurrence. Hence

(2.5) $$A \geqslant \frac{1}{\eta}\left\{\frac{(k+1)\, D}{(\varrho+k+1)\, \eta^{(k+1)/\varrho}} + C\, \frac{\eta^{(k+1)/\varrho}-1}{\eta^{(k+1)/\varrho}}\right\},$$

(2.6) $$B \geqslant \frac{1}{\eta}\left\{\frac{(k+1)\, D}{(\varrho+k+1)\, \eta^{(k+1)/\varrho}} + D\, \frac{\eta^{(k+1)/\varrho}-1}{\eta^{(k+1)/\varrho}}\right\}.$$

It can be seen after a long calculation that the maxima of the right-hand side expressions in (2.5) and (2.6) occur at

(2.7) $$\eta = \left\{\frac{(\varrho+k+1)\, C-(k+1)\, D}{C\varrho}\right\}^{\varrho/(k+1)},$$

and

(2.8) $$\eta = 1,$$

respectively. Substituting these values of η from (2.7) and (2.8) in (2.5) and (2.6), respectively, we get (2.1) and (2.2). Again

$$\log g_k(r\eta^{1/\varrho}) < \frac{A_1}{r^{k+1}} + \frac{(k+1)(C+\varepsilon)}{(r\eta^{1/\varrho})^{k+1}} \int_{r_0}^{r} x^{\varrho+k} dx + \frac{(k+1)\log G(r\eta^{1/\varrho})}{(r\eta^{1/\varrho})^{k+1}} \int_{r}^{r\eta^{1/\varrho}} x^k dx$$

$$\approx \frac{A_1}{r^{k+1}} + \frac{(k+1)(C+\varepsilon)r^\varrho}{(\varrho+k+1)\eta^{(k+1)/\varrho}} + \frac{\eta^{(k+1)/\varrho}-1}{\eta^{(k+1)/\varrho}} \log G(r\eta^{1/\varrho})$$

and so we have

$$A \leqslant \frac{1}{\eta}\left\{\frac{(k+1)C}{(\varrho+k+1)\eta^{(k+1)/\varrho}} + \frac{\eta(\eta^{(k+1)/\varrho}-1)}{\eta^{(k+1)/\varrho}} C\right\}, \tag{2.9}$$

$$B \leqslant \frac{1}{\eta}\left\{\frac{(k+1)C}{(\varrho+k+1)\eta^{(k+1)/\varrho}} + \frac{\eta(\eta^{(k+1)/\varrho}-1)}{\eta^{(k+1)/\varrho}} D\right\}. \tag{2.10}$$

It can also be seen that the minima of the right-hand side expressions in (2.9) and (2.10) occur at

$$\eta = 1 \tag{2.11}$$

and

$$\eta = \frac{C}{D}, \tag{2.12}$$

respectively. Substituting these values of η from (2.11) and (2.12) in (2.9) and (2.10) respectively, we get (2.3) and (2.4). This proves the theorem.

COROLLARY. *If $C = D$, then*

$$A = B = \frac{k+1}{\varrho+k+1} C .$$

The proof is straight forward.

THEOREM 2. *If A and B are defined as above and that $A = B$, then*

$$C = D = \frac{\varrho+k+1}{k+1} A . \tag{2.13}$$

Proof. Let $0 \leqslant \eta \leqslant 1$. Then

$$[(1+\eta)^{k+1}-1]\log G(r) < \frac{k+1}{r^{k+1}} \int_{r}^{(1+\eta)r} x^k \log G(x)\, dx$$

$$= \frac{k+1}{r^{k+1}} \left[\int_0^{(1+\eta)r} - \int_0^{r}\right] x^k \log G(x)\, dx$$

$$= (1+\eta)^{k+1} \log g_k((1+\eta)r) - \log g_k(r) .$$

But for $r > r_0$ and for every arbitrarily small $\varepsilon > 0$,

$$A-\varepsilon < \frac{\log g_k(r)}{r^\varrho} < A+\varepsilon .$$

And so,

$$[(1+\eta)^{k+1}-1]\log G(r) < [(1+\eta)^{\varrho+k+1}(A+\varepsilon)-(A-\varepsilon)]r^\varrho .$$

