ANNALES
POLONICI MATHEMATICI
XXIV (1971)

# On the continuous dependence of local analytic solutions of the functional equation in the non-uniqueness case

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In this paper we shall study the problem of continuous dependence on given functions of local analytic solutions of the equation

$$\Phi(z) = H\big(z, \Phi[f(z)]\big),$$

where $f(z)$ and $H(z, w)$ are given functions and $\Phi$ is the required function.

We consider the sequence of equations

$$\Phi(z) = H_n\big(z, \Phi[f_n(z)]\big) \tag{1}$$

supposing that for $n = 0, 1, 2, \ldots$ the following hypotheses are fulfilled:

(I) $f_n(z)$ is analytic in the disc $|z| \leqslant r$, $f_n(0) = 0$ and

$$|f_n'(0)| \leqslant \vartheta < 1;$$

(II) $H_n(z, w)$ is an analytic function of two complex variables $(z, w)$ for $|z| \leqslant r$, $|w| \leqslant R$ and $H_n(0, 0) = 0$;

(III) $f_n(z)$ tends to $f_0(z)$ and $H_n(z, w)$ tends to $H_0(z, w)$ uniformly for $|z| \leqslant r$, $|w| \leqslant R$.

By (I) and (III) there exists a positive integer $p$ such that

$$\left|[f_n'(0)]^p \frac{\partial H_n}{\partial w}(0, 0)\right| < 1 \quad \text{for } n = 0, 1, 2, \ldots \tag{2}$$

It follows by Smajdor's theorem [3] (also [1], p. 188) that if hypotheses (I) and (II) are fulfilled, then every formal solution

$$\Phi_n(z) = \sum_{k=1}^{\infty} \frac{\Phi_n^{(k)}(0)}{k!} z^k \tag{3}$$

of equation (1) has a positive radius of convergence. Moreover, there exists exactly one formal solution iff

$$\text{(IV)} \quad [f_n'(0)]^k \frac{\partial H_n}{\partial w}(0, 0) \neq 1 \quad \text{for } k = 1, 2, \ldots, p-1.$$

In [2] the following theorem on continuous dependence has been proved:

THEOREM I. *Under assumptions* (I)-(IV) *each of equations* (1) *for* $n = 0, 1, 2, \ldots$ *has exactly one solution* $\Phi_n(z)$ *analytic in a common neighbourhood of the origin and* $\Phi_n(z)$ *tends to* $\Phi_0(z)$ *in this neighbourhood.*

If hypothesis (IV) is not fulfilled, then either equation (1) has a one-parameter family of analytic solutions or there exist no analytic solutions (cf. [3], also [1], p. 188).

In this paper we consider the problem of continuous dependence in the case where (IV) is not fulfilled for some $n$.

**1.** We define the functions $H_{n,k}(z, w, w_1, \ldots, w_k)$ by the recurrent relations

$$H_{n,1}(z, w, w_1) = \frac{\partial H_n}{\partial z} + f_n'(z)\frac{\partial H_n}{\partial w} w_1, \tag{4}$$

$$H_{n,k+1}(z, w, w_1, \ldots, w_{k+1}) = \frac{\partial H_{n,k}}{\partial z} + f_n'(z)\left(\frac{\partial H_{n,k}}{\partial w} w_1 + \ldots + \frac{H_{n,k}}{w_k} w_{k+1}\right).$$

Denote by $C^n$ the Cartesian product of $n$ complex planes $C$ and by $D$ the set

$$D = \{(z, w) : |z| \leqslant r,\ |w| \leqslant R\}.$$

We have the following two lemmas (cf. [3], also [1], p. 188):

LEMMA 1. *Let hypotheses* (I) *and* (II) *be fulfilled. Then the expressions* $H_{n,k}$ *are defined and analytic in* $D \times C^k$, *and we have*

$$\begin{aligned} H_{n,k}(z, w, w_1, \ldots, w_k) &= G_{n,k}(z, w, w_1, \ldots, w_{k-1}) + \\ &+ [f_n'(z)]^k \frac{\partial H_n}{\partial w}(z, w) w_k, \qquad k = 1, 2, \ldots, \end{aligned} \tag{5}$$

*where* $G_{n,k}$ *is an analytic function in the set* $D \times C^{k-1}$.

