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The radius of close-to-convexity of $V_k(\varrho)$

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Abstract. Let $V_k(\varrho)$ denote the class of functions $f(z)$ which are analytic in the unit disc $E = \{z: |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$, $f'(z) \neq 0$ in E, and satisfying the condition

$$\int_0^{2\pi} \left| \operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right) - \varrho \right| d\theta \leqslant k(1-\varrho)\pi$$

for $z = re^{i\theta}$ in E, where $k \geqslant 2$ and $0 \leqslant \varrho < 1$. In this paper we determine the sharp radius of close-to-convexity of $V_k(\varrho)$.

Let $V_k(\varrho)$ denote the class of functions $f(z)$ which are analytic in the unit disc $E = \{z: |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$, $f'(z) \neq 0$ in E, and satisfying the condition

$$(1)\qquad \int_0^{2\pi} \left| \operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right) - \varrho \right| d\theta \leqslant k(1-\varrho)\pi \quad \text{for } z = re^{i\theta} \text{ in } E,$$

where $k \geqslant 2$ and $0 \leqslant \varrho < 1$.

When $\varrho = 0$, $V_k(\varrho)$ reduces to the well-known [4] class V_k of functions of boundary rotation bounded by $k\pi$. The class $V_k(\varrho)$ was introduced and studied by Padmanabhan and Parvatham [5]. In this paper we continue its study; in particular, we obtain the sharp radius of close-to-convexity of the class $V_k(\varrho)$, thus generalizing an earlier result due to Coonce and Ziegler [1]. We determine this radius using techniques similar to those employed by Krzyż [3].

In [2], Kaplan has shown that a function $f(z)$ regular in E and satisfying $f'(z) \neq 0$ in E maps $|z| = r < 1$ onto a close-to-convex curve if and only if

$$(2)\qquad \arg z_2 f'(z_2) - \arg z_1 f'(z_1) \geqslant -\pi,$$

for all z_1 and z_2 satisfying $|z_1| = r$, $z_2 = z_1 e^{i\theta}$, $0 < \theta < 2\pi$. Thus the radius of close-to-convexity of $V_k(\varrho)$ is the largest value of r for which (2) holds for all $f(z) \in V_k(\varrho)$.

The following results due to Padmanabhan and Parvatham [5] will be useful in the proof of our theorem.

LEMMA 1. *The radius of convexity of $V_k(\varrho)$ is the least positive root of the equation*

$$1-k(1-\varrho)r+(1-2\varrho)r^2 = 0. \tag{3}$$

The result is sharp.

LEMMA 2. *Let $f(z)\in V_k(\varrho)$. Then*

$$|\arg f'(z)| \leqslant k(1-\varrho)\sin^{-1}|z|. \tag{4}$$

The result is sharp.

THEOREM. *Let r_0 be the radius of convexity of $V_k(\varrho)$,*

$$\theta_0 = 2\cos^{-1}\frac{1+(1-2\varrho)r^2}{k(1-\varrho)r}, \quad 0\leqslant\theta_0<2\pi,$$

and

$$\Delta(r) = \theta_0+2(1-\varrho)\tan^{-1}\left\{\frac{r^2\sin\theta_0}{1-r^2\cos\theta_0}\right\}- -k(1-\varrho)\sin^{-1}\left\{r\left[\frac{2(1-\cos\theta_0)}{1-2r^2\cos\theta_0+r^4}\right]^{1/2}\right\}. \tag{5}$$

Then the radius of close-to-convexity of $V_k(\varrho)$ is the unique root r_1 of the equation $\Delta(r) = -\pi$ in the interval $(r_0, 1)$.

Proof. Let

$$\Delta(r,\theta) = \inf\arg\frac{z_2 f'(z_2)}{z_1 f'(z_1)}, \quad f(z)\in V_k(\varrho),$$

where z_1 and z_2 are any two points satisfying $|z_1| = r < 1$ and $z_2 = z_1 e^{i\theta}$, $0\leqslant\theta<2\pi$, and the argument is chosen to vary continuously from the initial value zero.

