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Random differential inclusions: Measurable selection approach

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Abstract. We study the Cauchy problem for the random differential inclusion $x' \in F(\omega, t, x)$. We prove the existence of an absolutely continuous solution which depends measurably on the random parameter.

1. Introduction. In this paper we study the Cauchy problem for the random differential inclusion $x' \in F(\omega, t, x)$, where ω is a random parameter and F a given set-valued mapping. We look for an absolutely continuous solution which depends measurably on ω.

It seems that the study of random differential inclusions was initiated by Castaing [1], [2] (see also [3]). There are three main approaches to this problem: a "parametrization" of proofs for deterministic differential inclusions, application of random fixed point theorems for set-valued random operators, and measurable selection approach.

The first approach was recently adopted by De Blasi and Myjak [5]. The second method was applied by Phan Van Chuong [4], Fryszkowski [7] and the author [11].

In this paper we adopt the measurable selection approach. In the main result we assume that for almost all values of the random parameter the corresponding differential inclusion has a deterministic solution and prove that this solution can be chosen in the measurable way. We illustrate the application of this theorem by two examples. Similar approach was already used by the author [12] in the study of continuously differentiable solutions of random differential inclusions.

2. Preliminaries. Throughout this paper, $(\Omega, \mathscr{A}, P)$ is a probability space and T an interval on the real line R. For a metric space X, $\mathscr{P}(X)$ denotes the family of all nonempty subsets of X. By $\mathscr{B}(X)$ and $\mathscr{L}(T)$ we mean, respectively, the Borel σ-field on X, and the Lebesgue σ-field on T.

Key words. *Random solutions, differential inclusions, measurable selection.*

Let $F\colon \Omega \to \mathscr{P}(X)$ be a set-valued mapping (i.e., multifunction). The *graph* of F is defined as

$$\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X\colon\ x \in F(\omega)\}.$$

A multifunction F is said to be *measurable* if for each open $G \subset X$

$$\{\omega \in \Omega\colon\ F(\omega) \cap G \neq \emptyset\} \in \mathscr{A}.$$

(Note that such a map is called weakly measurable by Himmelberg [9].)

2.1. Proposition. *Let X be separable. Then:*

(i) *F is measurable iff $\omega \to d(x, F(\omega))$ is a measurable function of ω for each $x \in X$.*

(ii) *If F is measurable and closed-valued, then $\operatorname{Gr} F$ is $\mathscr{A} \otimes \mathscr{B}(X)$-measurable.*

Proof. See Himmelberg [9], Theorem 3.3.

We shall use the following measurable selection theorem:

2.2. Theorem. *Suppose X is a Souslin space* (i.e., continuous image of a Polish space). *If $\operatorname{Gr} F \in \mathscr{A} \otimes \mathscr{B}(X)$, then there exists a measurable function $f\colon \Omega \to X$ such that $f(\omega) \in F(\omega)$ P-almost surely.*

Proof. See Sainte-Beuve [14], Theorem 3 or Himmelberg [9], Theorem 5.7.

Let Y be a metric space. A multifunction $H\colon X \to \mathscr{P}(Y)$ is *closed* if its graph is a closed subset of $X \times Y$.

The *generalized Hausdorff metric* on $\mathscr{F}(Y)$, the family of all closed and nonempty subsets of Y, is defined by

$$D(A, B) = \max\{\sup\{d(y, A)\colon\ y \in B\},\ \sup\{d(y, B)\colon\ y \in A\}\}.$$

We say $H\colon X \to \mathscr{F}(Y)$ is *continuous* if it is continuous with respect to the generalized Hausdorff metric.

A function $f\colon \Omega \times X \to Y$ is a *Carathéodory map* if, for each $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous, and for each $x \in X$, $f(\cdot, x)$ is measurable. It is well known that if X is separable, then such a mapping f is product-measurable. An analogous fact holds for multifunctions.

2.3. Proposition. *Suppose X is separable. If $F\colon \Omega \times X \to \mathscr{F}(Y)$ is measurable in ω and continuous in x, then F is product-measurable.*

Proof. It suffices to show that $f(\omega, x) = d(y, F(\omega, x))$ is a measurable function for each $y \in Y$. Because of Proposition 2.1(i), f is measurable in ω. Since

$$|d(y, F(\omega, x_1)) - d(y, F(\omega, x_2))| \leqslant D(F(\omega, x_1), F(\omega, x_2)), \qquad x_1, x_2 \in X,$$

f is continuous in x. Hence, f is product-measurable.

For $A \subset \boldsymbol{R}^n$, we introduce the notation $|A| = \sup\{|a|: a \in A\}$, where $|\cdot|$ is the Euclidean norm in $\boldsymbol{R}^n$.

Suppose T is a compact interval. Denote by $\mathrm{AC}(T)$ the space of all absolutely continuous functions $x: T \to \boldsymbol{R}^n$ endowed with the norm

$$\|x\| = \sup\{|x(t)|: t \in T\} + \int_T |x'(t)|\,dt.$$

By $L(T)$ we mean the space of all integrable functions $x: T \to \boldsymbol{R}^n$ with the norm

$$\|x\|_1 = \int_T |x(t)|\,dt$$

(we identify equivalent functions). $\mathrm{AC}(T)$ and $L(T)$ are separable Banach spaces.