Therefore

$$[(1+\eta)^{k+1}-1]\frac{\log G(r)}{r^\varrho} < [(1+\eta)^{\varrho+k+1}-1]A+[(1+\eta)^{\varrho+k+1}+1]\varepsilon .$$

Hence

$$[(1+\eta)^{k+1}-1]\lim_{r\to\infty}\frac{\log G(r)}{r^\varrho} \leqslant [(1+\eta)^{\varrho+k+1}-1]A ,$$

i.e.,

$$\lim_{r\to\infty}\frac{\log G(r)}{r^\varrho} \leqslant \frac{(1+\eta)^{\varrho+k+1}-1}{(1+\eta)^{k+1}-1}A .$$

But η is arbitrary and hence making $\eta\to 0$, we obtain

$$\lim_{r\to\infty}\frac{\log G(r)}{r^\varrho} \leqslant \frac{\varrho+k+1}{k+1}A . \tag{2.14}$$

Similarly by considering the inequality

$$[1-(1-\eta)^{k+1}]\log G(r) > \frac{k+1}{r^{k+1}}\int\limits_{(1-\eta)r}^{r} x^k \log G(x)\,dx ,$$

we can prove that

$$\lim_{r\to\infty}\frac{\log G(r)}{r^\varrho} \geqslant \frac{\varrho+k+1}{k+1}A , \tag{2.15}$$

and so (2.14) and (2.15) yield (2.13).

3. Comparative growth of $G(r)$ and $g_k(r)$. Let $S(r)$ be a real and non-negative function increasing for $r_0 < r < \infty$, where $r_0 > 0$.

The order μ and lower order λ of the function $S(r)$ are defined as:

$$\mu = \limsup_{r\to\infty}\frac{\log S(r)}{\log r}; \qquad \lambda = \liminf_{r\to\infty}\frac{\log S(r)}{\log r} ,$$

where $0 \leqslant \lambda \leqslant \mu \leqslant \infty$.

If $0 < \mu < \infty$, we set

$$\limsup_{r\to\infty}\frac{S(r)}{r^\mu} = C ,$$

and distinguish the following cases:

(a) $S(r)$ *has maximal type if* $C = +\infty$,

(b) $S(r)$ *has mean type if* $0 < C < \infty$,

(c) $S(r)$ *has minimal type if* $C = 0$,

(d) $S(r)$ *has convergence class or is of convergence class if*

$$\int_{r_0}^{\infty} \frac{S(x)}{x^{\mu+1}} dt \qquad \textit{converges} .$$

It is known ([1], p. 17) that if $S(r)$ is of order μ and of convergence class, then it is of minimal type, namely (c). We prove that the logarithms of $G(r)$ and $g_k(r)$ are of the class (a), (b), (c) or (d). First, let us note in view of Jensen's Theorem on the zeros of an entire function that if ϱ_1 and λ_1 are respectively the exponent convergence and lower exponent convergence of the zeros of $f(z)$, i.e.

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n(r)}{\log r} = \frac{\varrho_1}{\lambda_1}; \tag{3.1}$$

where $n(r)$ is the number of the zeros of $f(z)$ in $|z| \leqslant r$, then

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log G(r)}{\log r} = \frac{\varrho_1}{\lambda_1}. \tag{3.2}$$

We now state and prove the following

THEOREM 3. *If $f(z)$ is an entire function, then the order of the functions $S_1(r) = \log G(r)$ and $S_2(r) = \log g_k(r)$ are the same, i.e. equal to ϱ_1. Further, if $0 < \varrho_1 < \infty$, then $S_1(r)$ and $S_2(r)$ are of the same class* (a), *or* (b) *or* (c) *or* (d).