Let $f(z)\in V_k(\varrho)$. Define $F(z)$ by

$$F'(z) = \frac{f'\left(\dfrac{z+z_1}{1+\bar{z}_1 z}\right)}{f'(z_1)(1+\bar{z}_1 z)^{2(1-\varrho)}}. \tag{6}$$

Padmanabhan and Parvatham [5] have shown that $F(z)\in V_k(\varrho)$. Let $w_0 = (z_2-z_1)/(1-\bar{z}_1 z_2)$. Then

$$F'(w_0) = \frac{f'(z_2)}{f'(z_1)}\cdot\left[\frac{1-\bar{z}_1 z_2}{1-|z_1|^2}\right]^{2(1-\varrho)}$$

Hence it follows that

$$\arg \frac{z_2 f'(z_2)}{z_1 f'(z_1)} = \arg \frac{z_2}{z_1}\left[\frac{1-|z_1|^2}{1-\bar{z}_1 z_2}\right]^{2(1-\varrho)} + \arg F'(w_0). \tag{7}$$

We have

$$|w_0| = r\left\{\frac{2(1-\cos\theta)}{1-2r^2\cos\theta+r^4}\right\}^{1/2} \tag{8}$$

and

$$\arg \frac{z_2}{z_1}\left[\frac{1-|z_1|^2}{1-\bar{z}_1 z_2}\right]^{2(1-\varrho)} = \theta+2(1-\varrho)\tan^{-1}\left[\frac{r^2\sin\theta}{1-r^2\cos\theta}\right]. \tag{9}$$

Therefore, using (4), (8) and (9) in (7), we get

$$\begin{aligned} \Delta(r,\theta) = \theta+2(1-\varrho)\tan^{-1}&\left[\frac{r^2\sin\theta}{1-r^2\cos\theta}\right]- \\ &-k(1-\varrho)\sin^{-1}\left\{r\left[\frac{2(1-\cos\theta)}{1-2r^2\cos\theta+r^4}\right]^{1/2}\right\}. \end{aligned} \tag{10}$$

Further, there exists a function $f(z)$ in $V_k(\varrho)$ for which the infimum defining $\Delta(r,\theta)$ is actually attained at prescribed points z_1, z_2 and is equal to the expression on the right of (10). To see this, let $g(z)$ be the function in $V_k(\varrho)$ for which equality holds in (4) at the point w_0 and let $f(z)$ be defined by

$$f'(z) = \frac{g'\left(\dfrac{z-z_1}{1-\bar{z}_1 z}\right)}{g'(-z_1)(1-\bar{z}_1 z)^{2(1-\varrho)}}, \quad f(0)=0.$$

Then $f(z)$ is in $V_k(\varrho)$ and has the property asserted.

Let $\Delta(r) = \inf\limits_{0\le\theta<2\pi}\Delta(r,\theta)$. Then $\Delta(r)$ is a decreasing function of r and it follows from (2) that the radius of close-to-convexity of $V_k(\varrho)$ is the root r_1 of the equation $\Delta(r) = -\pi$.

If r_0 is the radius of convexity of $V_k(\varrho)$, then $\Delta(r)\geq 0$ for $r \leq r_0$ and so $r_1 > r_0$; hence we may assume that $r \in [r_0, 1)$ throughout the rest of the proof.

Differentiating (10) w.r.t. θ, we get, after a brief calculation

$$\frac{\partial\Delta(r,\theta)}{\partial\theta} = \frac{[1-k(1-\varrho)\cos\frac{1}{2}\theta r+(1-2\varrho)r^2](1-r^2)}{1-2r^2\cos\theta+r^4}.$$

Clearly, $\Delta(r,\theta)$ assumes its minimum value for a fixed r at $\theta=\theta_0$, where

$$\cos\frac{\theta_0}{2} = \frac{1+(1-2\varrho)r^2}{k(1-\varrho)r}.$$

Such a θ_0 exists in view of the fact that, for $r > r_0$,

$$\frac{1+(1-2\varrho)r^2}{k(1-\varrho)r} < 1.$$

Hence

$$\Delta(r) = \inf_{0 \leq \theta < 2\pi} \Delta(r, \theta) = \Delta(r, \theta_0).$$

Thus we get (5). We have $\Delta(r_0) = 0$ and $\Delta(r) \to -\infty$ as $r \to \bar{1}$. Since $\Delta(r)$ is a decreasing function of r, there exists a unique root r_1 of the equation $\Delta(r) = -\pi$ in the interval $(r_0, 1)$ and this root r_1 is the radius of close-to-convexity of $V_k(\varrho)$.

This completes the proof of the theorem.

For $\varrho = 0$ we obtain Theorem 1 of Coonce and Ziegler [1] as a particular case of the above theorem.

References

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