ANNALES
POLONICI MATHEMATICI
XXI (1969)

On the total curvature of a closed curve

by M. Rochowski (Katowice)

1. Preliminaries. Let

(1) $$\varphi: C \to E^n$$

be an imbedding of the closed curve

(2) $$C = \{s, \operatorname{mod} L\}, \quad L > 0$$

in the Euclidean space E^n. The function φ is supposed to be of class C^∞. By $\varphi(s)e_1 e_2 \ldots e_n$ we denote the Frenet frame of the curve at the point $\varphi(s)$ and by $k_1, k_2, \ldots, k_{n-1}$ the consecutive curvatures of $\varphi(C)$. Then the Frenet formulas can be written as

(3) $$\frac{de_p}{ds} = -k_{p-1}e_{p-1} + k_p e_{p+1}, \quad k_0 = k_n = 0, \; p = 1, 2, \ldots, n.$$

Fenchel [4] shows that for $\varphi(C) \subset E^3$ we have

(4) $$\int_C k_1 ds \geqslant 2\pi,$$

and equality holds only for convex plane curves. Borsuk [1] proved (4) in the case of $\varphi(C) \subset E^n$. In the same paper the author raises the question whether or not

(5) $$\int_C k_1 ds \geqslant 4\pi$$

holds for each knotted curve $\varphi(C) \subset E^3$? This problem was given a positive answer by Fary [3] and Milnor [5].

Fenchel's proof is synthetic and other proofs are based on a construction of a sequence of polygons inscribed in $\varphi(C)$. In this note we give a new proof of (4) if $\varphi(C) \subset E^n$ and (5) if $\varphi(C) \subset E^3$ in which the polygonal construction is omitted.

2. Proof of the inequality (4). Let B_ν be the normal bundle of the curve (2) induced by the imbedding (1). This means that

$$B_\nu = \left\{(s, \nu) \,|\, \nu \cdot \frac{d\varphi(s)}{ds} = 0\right\},$$

where $\nu(s, t)$ is an arbitrary unit vector normal to $\varphi(C)$ at $\varphi(s)$. The vector ν can be written as

$$(6)\qquad \nu(s, t) = t^a e_a\,, \qquad t = (t^2, t^3, \ldots, t^n)\,, \qquad \sum_{a=2}^{n}(t^a)^2 = 1 \qquad (a = 2, \ldots, n)\,.$$

Let

$$(7)\qquad f\colon B_\nu \to E^n$$

be the mapping defined by

$$(8)\qquad f(s, t) = \varphi(s) + \varepsilon\nu(s, t) = \varphi(s) + \varepsilon t^a e_a\,, \qquad \varepsilon = \text{const}\,.$$

If $|\varepsilon|$ is sufficiently small, then (8) is an imbedding. First we prove that (8) is an immersion for small $|\varepsilon|$.

In fact, if we take, for example, $t^n = \pm\sqrt{1-(t^2)^2 - \ldots - (t^{n-1})^2}$, then

$$\frac{\partial\varphi}{\partial s} = \frac{d\varphi}{ds} + \varepsilon t^a \frac{de_a}{ds} = \frac{d\varphi}{ds} + \varepsilon t^a(-k_{a-1}e_{a-1} + k_a e_{a+1})\,,$$

$$\frac{\partial\varphi}{\partial t^a} = \varepsilon\left(e_a + \frac{\partial t^n}{\partial t^a}e_n\right), \qquad (a = 2, \ldots, n-1)\,.$$

The coefficient of $[e_1 e_2 \ldots e_{n-1}]$ in the development of the vector product

$$\left[\frac{\partial\varphi}{\partial s}\frac{\partial\varphi}{\partial t^2}\cdots\frac{\partial\varphi}{\partial t^{n-1}}\right]$$

is

$$(9)\qquad \varepsilon^{n-2}(1-\varepsilon t^2 k_1)\,.$$

Because C is compact, the curvature k_1 is bounded and (9) is different from zero for sufficiently small $|\varepsilon|$. This proves our statement.