The proof makes use of the following

LEMMA. *This is*

$$\log g_k(r) \leqslant \log G(r) \leqslant \frac{R^{k+1}}{R^{k+1} - r^{k+1}} \log g_k(R) , \qquad R > r . \tag{3.3}$$

Proof of the lemma. We have

$$\log g_k(r) = \frac{k+1}{r^{k+1}} \int_0^r \log G(x) x^k dx \leqslant \log G(r) . \tag{3.4}$$

Further,

$$\log g_k(R) = \frac{k+1}{R^{k+1}} \int_0^R x^k \log G(x) dx \geqslant \frac{k+1}{R^{k+1}} \int_r^R x^k \log G(x) dx \tag{3.5}$$

$$\geqslant \frac{R^{k+1} - r^{k+1}}{R^{k+1}} \log G(r) .$$

From (3.4) and (3.5), the lemma follows:

Proof of Theorem 3. Putting $R = 2r$ in (3.3), we get

$$S_2(r) \leqslant S_1(r) \leqslant \frac{2^{k+1}}{2^{k+1}-1} S_2(2r) . \tag{3.6}$$

Now one can easily see from (3.6) that the orders of the functions $S_1(r)$ and $S_2(r)$ are the same and each is equal to ϱ_1 (incidently the lower orders of these functions are also the same and are equal to λ_1, see (3.1) and (3.2)). Suppose now for instance that $S_2(r)$ has convergence class of order ϱ_1, then

$$\int_{r_0}^{\infty} \frac{S_1(r)}{r^{\varrho_1+1}} dr \leqslant \frac{2^{k+1}}{2^{k+1}-1} \int_{r_0}^{r} \frac{S_2(2r)}{(2r)^{k+1}} dr , \qquad 0 < r < \infty .$$

Thus $S_1(r)$ has also the convergence class. The converse is also obvious. The other results are also proved similarly.

THEOREM 4. *Let $n(r)$ be the number of zeros of $f(z)$ in $|z| \leqslant r$ and $n(r) \approx r^{\varrho_1}L(r)$, where $\varrho_1 > 1$, $L(r)$ is an increasing function of r (however, slowly) such that $L(ar) \approx L(r)$, for any $a > 0$ and*

$$\int_0^r x^c L(x)\, dx \approx \frac{L(r) r^{c+1}}{c+1} ,$$

for any $c > 0$ and if

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{gk(r)}{G(r)}\right\}^{1/N(r)} = \frac{H}{h} ;$$

then

(i) $H = h = \exp\left\{\dfrac{-\varrho_1}{\varrho_1+k+1}\right\}$;

(ii) $\lim\limits_{r\to\infty}\left\{\dfrac{G(r)}{g_k(r)}\right\}^{(k+1)/[\log g_k(r)]} = e^{\varrho_1}$.

Proof. By Jensen's Theorem ([9], p. 125)

$$N(r) = \int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta .$$

Therefore

$$\left\{\frac{g_k(r)}{G(r)}\right\}^{1/N(r)} = \exp\left[\frac{1}{N(r)} \left\{\frac{k+1}{r^{k+1}} \int_0^r x^k N(x)\, dx - N(r)\right\}\right] \tag{3.7}$$

$$= \exp\left\{-\frac{1}{N(r) r^{k+1}} \int_0^r x^k n(x)\, dx\right\} . \tag{3.8}$$

Now let $n(x) = 0$ for $x \leqslant 1$. Then, following Kamthan [2], we get

$$\int_0^r x^k n(x)\,dx \approx \frac{r^{k+1} n(r)}{\varrho_1 + k + 1}, \tag{3.9}$$

and

$$N(r) = \int_0^r \frac{n(x)}{x}\,dx \approx \frac{n(r)}{\varrho_1}. \tag{3.10}$$

The first part of the theorem now follows from (3.8) through (3.10).

To prove the second part, we notice that

$$\frac{k+1}{\log g_k(r)}\log\frac{G(r)}{g_k(r)} = (k+1)\left\{\frac{\log G(r)}{\log g_k(r)} - 1\right\} = (k+1)\left\{\frac{r^{k+1}N(r)}{(k+1)\int_0^r N(x)x^k dx} - 1\right\}.$$

But

$$N(r) \approx \frac{n(r)}{\varrho_1};$$

$$\frac{k+1}{r^{k+1}}\int_0^r N(x)x^k dx \approx \frac{(k+1)n(r)}{(\varrho_1+k+1)\varrho_1}.$$

The result is now proved.

