A MOST BIAS-ROBUST LINEAR ESTIMATE OF THE SCALE PARAMETER OF THE EXPONENTIAL DISTRIBUTION

If the original statistical exponential model is violated in such a way that the random variable under consideration is distributed with pdf 
\((1/\lambda\Gamma(1+1/p))\exp\{-\langle x/\lambda \rangle^p\}\) rather than \((1/\lambda)\exp\{-\langle x/\lambda \rangle\}\), then the sample mean, being MVUE in the original model, is a bias estimate of \(\lambda\). An estimate which is uniformly most bias-robust in the class of all linear estimates in such an extension of the model is constructed.

**Introduction and results.** Consider the statistical model

\[ M_1 = \left( R^+_1, \mathcal{B}^+_1, \{ P_{\lambda,1}, \lambda > 0 \} \right), \]

where \( R^+_1 \) is the real half-line, \( \mathcal{B}^+_1 \) is the family of Borel subsets of \( R^+_1 \), and \( P_{\lambda,1} \) is the exponential distribution with probability density function (pdf) \( f_{\lambda,1}(x) = (1/\lambda)\exp\{-\langle x/\lambda \rangle\} \). Consider the extension

\[ M_{p_1,p_2} = \left( R^+_1, \mathcal{B}^+_1, \{ P_{\lambda,p}, \lambda > 0, p_1 \leq p \leq p_2 \} \right) \]

of the model \( M_1 \), where \( 0 < p_1 \leq 1 \leq p_2 \leq 2.16 \) and \( P_{\lambda,p} \) is the exponential power distribution (a special case of the generalized gamma distribution) with pdf

\[ f_{\lambda,p}(x) = (1/\lambda\Gamma(1+1/p))\exp\{-\langle x/\lambda \rangle^p\}. \]

The reason for introducing a rather strange-looking number 2.16 will become clear in the sequel.

Some asymptotic problems of the robustness of confidence intervals in the extension \( M_{1,\infty} \) of \( M_1 \) have been considered by Pollock [1]. We confine ourselves to considerations of the robustness with respect to the bias of estimates of the scale parameter \( \lambda \).

Let \( X_1, X_2, \ldots, X_n \) be a sample from the underlying distribution. It is well known that the sample mean \( \bar{X}_n = \frac{\sum X_i}{n} \) is a minimum variance unbiased estimate of \( \lambda \) in the original model \( M_1 \). If the "true" distribution is \( P_{\lambda,p} \), then the bias of \( \bar{X}_n \) is equal to \( \mathbb{E}_{\lambda,p} \bar{X}_n - \lambda \), where \( \mathbb{E}_{\lambda,p} \bar{X}_n \) is the expected value of \( \bar{X}_n \) under the distribution \( P_{\lambda,p} \). Following a gen-
eral concept presented in [2] we define the function
\[ b_{\overline{X}_n}(\lambda) = \sup_{p_1 \leq p < p_2} (E_{1,p} \overline{X}_n - \lambda) - \inf_{p_1 < p < p_2} (E_{1,p} \overline{X}_n - \lambda) \]
which describes how much the bias of $\overline{X}_n$ changes when, given $\lambda$, the parameter $p$ runs over the interval $[p_1, p_2]$. Let $T$ be another estimate of $\lambda$. The estimate $T$ is more (bias-) robust than $\overline{X}_n$ at $\lambda = \lambda_0$ if $b_T(\lambda_0) < b_{\overline{X}_n}(\lambda_0)$ and is uniformly more robust than $\overline{X}_n$ if $b_T(\lambda) < b_{\overline{X}_n}(\lambda)$ for all $\lambda > 0$.

Given a sample size $n$ and a real-valued vector $a = (a_1, a_2, \ldots, a_n)$ consider the estimate
\[ T_n(a) = \sum_{j=1}^{n} a_j X_j^{(n)}, \]
where $X_j^{(n)}$ ($j = 1, 2, \ldots, n$) are order statistics. (Remind that the set of order statistics forms a minimal sufficient statistic in the model $M_{p_1, p_2}$, $p_1 < p_2$.) Of course, the sample mean $\overline{X}_n$ is a special case of $T_n(a)$. We prove the following

**Proposition.** $X_1^{(n)}/E_{1,1} X_1^{(n)}$ is the uniformly most robust estimate of $\lambda$ in every extension $M_{p_1, p_2}$ ($0 < p_1 \leq 1 \leq p_2 < 2.16$) of the model $M_1$, in the class of linear estimates $T_n(a)$, $a \geq 0$, which are unbiased in the original model $M_1$.