LEMMA 2. *Let hypotheses* (I) *and* (II) *be fulfilled. If* (3) *is a formal solution of equation* (1), *then we have*

$$\Phi_n^{(k)}(0) = H_{n,k}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(k)}(0)\big), \qquad k = 1, 2, \ldots \tag{6}$$

By induction from (III) and (4) we obtain

LEMMA 3. *If hypotheses* (I)-(III) *are fulfilled, then* $H_{n,k}$ *tends to* $H_{0,k}$ *and* $G_{n,k}$ *tends to* $G_{0,k}$ *uniformly in* $D \times C^k$ *resp.* $D \times C^{k-1}$.

Lemmas 1 and 2 together with W. Smajdor's theorem imply

LEMMA 4. *Let hypotheses* (I) *and* (II) *be fulfilled and suppose that there exists a positive integer* $k = k(n)$ *such that*

$$[f_n'(0)]^k \frac{\partial H_n}{\partial w}(0, 0) = 1. \tag{7}$$

*Then equation* (1) *has an analytic solution of form* (3) *if and only if*

$$G_{n,k}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(k-1)}(0)\big) = 0, \tag{8}$$

*where the numbers* $\Phi_n'(0), \ldots, \Phi_n^{(k-1)}(0)$ *are uniquely determined by the condition*

$$\Phi_n^{(s)}(0)\left(1-[f_n'(0)]^s \frac{\partial H_n}{\partial w}(0, 0)\right) = G_{n,k}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(s-1)}(0)\big), \tag{9}$$

$$s = 1, 2, \ldots$$

*If* (7) *and* (8) *are fulfilled, then there exists a one-parameter family of analytic solutions of equation* (1) *depending on the parameter* $t = \Phi_n^{(k)}(0)$.

Now we prove

LEMMA 5. *Suppose that hypotheses* (I)-(III) *are fulfilled and that for* $n = 1, 2, \ldots$ *relations* (7) *hold with* $k(n) = k(1) = k$ (*i.e.,* $k$ *is independent of* $n$). *Then* (7) *holds for* $n = 0$. *Moreover, if we assume that*

$$\Phi_n^{(k)}(0) = t_n; \quad n = 0, 1, 2, \ldots; \quad \lim_{n\to\infty} t_n = t_0, \tag{10}$$

*then we have*

$$\lim_{n\to\infty} \Phi_n^{(s)}(0) = \Phi_0^{(s)}(0), \quad s = 1, 2, \ldots \tag{11}$$

Proof. From Lemma 3, (9) and (10) we obtain relation (11) by induction. Now in view of (III) and Lemma 3 we obtain relation (7) for $n = 0$. This completes the proof.

LEMMA 6. *Suppose that hypotheses* (I)-(III) *are fulfilled and that for every* $n, n = 1, 2, \ldots,$ *there exists a positive integer* $k(n)$ *such that*

$$f_n'(0)^{k(n)} \frac{\partial H_n}{\partial w}(0, 0) = 1; \tag{12}$$

*then there exists a positive integer* $n_0$ *such that* $k(n) = k(n_0) = k$ *for* $n \geqslant n_0$ *and*

$$[f_0'(0)]^k \frac{\partial H_0}{\partial w}(0, 0) = 1. \tag{13}$$

Proof. According to (III) there exists an $A > 0$ such that

$$\left|\frac{\partial H_n}{\partial w}(0, 0)\right| < A \quad \text{for } n = 1, 2, \ldots$$

Hence and on account of hypothesis (I) we obtain

$$1 = \left|[f_n'(0)]^{k(n)} \frac{\partial H_n}{\partial w}(0, 0)\right| < A\vartheta^{k(n)}.$$

Since $0 < \vartheta < 1$, the sequence $k(n)$ is bounded. If the lemma were not true, then there would exist two positive integers $l$ and $m$, $l \neq m$ and two sequences of positive integers $m_\nu$ and $l_\nu$ such that

$$[f'_{l_\nu}(0)]^l \frac{\partial H_{l_\nu}}{\partial w}(0,0) = 1, \quad [f'_{m_\nu}(0)]^m \frac{\partial H_{m_\nu}}{\partial w}(0,0) = 1.$$

Letting $\nu \to \infty$ in the above relations, we obtain

$$[f'_0(0)]^l \frac{\partial H_0}{\partial w}(0,0) = 1, \quad [f'_0(0)]^m \frac{\partial H_0}{\partial w}(0,0) = 1.$$

Hence $[f'_0(0)]^{l-m} = 1$. This contradicts hypothesis (I) and the lemma is proved.

**2.** In this section we consider the problem of continuous dependence supposing that in the sequence of equations (1) some have a solution containing the parameter.