THEOREM 5. *For an entire function of exponent of convergence* ϱ_1, $0 < \varrho_1 < \infty$, *we have for large* r

$$\frac{\log G(r)}{\log g_k(r)} < \left\{1 + \frac{r^{\varrho_1+\varepsilon}}{k+1} + O(r^{-\varrho_1-\varepsilon})\right\}\left(1 + o(1)\right), \qquad \varepsilon > 0.$$

Proof. We have for $R > r$

$$\log g_k(R) = \log g_k(r) + \int_r^R \frac{n(x)}{x}\,dx - \frac{1}{R^{k+1}}\int_r^R n(x)x^k dx + \tag{3.11}$$

$$+\left\{\frac{1}{r^{k+1}} - \frac{1}{R^{k+1}}\right\}\int_0^r n(x)x^k dx$$

$$= \log g_k(r) + W_k(R, r) \quad \text{(say)}.$$

Since $(1/r^{k+1} - 1/R^{k+1}) > 0$, one finds

$$W_k(R, r) \leqslant n(R)\log\left(\frac{R}{r}\right) = n(R)\log\left(1 + \frac{R-r}{r}\right) \tag{3.12}$$

$$< R^{\varrho_1+\varepsilon}\frac{R-r}{r}, \quad R > r > R_0, \quad \varepsilon > 0.$$

Choose R to satisfy the equality:

$$\frac{R-r}{r}=\frac{1}{r^{\varrho_1+\varepsilon}}.$$

Then

$$\frac{R^{k+1}-r^{k+1}}{R^{k+1}}=1-\left(\frac{r}{R}\right)^{k+1}=1-\{1+r^{-\varrho_1-\varepsilon}\}^{-(k+1)}$$

$$=(k+1)\left\{1-\frac{k+2}{2!}r^{-\varrho_1-\varepsilon}+O(r^{-2\varrho_1-2\varepsilon})\right\}r^{-\varrho_1-\varepsilon}.$$

Therefore

$$\frac{R^{k+1}}{R^{k+1}-r^{k+1}}=\frac{r^{\varrho_1+\varepsilon}}{k+1}\left\{1+\frac{k+2}{2!}r^{-\varrho_1-\varepsilon}+O(r^{-2\varrho_1-2\varepsilon})\right\}. \tag{3.13}$$

Also from (3.12)

$$W_k(R,r)<\left(\frac{R}{r}\right)^{\varrho_1+\varepsilon}=\{1+r^{-\varrho_1-\varepsilon}\}^{\varrho_1+\varepsilon}<2\,, \quad \text{for large } r\,. \tag{3.14}$$

Theorem now follows from (3.3) and (3.11) through (3.14).

In this final result we single out a subclass of a class of entire functions having the lower exponent of convergence of zeros, i.e. λ_1, equal to infinity (cf. (3.1)). For such functions we have:

THEOREM 6. *For a class of entire functions for which* $\log\log G(r)$ *is a convex function of* $\log r$, *suppose*

$$\lim_{r\to\infty}\frac{\sup}{\inf}\left\{\frac{\log G(r)}{\log g_k(r)}\right\}^{1/\log r}=\frac{L}{l^*};$$

then

$$\lim_{r\to\infty}\frac{\sup}{\inf}\frac{\log_3 g_k(r)}{\log r}=\frac{\log L}{\log l^*};\qquad \log_3=\log\log\log\,.$$

To prove this result we require a lemma

LEMMA. *Under the hypothesis of the theorem* $r^{k+1}\log G(r)$ *is a convex function of* $r^{k+1}\log g_k(r)$.

Proof of the lemma. Since $\log\log G(r)$ is a convex function of $\log r$, we have (cf. [3], p. 4.)

$$\log\log G(r)=\log\log G(r_0)+\int_{r_0}^{r}\omega(x)\,d(\log x)\,,$$

where $\omega(x)$ is non-decreasing and $\to\infty$ with x. Therefore (for details see Lemma 7, p. 108, [4])

$$r^{k+1}\log G(r)=O(1)+\int_{\delta}^{r}\omega^*(x)\,d\big(x^{k+1}\log g_k(r)\big)\,,$$

where

$$\omega^*(x) = 1 + \frac{\omega(x)}{k+1}.$$

The lemma is now proved. Now following Kamthan (Theorem 11, [4]), the proof of Theorem 6 is obtained.

Remark. Theorems 1 and 2 of this paper can be generalized to proximate order similar to Theorems 1 and 2 of [7].

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