**Proof.** The proof consists in constructing an appropriate $T_n(a)$, $a \geq 0$.

The bias-robustness of $T_n(a)$ is described by the function
\[ b_{T_n(a)}(\lambda) = \lambda \left( \sup_{p_1 < p < p_2} \sum_{j=1}^{n} a_j E_{1,p} X_j^{(n)} - \inf_{p_1 < p < p_2} \sum_{j=1}^{n} a_j E_{1,p} X_j^{(n)} \right), \]
where $a = (a_1, a_2, \ldots, a_n)$ is a vector such that $T_n(a)$ is an unbiased estimate of $\lambda$ in $M_1$, i.e.,
\[ \sum_{j=1}^{n} a_j E_{1,1} X_j^{(n)} = 1. \]

The problem of constructing the uniformly most bias-robust estimate $T_n(a)$ reduces to finding such an $a$ which minimizes
\[ \sup_{p_1 < p < p_2} \sum_{j=1}^{n} a_j E_{1,p} X_j^{(n)} - \inf_{p_1 < p < p_2} \sum_{j=1}^{n} a_j E_{1,p} X_j^{(n)} \]
subject to (2) and to the condition
\[ a_j \geq 0, \quad j = 1, 2, \ldots, n. \]

Given $(j, n)$, the expectation $E_{1,p} X_j^{(n)}$ is a decreasing function of $p \in (0, 2.16)$. The upper bound $2.16$ is important because $\Gamma(1+1/p)$ is
strictly monotone in $p$ in this interval, and so is $F_{1,p}(x)$ for any fixed $x > 0$. A more exact upper bound for the interval of monotonicity is 2.1662276.

Now, we consider the following linear programming problem: minimize

$$\sum_{j=1}^{n} a_j E_{1,p_1} X_j^{(n)} - \sum_{j=1}^{n} a_j E_{1,p_2} X_j^{(n)}$$

under conditions (2) and (3).

All vertices of the polyhedron of $a$'s which satisfies (2) and (3) are of the form

$$\left(a_1 = 0, \ldots, a_{j-1} = 0, a_j = \frac{1}{E_{1,1} X_j^{(n)}}, a_{j+1} = 0, \ldots, a_n = 0\right),$$

$$j = 1, 2, \ldots, n.$$

We conclude that all but one coordinates of the optimal vector $a$ are equal to zero, and the index $j_0$ of the non-zero coordinate of the optimal vector is that which minimizes

$$\gamma_j^{(n)}(p_1, p_2) = \frac{E_{1,p_1} X_j^{(n)} - E_{1,p_2} X_j^{(n)}}{E_{1,1} X_j^{(n)}}.$$

Using the formula

$$E_{1,p} X_j^{(n)} = \frac{j \int_{0}^{1} F_{1,p}^{-1}(t) t^{j-1} (1-t)^{n-j} dt}{\int_{0}^{1} F_{1,1}^{-1}(t) t^{j-1} (1-t)^{n-j} dt},$$

where $F_{1,p}(x) = \int_{0}^{x} f_{1,p}(u) du$, we obtain

$$\gamma_j^{(n)}(p_1, p_2) = \int_{0}^{1} \frac{F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t)}{F_{1,1}^{-1}(t)} g_j^{(n)}(t) dt,$$

where

$$g_j^{(n)}(t) = \frac{F_{1,1}^{-1}(t) t^{j-1} (1-t)^{n-j}}{\int_{0}^{1} F_{1,1}^{-1}(t) t^{j-1} (1-t)^{n-j} dt}.$$

It is easy to see that for an appropriate number $t_{n,j} \in (0, 1)$ we have $g_j^{(n)}(t) > g_{j+1}^{(n)}(t)$ for $t < t_{n,j}$ and $g_j^{(n)}(t) < g_{j+1}^{(n)}(t)$ for $t > t_{n,j}$. It follows that if $(F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t))/F_{1,1}^{-1}(t)$ were an increasing function, we would have $\gamma_j^{(n)} < \gamma_{j+1}^{(n)}$. We show that this is the case. To this end it suffices to prove that $F_{1,q}^{-1}(t)/F_{1,1}^{-1}(t)$ is an increasing function in $t$ for $p < 1$ and a decreasing one for $p > 1$ or, letting $t = F_{1,p}(x)$ and $s_p(x) = F_{1,1}^{-1}(F_{1,p}(x))$, to prove that $x/s_p(x)$ increases with $x$ for $p < 1$ and decreases for $p > 1$. 
Consider the derivative
\[ \frac{d}{dx} \left( \frac{x}{s_p(x)} \right) = s_p^{-2}(x)\left[ s_p(x) - xs_p'(x) \right]. \]

Differentiating the identity
\[ \frac{1}{\Gamma(1+1/p)} \int_0^x e^{-u^p} \, du = \int_0^x e^{-u} \, du, \quad x \geq 0, \]
we obtain
\[ s_p'(x) = \frac{1}{\Gamma(1+1/p)} \exp\{s_p(x) - x^p\}, \quad s_p(0) = 0, \]
and
\[ s_p(x) = \frac{1}{\Gamma(1+1/p)} \int_0^x \exp\{s_p(u) - u^p\} \, du. \]