The abbreviation *a.e.* is used for *almost everywhere*.

3. Main result. Let T be a compact interval on R and U an open subset of $\boldsymbol{R}^n$. We shall study the Cauchy problem for the random differential inclusion

$$\text{(1)} \qquad \frac{d}{dt}x(\omega, t) \in F(\omega, t, x(\omega, t)),$$

$$\text{(2)} \qquad x(\omega, t_0) = v(\omega),$$

where the set-valued mapping $F: \Omega \times T \times U \to \mathscr{P}(\boldsymbol{R}^n)$, the map $v: \Omega \to U$ and $t_0 \in T$ are given.

3.1. DEFINITION. A function $x: \Omega \times T \to U$ is called a *random solution* of problem (1)–(2) if it is measurable in ω, absolutely continuous in t, and for P-almost all $\omega \in \Omega$ conditions (1)–(2) hold a.e. in T.

We shall prove a general existence theorem for problem (1)–(2). Assume:

(A) For P-almost all $\omega \in \Omega$ the deterministic Cauchy problem

$$(\mathrm{A}_1) \qquad y'(t) \in F(\omega, t, y(t)),$$

$$(\mathrm{A}_2) \qquad y(t_0) = v(\omega)$$

has an absolutely continuous solution defined on T.

(B) F is $\mathscr{A} \otimes \mathscr{L}(T) \otimes \mathscr{B}(U)$-measurable and closed-valued.

(C) v is measurable.

3.2. THEOREM. *If the Cauchy problem* (1)–(2) *satisfies assumptions* (A), (B) *and* (C), *then it has at least one random solution.*

Proof. We can assume that for each $\omega \in \Omega$ the deterministic problem (A_1)–(A_2) has a solution. Define the set-valued mapping

$$H(\omega) = \{y \in \mathrm{AC}(T): y \text{ satisfies } (\mathrm{A}_1)\text{–}(\mathrm{A}_2) \text{ a.e. in } T\}, \qquad \omega \in \Omega.$$

In order to apply the measurable selection theorem, we shall prove that H has a measurable graph.

Write

$$Y = \{y \in AC(T): y(t) \in U \text{ for all } t \in T\}.$$

It is immediate that Y is an open subset of $AC(T)$. Note that

$$\mathrm{Gr}\, H = \{(\omega, y) \in \Omega \times Y: \int_T d\big(y'(t), F(\omega, t, y(t))\big)\, dt + |y(t_0) - v(\omega)| = 0\}.$$

Let the functions $f: \Omega \times Y \times L(T) \to \bar{R}$ and $g: \Omega \times Y \to \bar{R}$ be defined by

$$f(\omega, y, z) = \int_T d\big(z(t), F(\omega, t, y(t))\big)\, dt + |y(t_0) - v(\omega)|,$$

$$g(\omega, y) = f(\omega, y, y').$$

We prove that f and g are product-measurable.

Since F is measurable, for each $w \in \boldsymbol{R}^n$ the map $(\omega, t, u) \to d\big(w, F(\omega, t, u)\big)$ is measurable with respect to the σ-field $\mathscr{A} \otimes \mathscr{L}(T) \otimes \mathscr{B}(U)$. Note that $(t, y) \to y(t)$ is a continuous mapping from $T \times Y$ into U. Thus the function $(\omega, t, y) \to d\big(w, F(\omega, t, y(t))\big)$ is $\mathscr{A} \otimes \mathscr{L}(T) \otimes \mathscr{B}(Y)$-measurable. Consequently, for each $z \in L(T)$, $f(\cdot, z)$ is $\mathscr{A} \otimes \mathscr{B}(Y)$-measurable. Now, let $(\omega, y) \in \Omega \times Y$ be arbitrary but fixed. It is an immediate consequence of definition that either $f(\omega, y, \cdot) = +\infty$ or $f(\omega, y, \cdot)$ has finite values. In the second case we have

$$\begin{aligned} |f(\omega, y, z_1) - f(\omega, y, z_2)| &\leqslant \int_T \Big| d\big(z_1(t), F(\omega, t, y(t))\big) - d\big(z_2(t), F(\omega, t, y(t))\big)\Big|\, dt \\ &\leqslant \int_T |z_1(t) - z_2(t)|\, dt = \|z_1 - z_2\|_1 \end{aligned}$$

for any $z_1, z_2 \in L(T)$. Hence, for each $(\omega, y) \in \Omega \times Y$, $f(\omega, y, \cdot)$ is continuous as an extended real-valued function. Being a Carathéodory map, f is $\mathscr{A} \otimes \mathscr{B}(Y) \otimes \mathscr{B}(L(T))$-measurable. The operator $y \to y'$ from Y to $L(T)$ is continuous. Thus g is measurable as a composition of measurable functions f and $(\omega, y) \to (\omega, y, y')$.