Suppose, to the contrary, that the mapping (8) is not one to one for each ε. Then we can choose such convergent sequences

$$(s'_k, \nu'_k),\ (s''_k, \nu''_k) \in B_\nu\,,$$

that

$$(10)\qquad f_k(s'_k, t'_k) = f_k(s''_k, t''_k)$$

if ε_k tends to zero with $k\to\infty$, and f_k denotes the mapping (8), in which $\varepsilon = \varepsilon_k$ and $\nu'_k = \nu(s'_k, t'_k)$, $\nu''_k = \nu(s''_k, t''_k)$, and if $s'_k \to s'_0$, $s''_k \to s''_0$. Then f_k tends to φ and we have

$$(11)\qquad \varphi(s'_0) = \varphi(s''_0)\,.$$

There follows $s'_0 = s''_0$. Indeed, $s'_0 \neq s''_0$ and (11) would contradict the fact that φ is an imbedding. On the other hand, (8) is an immersion for small $|\varepsilon|$. This means that for small $|\varepsilon|$ and $\delta > 0$ the points $f(s', t')$, $f(s'', t'')$ are distinct for $0 < |s'-s''| < \delta$ and all t', t''. But this contradicts (10).

From now on we suppose that ε is a constant for which (8) is an imbedding.

The imbedding (8) leads to the spherical mapping

$$g: B_\nu \to S^{n-1} \tag{12}$$

defined by the formula

$$(s, \nu) \to \varphi(s) + \varepsilon\nu(s, t) \to \nu(s, t) . \tag{13}$$

From (6) we have

$$d\nu = t^\alpha de_\alpha + e_\alpha dt^\alpha = t^\alpha(-k_{\alpha-1}e_{\alpha-1} + k_\alpha e_{\alpha+1})ds + e_\alpha dt^\alpha .$$

In S^{n-2} we introduce the spherical coordinates

$$t^2 = \cos\theta^2 ,$$
$$t = \sin\theta^2 \sin\theta^3 \ldots \sin\theta^{\alpha-1}\cos\theta^\alpha , \qquad \alpha = 3, \ldots, n-1 ,$$
$$t^n = \sin\theta^2 \ldots \sin\theta^{n-1} , \qquad 0 < \theta^\beta \leqslant \pi , \ \beta = 2, \ldots, n-2 , \ 0 \leqslant \theta^{n-1} < 2\pi .$$

Then we can write

$$d_1\nu \overset{\mathrm{df}}{=} t^\alpha de_\alpha = t^\alpha(-k_{\alpha-1}e_{\alpha-1} + k_\alpha e_{\alpha+1})ds ,$$

$$d_\beta\nu \overset{\mathrm{df}}{=} \frac{\partial t^\alpha}{\partial\theta^\beta}e_\alpha d\theta^\beta , \qquad \beta = 2, \ldots, n-1, \ \alpha = 2, \ldots, n .$$

Hence, for the vector product of the above vector we have

$$[d_1\nu d_2\nu \ldots d_{n-1}\nu]$$
$$= k_1 \sin^{n-3}\theta^2 \cos\theta^2 \sin^{n-4}\theta^3 \ldots \sin\theta^{n-2} t^\alpha e_\alpha ds d\theta^2 \ldots d\theta^{n-1} , \qquad \alpha = 2, \ldots, n .$$

Let $d\sigma_k$ denote the surface element of S^k. Then

$$d\sigma_{n-1} = |[d_1\nu d_2\nu \ldots d_{n-1}\nu]| = k_1 \sin^{n-3}\theta^2 |\cos\theta^2| d\sigma_{n-3} ds d\theta^2 .$$

We assume $d\sigma_0 = 1$.

The point $(s_0, \nu_0) \in B_\nu$ is said to be a critical point of the mapping (12) if its Jacobian determinant vanishes at this point. By Sard's theorem [6] it is known that the image of the set of critical points is of measure zero in S^{n-1}. The Jacobian determinant vanishes if and only if the scalar product $\nu_0 \cdot d^2\varphi/ds^2 = k_1 \sin\theta^2 \cos\theta^2$ vanishes. On the other hand, the point $(s_0, \nu_0) \in B_\nu$ is said to be a critical point [7] of the real-valued function $\nu_0 \cdot \varphi(s)$ ($\nu_0 = \text{const}$) if $\nu_0 d\varphi = 0$. The point is said to be a degenerated critical point if $\nu_0 \cdot d\varphi = \nu_0 d^2\varphi = 0$. Thus, if $(s_0, \nu_0) \in B_\nu$ is a degenerated critical point of $\nu_0 \cdot \varphi(s)$, then it is a critical point of (12).

Hence, for each vector $\nu \in S^{n-1}$, except a set of measure zero, the function $\nu \cdot \varphi(s)$ has only non-degenerated critical points. From the Morse equality [7] it is known that there are an even number of such points for each vector ν.