If relation (IV) holds for $n = 0$, then it follows by (III) that (IV) holds also for $n \geqslant N$, i.e. equation (1) has for $n \geqslant N$ exactly one solution $\Phi_n(z)$ having form (3) and analytic in a neighbourhood of the origin. Then it follows by Theorem I that the sequence $\Phi_n(z)$ coverges to $\Phi_0(z)$ uniformly in a neighbourhood of the origin.

Now suppose that (IV) is not fulfilled for $n = 0$, i.e. that there exists a positive integer $k$ such that relation (7) holds for $n = 0$. According to Lemma 4, if for $n = 0$ there exists an analytic solution of equation (1), then it depends on the parameter $t = \Phi_0^{(k)}(0)$.

Evidently, it is sufficient to consider the following two cases.

A. For every $n$, $n = 1, 2, \ldots$, equation (1) has a one-parameter family of analytic solutions, i.e. relations (7) and (8) hold for every $n$ with $k = k(n)$.

B. For every $n$, $n = 1, 2, \ldots$, equation (1) has exactly one analytic solution, i.e. (IV) holds for $n = 1, 2, \ldots$

We shall prove the following

THEOREM 1. *Let hypotheses* (I)-(III) *be fulfilled and suppose that case* A *occurs. Then equation* (1) *has for* $n = 0, 1, 2, \ldots$ *a one-parameter family of analytic solutions* $\Phi_n(z, t)$, *where* $t$ *is a parameter. These solutions exist in a common neighbourhood of the origin, independent of* $n$ *and* $t$. *Moreover, if we fix* $t = t_n$ *for* $n = 0, 1, 2, \ldots$ *such that*

$$\lim_{n\to\infty} t_n = t_0, \tag{14}$$

*then the solution* $\Phi_n(z) \stackrel{\mathrm{df}}{=} \Phi_n(z, t_n)$ *converges to* $\Phi_0(z) = \Phi_0(z, t_0)$ *uniformly in this neighbourhood.*

Proof. It follows by Lemma 4 that relations (7) and (8) hold for $n = 1, 2, \ldots$ By Lemma 6 there exists an $n_0$ such that for $n \geqslant n_0$ we have $k(n) = k(n_0) = k$. It is no restriction to suppose that $n_0 = 1$. Now, according to Lemma 3 and (III) relations (7) and (8) hold also for $n = 0$ ($k(0) = k$), which means that equation (1) for $n = 0$ has a one-parameter family of analytic solutions in a neighbourhood of the origin. Evidently, we have $k < p$, where $p$ is defined by (2), and in view of Lemma 4 we can write

$$\Phi_n(z, t_n) = P_n(z) + z^p \varphi_n(z), \tag{15}$$

where

$$P_n(z) = \sum_{\substack{s=1 \\ s \neq k}}^{p} \frac{\Phi_n^{(s)}(0)}{s!} z^s + \frac{t_k}{k!} z^k, \tag{16}$$

$\varphi_n(z)$ is analytic in a neighbourhood of the origin and $\varphi_n(0) = 0$. It follows by Lemma 5 and (14) that $P_n(z)$ converges to $P_0(z)$ uniformly on every compact set contained in $C$. Thus for the proof of our theorem it is sufficient to show that $\varphi_n(z)$ converges to $\varphi_0(z)$ uniformly in a neighbourhood of the origin.

Let us define the functions

$$h_n(z, v) = \frac{H_n\big(z, P_n[f_n(z)] + [f_n(z)]^p v\big) - P_n(z)}{z^p}. \tag{17}$$

By (I) and (II) the partial derivative

$$\frac{\partial h_n}{\partial v}(z, v) = \frac{\partial H_n}{\partial w}(z, w)\left(\frac{f(z)}{z}\right)^p, \quad w = P_n[f_n(z)] + [f_n(z)]^p v, \tag{18}$$

is analytic function in a neighbourhood of the point $(z, v) = (0, 0)$.