The measurability of g implies $\mathrm{Gr}\, H \in \mathscr{A} \otimes \mathscr{B}(Y)$. In virtue of Theorem 2.2, there exists a measurable map $h: \Omega \to Y$ such that $h(\omega) \in H(\omega)$ a.s. The function $x(\omega, t) = h(\omega)(t)$ is a required random solution of our problem. It completes the proof.

3.3. Remark. The same measurable selection approach can be applied to the random functional-differential inclusion

$$\frac{d}{dt} x(\omega, t) \in F(\omega, t, x(\omega, \cdot)), \qquad t \in T,$$

$$x(\omega, t) = v(\omega, t), \qquad t \in \overline{S \setminus T},$$

where $F\colon \Omega \times T \times C(S) \to \mathscr{P}(\boldsymbol{R}^n)$ and $v\colon \Omega \times \overline{(S \setminus T)} \to \boldsymbol{R}^n$ are given, S and T are intervals such that $T \subset S$, and $C(S)$ stands for the space of continuous functions from S to $\boldsymbol{R}^n$. The existence of random solutions for this problem was studied by Fryszkowski [7], [8].

4. Applications. By use of Theorem 3.2, for each existence result for a differential inclusion we can obtain its random analogue. As the first example we give a probabilistic version of the global result of Lasota and Opial [10].

4.1. THEOREM. *Let T be a compact interval on R and $F\colon \Omega \times T \times \boldsymbol{R}^n \to \mathscr{P}(\boldsymbol{R}^n)$ a set-valued map. Suppose*:

(i) *F is convex-valued, for each $(\omega, t) \in \Omega \times T$ the multifunction $F(\omega, t, \cdot)$ is closed, and for each $u \in \boldsymbol{R}^n$ the multifunction $F(\cdot, u)$ is $\mathscr{A} \otimes \mathscr{L}(T)$-measurable.*

(ii) *For each $\omega \in \Omega$ there exist integrable functions $a_\omega, b_\omega\colon T \to R$ such that*

$$|F(\omega, t, u)| \leqslant a_\omega(t) + b_\omega(t)|u|.$$

Then for each $t_0 \in T$ and each measurable $v\colon \Omega \to \boldsymbol{R}^n$ the Cauchy problem (1)–(2) *has a random solution.*

Proof. We cannot apply Theorem 3.2 immediately since F is not necessarily product-measurable. By [13], Theorem 1, there exists a convex-valued multifunction $\tilde{F}\colon \Omega \times T \times \boldsymbol{R}^n \to \mathscr{P}(\boldsymbol{R}^n)$ which is $\mathscr{A} \otimes \mathscr{L}(T) \otimes \mathscr{B}(\boldsymbol{R}^n)$-measurable and such that $\tilde{F}(\omega, t, u) \subset F(\omega, t, u)$, and $u \to \tilde{F}(\omega, t, u)$ is closed for each $(\omega, t) \in \Omega \times T$. Because of the result of Lasota and Opial [12], for each $\omega \in \Omega$ the deterministic Cauchy problem

$$y'(t) \in \tilde{F}(\omega, t, y(t)), \qquad y(t_0) = v(\omega)$$

has an absolutely continuous solution on T. An application of Theorem 3.2 with $\tilde{F}$ completes the proof.

This is a slight generalization of Theorem 5.2 from [11], which was obtained by an application of the random Kakutani–Ky Fan fixed point theorem.

Now we derive from Theorem 3.2 a local existence result for a random differential inclusion with nonconvex right-hand side.

4.2. THEOREM. *Let T be an open interval, U an open subset of $\boldsymbol{R}^n$, and $F\colon \Omega \times T \times U \to \mathscr{P}(\boldsymbol{R}^n)$ a set-valued mapping. Suppose*:

(i) *F is compact-valued and there is a constant $M > 0$ such that $|F(\omega, t, u)| \leqslant M$ for all $\omega \in \Omega$, $t \in T$, $u \in U$.*

(ii) *For each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous, and for each $(t, u) \in T \times U$, $F(\cdot, t, u)$ is measurable.*

Then for each $t_0 \in T$ and each measurable $v\colon \Omega \to \boldsymbol{R}^n$ satisfying

$$\{u \in \boldsymbol{R}^n\colon |u - v(\omega)| \leqslant b\} \subset U, \qquad \omega \in \Omega,$$

for some positive b, the Cauchy problem (1)–(2) *has at least one random solution defined on* $\Omega \times [t_0 - d, t_0 + d]$, *where* $d > 0$.

Proof. In virtue of Proposition 2.3, F is product-measurable. By the result of Filippov [6] for each $\omega \in \Omega$ the deterministic Cauchy problem

$$y'(t) \in F(\omega, t, y(t)), \qquad y(t_0) = v(\omega)$$

has an absolutely continuous solution on the interval $I = [t_0 - d, t_0 + d]$, where $d = \min\{a, b/M\}$, and $[t_0 - a, t_0 + a] \subset T$. Now the application of Theorem 3 2 completes the proof.

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