Now let $m = m(t^1, \ldots, t^n)$ denote the number, divided by two, of non-degenerated critical points of the function $\nu \cdot \varphi(s)$ for each vector $\nu = \nu(t^1, \ldots, t^n) \in E^n$ (except a set of measure zero). To each critical point of C there corresponds for the same vector ν two critical points of B_ν. Hence the sphere S^{n-1} is covered by the mapping (12) $2m$ times. Therefore

$$\int_{B_\nu} k_1 \sin^{n-3}\theta^2 |\cos\theta^2|\, d\sigma_{n-3}\, ds\, d\theta^2 = \int_{S^{n-1}} 2m\, d\sigma_{n-1}\,.$$

If $m(t^1, \ldots, t^n) \geqslant m_0$ ($\geqslant 1$), then for $n > 2$ we can write

$$\int_0^L k_1\, ds \int_0^{\pi} \sin^{n-3}\theta^2 |\cos\theta^2|\, d\theta^2 \int_{S^{n-3}} d\sigma_{n-3} \geqslant 2m_0 \int_{S^{n-1}} d\sigma_{n-1}\,,$$

and for $n = 2$

$$\int_0^L k_1\, ds \int_0^{2\pi} |\cos\theta^2| d\theta^2 \geqslant 2m_0 \int_{S^2} d\sigma_2\,.$$

Denoting by ω_k the surface area of S^k and using the equality $\omega_{n-1}/\omega_{n-3} = 2\pi/n-2$ we have

$$\int_0^L k_1\, ds \geqslant 2\pi m_0\,.$$

This proves (4).

3. The imbedded curve $\varphi(C) \subset E^3$ is said to be knotted if there exists no homeomorphism $h: E^3 \to E^3$ transforming $\varphi(C)$ onto a plan circle. Tne quality (5) will be proved if we show that the condition $m_0 \geqslant 2$ (except a set of measure zero of vectors $\nu \in S^2$) is necessary for a curve to be knotted.

Suppose, to the contrary, that there exists a vector ν for which the function $\nu \cdot \varphi(s)$ has only two non-degenerated critical points, i.e. $m_0 = 1$. Since $\nu \cdot d^2\varphi/ds^2$ is continuous, a small change of ν leaves m_0 constant. Therefore we can assume that $\nu \cdot \varphi(s)$ has only non-degenerated critical points. Let $s_1, s_2 \in C$ denote the critical points of $\nu \cdot \varphi(s)$. These points divide C, just as $\varphi(s_1), \varphi(s_2) \in \varphi(C)$ divides $\varphi(C)$, into two arcs l_1, l_2 which have only the end-points in common. On each of this arcs the function $\nu \cdot \varphi(s)$ is strictly monotone. Therefore each plane $\nu x = c$, where $x \in E^3$ and $\nu \cdot \varphi(s_2) < c < \nu \cdot \varphi(s_1)$, intersects $\varphi(C)$ at exactly two points. These points we join by a line segment and construct its symmetry line. Through $\varphi(s_1)$, $\varphi(s_2)$ we drew lines which are the limit positions of the symmetry lines. Since φ is a regular function, such lines exist. These two lines we take as the boundaries of two halfplanes H_1, H_2, such that H_1 lies in the halfspace

$$H_1 \subset \{x \in E^3 | \nu x > \nu\varphi(s_1)\} ,$$

$$H_2 \subset \{x \in E^3 | \nu x < \nu\varphi(s_2)\} .$$

The orientation of E^3 and the vector ν determines the orientations of all planes $\nu x = c, -\infty < c < \infty$. Therefore, since C is compact, we can choose such a finite open covering by intervals $\{U_i\}_{1 \leqslant i \leqslant l}$ of the closed interval $[\nu \cdot \varphi(s_2), \nu \cdot \varphi(s_1)]$ that for each i and arbitrary $c', c'' \in U_i$ the absolute value of the angle between the projection of the positively oriented symmetry line of $\nu x = c''$ onto the plane $\nu x = c'$ and the positively oriented symmetry line of the last is less than $\pi/2$.