Next we put

$$g(z) = H_n\big(z, P_n[f_n(z)]\big) - P_n(z). \tag{19}$$

We shall show that

$$g^{(l)}(z) = H_{n,l}\big(z, P_n[f_n(z)], P_n'[f_n(z)], \ldots, P_n^{(l)}[f_n(z)]\big) - P_n^{(l)}(z). \tag{20}$$

In fact, we have by (4)

$$\begin{aligned} g'(z) &= \frac{\partial H_n}{\partial z}\big(z, P_n[f_n(z)]\big) + \frac{\partial H_n}{\partial w}\big(z, P_n[f_n(z)]\big) f_n'(z) P_n'[f_n(z)] - P_n'(z) \\ &= H_{n,1}\big(z, P_n[f_n(z)], P_n'[f_n(z)]\big) - P_n'(z). \end{aligned}$$

Thus (20) holds for $l = 1$. We assume that (20) holds for an $l \geqslant 1$. Hence we get

$$g^{(l+1)}(z) = \frac{\partial H_{n,l}}{\partial z} + f_n'(z)\left(\frac{\partial H_{n,l}}{\partial w} P_n'[f_n(z)] + \ldots + \frac{\partial H_{n,l}}{\partial w_l} P_n^{(l+1)}[f_n(z)]\right) - $$
$$-P_n^{(l+1)}(z)$$
$$= H_{n,l+1}\big(z, P_n[f_n(z)], P_n'[f_n(z)], \ldots, P_n^{(l+1)}[f(z)]\big) - P_n^{(l+1)}(z)$$

and (20) is proved.

From (15) we obtain $P_n^{(l)}(0) = \Phi_n^{(l)}(0)$, $l = 1, 2, \ldots, p$ and $\Phi_n^{(k)}(0) = t_n$. Now, putting $z = 0$ in (20), we have by (6)

$$g^{(l)}(0) = H_{n,l}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(l)}(0)\big) - \Phi_n^{(l)}(0) = 0$$

for $l = 1, \ldots, p$, so $h_n(z, 0)$ is an analytic function of $z$ at the origin. By (18) we have

$$h_n(z, v) = \int_0^v \frac{\partial h_n}{\partial u}(z, u)\,du + h_n(z, 0);$$

thus $h_n(z, v)$ is an analytic function in a neighbourhood of the point $(0, 0)$. Moreover, we have

$$h_n(0, 0) = 0 \tag{21}$$

and $\varphi_n(z)$ defined by relation (15) satisfies the equation

$$\varphi(z) = h_n\big(z, \varphi[f_n(z)]\big). \tag{22}$$

Let us take an arbitrary $R_1 > 0$ and let $|v| \leqslant R_1$. Since $P_n(0) = f_n(0) = 0$, there is a $\sigma_1 > 0$, $\sigma_1 \leqslant r$, such that

$$|P_0[f_0(z)]| < \frac{R}{2} \quad \text{and} \quad |f_0(z)|^p < \frac{R}{2R_1} \quad \text{for } |z| \leqslant \sigma_1.$$

In virtue of the uniform convergence of $P_n(z)$ to $P_0(z)$ and $f_n(z)$ to $f_0(z)$ there exists a positive integer $N$ such that

$$|P_n[f_n(z)]| \leqslant \frac{R}{2} \quad \text{and} \quad |f_n(z)|^p \leqslant \frac{R}{2R_1} \quad \text{for } |z| \leqslant \sigma_1,\ n \geqslant N.$$

From the continuity of $P_n[f_n(z)]$ and $f_n(z)$ there exists a $\sigma_2 > 0$ such that these inequalities are valid for $n = 1, \ldots, N-1$ and $|z| \leqslant \sigma_2$. Taking $r_1 = \min(\sigma_1, \sigma_2)$ we have

$$|P_n[f_n(z)] + [f_n(z)]^p v| \leqslant \frac{R}{2} + \frac{R}{2R_1} R_1 = R$$

for $|z| \leqslant r_1$, $|v| \leqslant R_1$ $(n = 0, 1, 2, \ldots)$ and, moreover,

(23) $h_n(z, v)$ tends uniformly to $h_0(z, v)$ for $|z| \leqslant r_1$, $|v| \leqslant R_1$.

Functions $h_n(z, v)$ fulfil hypotheses (II) and (III). Since by (18) we have

$$\left|\frac{\partial h_n}{\partial v}(0,0)\right| = \left|[f_n'(0)]^p \frac{\partial H_n}{\partial w}(0,0)\right| < 1,$$

$h_n(z, v)$ satisfies also hypotheses (IV). Thus $\varphi_n(z)$ is the unique analytic solution of equation (22) such that $\varphi_n(0) = 0$. We may apply Theorem I and this completes the proof.