Hence
\[ \frac{d}{dx} \left( \frac{x}{s_p(x)} \right) = \frac{\int_0^x \exp\{s_p(u) - u^p\} \, du - x \exp\{s_p(x) - x^p\}}{\Gamma(1+1/p)s_p^2(x)}. \]

The integrand in the last formula equals 1 for \( u = 0 \) and is a decreasing (increasing) function for \( p < 1 \) (\( p > 1 \)); this follows from the inequalities
\[ px^{p-1} \int_0^\infty e^{-u^p} \, du \geq \int_0^\infty \frac{d(-e^{-u})}{u^p} = e^{-x^p} \quad \text{for} \quad p \leq 1 \]
when applied to
\[ \frac{d}{dx} \exp\{s_p(x) - x^p\} = \frac{f_{1,p}(x)}{[1 - F_{1,p}(x)]^2} \left( e^{-x^p} - px^{p-1} \int_x^\infty e^{-u^p} \, du \right). \]

As a consequence we obtain
\[ \frac{d}{dx} \left( \frac{x}{s_p(x)} \right) \geq 0 \quad \text{for} \quad p \leq 1, \]
which proves the monotonicity of \( x/s_p(x) \), and hence the monotonicity of \( F_{1,p}^{-1}(t)/F_{1,1}^{-1}(t) \). It follows that \( T_n = X_1^n/E_{1,1}X_1^n \) is the most bias-robust statistic in the class of all linear estimates \( T_n(a), \ a \geq 0 \), which are unbiased in the original model \( M_1 \).

**Some numerical results.** The bias-robustness of the estimate \( T_n \) in the model \( M_{p_1,p_2} \) is described by the function
\[ b_{T_n}(\lambda) = \lambda y_{1}^{(n)}(p_1, p_2). \]
For the exponential distribution with pdf \( f_{1,1}(x) \) we have \( E_{1,1}x_1^{(n)} = 1/n \) and by (4) and (5) we obtain

\[
\gamma_1^{(n)}(p_1, p_2) = n^2 \int_0^1 \left( F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t) \right) (1-t)^{n-1} \, dt.
\]

Given \((p_1, p_2), \ 0 < p_1 \leq 1 \leq p_2 \leq 2.16, \ F_{1,p_1}^{-1}(t) - F_{1,p_2}^{-1}(t)\) is an increasing function so that \(\gamma_1^{(n)}(p_1, p_2)\) is a decreasing function in \(n\). The values of \(\gamma_1^{(n)}(p_1, p_2)\) for some small \(n\) and some supermodels \(M_{p_1,p_2}\) are given in Table 1.

**Table 1**

<table>
<thead>
<tr>
<th>Values of (p_1) and (p_2)</th>
<th>(p_1 = 0.9)</th>
<th>(p_1 = 1)</th>
<th>(p_1 = 0.9)</th>
<th>(p_1 = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b(p_1, p_2))</td>
<td>0.178</td>
<td>0.118</td>
<td>0.296</td>
<td>0.436</td>
</tr>
<tr>
<td>(\gamma_1^{(n)}(p_1, p_2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(n = 2)</td>
<td>0.139</td>
<td>0.093</td>
<td>0.231</td>
<td>0.339</td>
</tr>
<tr>
<td>(n = 3)</td>
<td>0.121</td>
<td>0.081</td>
<td>0.201</td>
<td>0.290</td>
</tr>
<tr>
<td>(n = 4)</td>
<td>0.109</td>
<td>0.074</td>
<td>0.183</td>
<td>0.259</td>
</tr>
<tr>
<td>(n = 5)</td>
<td>0.101</td>
<td>0.070</td>
<td>0.171</td>
<td>0.238</td>
</tr>
<tr>
<td>(n = \infty)</td>
<td>0.052</td>
<td>0.035</td>
<td>0.087</td>
<td>0.114</td>
</tr>
</tbody>
</table>

The bias-robustness function (1) of the sample mean takes the form

\[
b_{\bar{x}_n}(\lambda) = \lambda b(p_1, p_2), \quad \text{where} \quad b(p_1, p_2) = \frac{\Gamma(2/p_1)}{\Gamma(1/p_1)} - \frac{\Gamma(2/p_2)}{\Gamma(1/p_2)}.
\]

The values of \(b(p_1, p_2)\) are given in Table 1. Note that the robustness of the sample mean does not depend on \(n\).

For large \(n\) the random variable \(2nF_{1,p}(X_1^{(n)})\) is distributed as \(\chi^2\) with 2 degrees of freedom so that \(E[F_{1,p}(X_1^{(n)})] \approx 1/n\). Moreover, \(X_1^{(n)}\) is small for \(n\) large enough so that

\[
F_{1,p}(X_1^{(n)}) \approx X_1^{(n)}/\Gamma(1+1/p).
\]

It follows that

\[
\gamma_1^{(n)}(p_1, p_2) \approx \Gamma(1+1/p_1) - \Gamma(1+1/p_2)
\]

asymptotically. The asymptotic values of this coefficient are given in the last row of Table 1.

**References**


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