Now we define a homeomorphic mapping $h: E^3 \to E^3$ in the following manner: In E^3 we choose such a coordinate system that $\varphi(s_2) = (0, 0, 0)$, the limit line of symmetry lines which passes through $\varphi(s_2)$ is identical with the coordinate line $0x^1$ and $\nu x = \nu \cdot \varphi(s_2)$ is the $0x^1x^2$ plane. Assuming $U_i \cap U_{i+1} \neq \emptyset$ $(i = 1, 2, \ldots, l-1)$ let $\nu \cdot \varphi(s_2) \in U_1$ and suppose that h is defined for $k = 1, 2, \ldots, i-1 < l-1$ in such a way that the image of $\{x \in E^3 | \nu x = c\}, c \in \bigcup_{j=1}^{i-1} U_j$ is the plane itself and the image of the symmetry line of this plane is the line $x^2 = 0, x^3 = c$. If $c \in U_{i-1} \cap U_i$ and $c' \in U_i$ is arbitrary, then there exists such a rotation of the plane $\{x \in E^3 | \nu x = c'\}$ onto itself about an angle $-\pi/2 < \alpha < \pi/2$ that the symmetry line in $\{x \in E^3 | \nu x = c'\}$ assumes a position parallel to the symmetry line in $\{x \in E^3 | \nu x = c\}$ after it. Up to a translation of $\{x \in E^3 | \nu x = c'\}$ onto itself we may assume that the symmetry lines corresponds by orthogonal projection of this plane onto the other. Therefore, since h is defined for c, it can be extended for the values $c' \in U_i$ by setting that the image of $\{x \in E^3 | \nu x = c'\}$ is the plane itself with the same orientation and the image of the symmetry line is the line $x^2 = 0, x^3 = c'$. Thus, by induction, h is defined for $1 \leqslant i \leqslant l$. The extension of h to the whole space is now easy. Namely, since h is defined for the limit lines of the symmetry lines which pass through $\varphi(s_2)$ and $\varphi(s_1)$, it is defined for each plane $\{x \in E^3 | \nu x = c\}$ $c > \nu \cdot \varphi(s_1)$ or $c < \nu \cdot \varphi(s_2)$ by setting that the image of the intersection of the plane $\{x \in E^3 | \nu x = c\}$ with H_1 or H_2 is the line $x^2 = 0, x^3 = c$ (the plane $\{x \in E^3 | \nu x = c\}$ is displaced as above into itself).

Under the mapping h the arcs l_1, l_2 are mapped onto arcs $h\varphi(l_1)$, $h\varphi(l_2)$ which lie on different sides of the plane $0x^1x^3$ and each plane $\{x \in E^3 | \nu x = c\}$, $\nu \cdot \varphi(s_2) < c < \nu \cdot \varphi(s_1)$, intersects $h\varphi(l_1) \cup h\varphi(l_2) = h\varphi(C)$ at exactly two points. The orthogonal projection p of $h\varphi(C)$ onto the plane $0x^1x^2$ can easily be extended to a homeomorphic mapping p: $E^3 \to E^3$ and $ph\varphi(C)$ is a closed plane curve homeomorphic with the circle.

Thus the assumption that there exists a direction ν for which $\nu\cdot\varphi(s)$ has only two non-degenerated critical points yields a contradiction.

Remark. If $\varphi(C)\subset E^n$ does not lie in a plane, then there exists such a vector ν that $\nu\cdot\varphi(s)$ has at least four non-degenerated critical points. This fact can be shown by an argument similar to that used in the proof of lemma 1 in [2]. From this it follows that

$$\int_0^L k_1\,ds > 2\pi\,,$$

if $\varphi(C)\not\subset E^2$, and equality in (4) holds only for the convex plane curves.

References

[1] K. Borsuk, *Sur la courbure totale des courbes*, Ann. Soc. Polon. Math. 20 (1947), pp. 251-265.

[2] S. S. Chern and R. K. Lashof, *On the total curvature of immersed manifolds*. Amer. Journ. Math. 79 (1957), pp. 306-318.

[3] M. I. Fary, *Sur la courbure totale d'une courbe gauche faisant un noeud*, Bull. Soc. Math. France 77 (1949), pp. 128-138.

[4] W. Fenchel, *Über Krümmung und Windung geschlossener Raumkurven*, Math. Ann. 101 (1929), pp. 238-252.

[5] J. W. Milnor, *On the total curvature of knots*, Ann. of Math. 52 (1950), pp. 248-257.

[6] — *Topology from the differentiable viewpoint*, 1965.

[7] H. Seifert und W. Threlfall, *Variations rechnung im grossen*, 1948.

Reçu par la Rédaction le 12. 3. 1968