In case B we have the following

THEOREM 2. *Let hypotheses* (I)-(III) *be fulfilled. Suppose that equation* (1) *has for $n = 0$ an analytic solution $\Phi_0(z, t)$ containing the parameter $t$ (i.e. relations* (7) *and* (8) *hold for $n = 0$) and suppose that equation* (1) *has for $n = 1, 2, \ldots$ exactly one analytic solution $\Phi_n(z)$ in a neighbourhood of the origin. Then the sequence $\Phi_n(z)$ converges uniformly in a neighbourhood of the origin if and only if there exist a limit*

$$c = \lim_{n\to\infty} \frac{G_{n,k}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(k-1)}(0)\big)}{1-[f_n'(0)]\dfrac{\partial H_n}{\partial w}(0,0)}, \qquad k = k(0). \tag{24}$$

*Moreover, if limit* (24) *exists, then $\Phi_n(z)$ tends to $\Phi_0(z) \overset{\mathrm{df}}{=} \Phi_0(z, c)$ uniformly in a neighbourhood of the origin.*

Proof. If $\Phi_n(z)$ converges uniformly in a neighbourhood of the origin, then the sequence $\Phi_n^{(k)}(0)$ converges as $n \to \infty$. By relation (9) and (7) we have

$$\Phi_n^{(k)}(0) = \frac{G_{n,k}\big(0, 0, \Phi_n'(0), \ldots, \Phi_n^{(k-1)}(0)\big)}{1-[f_n'(0)]^k\dfrac{\partial H_n}{\partial w}(0,0)}.$$

This completes the proof of the necessity.

Now suppose that (24) holds. Let us write the solution $\Phi_n(z)$ in form (15), where

$$P_n(z) = \sum_{s=1}^{p} \frac{\Phi_n^{(s)}(0)}{s!} z^s, \qquad n = 1, 2, \ldots,$$

and

$$P_0(z) = \sum_{\substack{s=1 \\ s=k}}^{p} \frac{\Phi_0^{(s)}(0)}{s!} z^s + \frac{c}{k!} z^k.$$

It is seen by (9), Lemma 3 and (24) that $\Phi_n^{(s)}(0)$ tends to $\Phi_0^{(s)}(0)$ for $s \neq k$ and that $\Phi_n^{(k)}(0)$ tends to $c$. Thus $P_n(z)$ tends uniformly on every compact set in $C$ to $P_0(z)$. Now, similarly to the proof of Theorem 1, we can prove that $\varphi_n(z)$ tends to $\varphi_0(z)$ uniformly in a neighbourhood of the origin. This completes the proof.

EXAMPLE. Take the sequence of equations

$$\Phi(z) = (2-1/n)\Phi[(\tfrac{1}{2}-1/n)z]+z/n, \quad n = 1, 2, \ldots \tag{25}$$

Here

$$f_n(0) = (\tfrac{1}{2}-1/n)z, \quad H_n(z, w) = (2-1/n)w+z/n$$

and

$$f_n'(0)\frac{\partial H_n}{\partial w}(0, 0) = (\tfrac{1}{2}-1/n)(2-1/n) < 1, \quad G_{n,1}(0, 0) = 1/n. \tag{26}$$

Thus, for every $n$, $n = 1, 2, \ldots$, equation (25) has exactly one analytic solution $\Phi_n(z)$ in a neighbourhood of the origin. Let us note that $f_n(z)$ tends to $f_0(z) = z/2$ and $H_n(z, w)$ tends to $H_0(z, w) = 2w$, uniformly on every compact subset of $C$, and the limit equation ($n = 0$) has the form

$$\Phi(z) = 2\Phi(z/2). \tag{27}$$

Hence we obtain

$$f_0'(0)\frac{\partial H_0}{\partial w}(0, 0) = 1; \quad G_{0,1}(0, 0) = 0,$$

i.e. equation (27) has a one-parameter family of analytic solutions $\Phi_0(z, t)$. It is easy to verify that $\Phi_0(z, t) = tz$. By (26) we have

$$c = \lim_{n\to\infty} \frac{G_{n,1}(0, 0)}{1-f_n'(0)\dfrac{\partial H_n}{\partial w}(0, 0)} = \tfrac{2}{5}.$$

According to Theorem 2 $\Phi_n(z)$ tends uniformly in a neighbourhood of the origin to $\Phi_0(z, \tfrac{2}{5}) = \tfrac{2}{5}z$.

## References

[1] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne 46, Warszawa 1968.

[2] J. Matkowski, *On the continuous dependence of local analytic solutions of a functional equations on the given functions*, Ann. Polon. Math. 23 (1970), p. 37-40

[3] W. Smajdor, *On the existence and uniqueness of analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$, ibidem 19 (1967), p. 37-45.

*Reçu par la Rédaction le 3. 1. 1970*