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Certain formulas associated with generalized Rice polynomials*

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In the present note, some recent expansions involving the generalized Rice polynomial

$$H_n^{(\alpha,\beta)}[\zeta,p,v]=\binom{\alpha+n}{n}{}_3F_2\left[\begin{matrix}-n,\ \alpha+\beta+n+1,\ \zeta;\\ \alpha+1,\ p;\end{matrix}\ v\right]$$

are first exhibited as specialized or limiting forms of certain known results due to Brafman, Chaundy, and the author, and then extended to hold for various classes of generalized hypergeometric polynomials. The bilinear generating functions derived here involve a general class of hypergeometric functions in three arguments which the author introduced a couple of years ago.

1. In a recent paper (see [8]) certain results have been derived for the generalized Rice polynomial

$$H_n^{(\alpha,\beta)}[\zeta,p,v]=\binom{\alpha+n}{n}{}_3F_2\left[\begin{matrix}-n,\ \alpha+\beta+n+1,\ \zeta;\\ \alpha+1,\ p;\end{matrix}\ v\right],\tag{1.1}$$

which, when $\alpha=\beta=0$, reduces to the original form ([10], p. 108)

$$H_n[\zeta,p,v]={}_3F_2\left[\begin{matrix}-n,\ n+1,\ \zeta;\\ 1,\ p;\end{matrix}\ v\right].\tag{1.2}$$

When expressed in terms of the generalized hypergeometric function

$${}_AF_B[z]\equiv{}_AF_B\left[\begin{matrix}(a);\\ (b);\end{matrix}\ z\right]=\sum_{m=0}^{\infty}\frac{[(a)]_m}{[(b)]_m}\frac{z^m}{m!}$$

$$=\sum_{m=0}^{\infty}\frac{[a_1]_m[a_2]_m\dots[a_j]_m\dots[a_A]_m}{[b_1]_m[b_2]_m\dots[b_k]_m\dots[b_B]_m}\frac{z^m}{m!},$$

where

$$[a_j]_m=\frac{\Gamma[a_j+m]}{\Gamma[a_j]}=a_j(a_j+1)(a_j+2)\dots(a_j+m-1),$$

* See Abstract 69T-B27, Amer. Math. Soc. Notices 16 (1969), p. 411.

(a) denotes the sequence of A parameters

$$a_1, a_2, \ldots, a_j, \ldots, a_A,$$

that is, unless otherwise stated, there are always A of the a parameters, and $[(a)]_m$ has the interpretation

$$\prod_{j=1}^{A} [a_j]_m,$$

and so on, the first formula (5) in ([8], p. 431) assumes the form

$$(1.3)\qquad \sum_{n=0}^{\infty} \frac{[\lambda]_n}{n!}\, {}_3F_2\left[\begin{matrix} -m,\ -n,\ a+\beta+m+1; \\ a+1,\ p; \end{matrix}\, x\right] t^n$$
$$= (1-t)^{-\lambda}\, {}_3F_2\left[\begin{matrix} -m,\ a+\beta+m+1,\ \lambda; \\ a+1,\ p; \end{matrix}\, -\frac{xt}{1-t}\right].$$

We now recall the following theorem on generating functions which we proved quite sometime ago (see [14], § 3).

THEOREM. *Let the coefficients* $\{\Psi_n^{(\lambda)}(x)\}$ *be generated by*

$$(1.4)\qquad E[M(x)t^m]G[Q(x)t^p] = \sum_{n=0}^{\infty} \frac{t^n}{[\lambda+1]_n} \Psi_n^{(\lambda)}(x),$$

where

$$E[z] = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad G[z] = \sum_{n=0}^{\infty} g_n z^n \qquad (g_n \neq 0),$$

$M(x) \neq 0$, $Q(x) \neq 0$ *are real functions, and* m, p *are positive integers.*
Then for arbitrary ν,

$$(1.5)\qquad [1-M(x)t^m]^{-\nu} H\left(Q(x)\left[\frac{t^m}{1-M(x)t^m}\right]^{p/m}\right)$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma\left[\nu+\dfrac{n}{m}\right]}{[\lambda+1]_n} \Psi_n^{(\lambda)}(x)t^n,$$

provided

$$H(z) = \sum_{n=0}^{\infty} \Gamma\left[\nu+\frac{np}{m}\right] g_n z^n.$$

The rôle of the theorem lies in the fact that, if $G[z]$ is a specified hypergeometric function, it gives for $\Psi_n^{(\lambda)}(x)$ a class (ν being arbitrary) of generating functions involving a hypergeometric function of the superior order. For instance, if we give $G[z]$ the hypergeometric form

$$G[z] = {}_AF_B\left[\begin{matrix}(a); \\ (b);\end{matrix}\; z\right],$$

and for the sake of simplicity, let $m = 1$, so that (1.4) becomes

$$(1.6)\qquad E[M(x)t]\,{}_AF_B\left[\begin{matrix}(a); \\ (b);\end{matrix}\; Q(x)t^p\right] = \sum_{n=0}^{\infty} \frac{[M(x)t]^n}{n!}\times$$

$$\times\,{}_{A+p}F_B\left[\begin{matrix}-\dfrac{n}{p}, -\dfrac{(n-1)}{p}, \ldots, -\dfrac{(n-p+1)}{p}, (a); \\ (b);\end{matrix}\; \left(-\frac{p}{M(x)}\right)^p Q(x)\right],$$

then our theorem gives us

(1.7)

$$[1-M(x)t]^{-\nu}\,{}_{A+p}F_B\left[\begin{matrix}\dfrac{\nu}{p}, \dfrac{\nu+1}{p}, \ldots, \dfrac{\nu+p-1}{p}, (a); \\ (b);\end{matrix}\; \left(\frac{pt}{1-M(x)t}\right)^p Q(x)\right]$$

$$= \sum_{n=0}^{\infty} \frac{[\nu]_n}{n!}[M(x)t]^n \times$$

$$\times\,{}_{A+p}F_B\left[\begin{matrix}-\dfrac{n}{p}, -\dfrac{(n-1)}{p}, \ldots, -\dfrac{(n-p+1)}{p}, (a); \\ (b);\end{matrix}\; \left(-\frac{p}{M(x)}\right)^p Q(x)\right]$$

for arbitrary parameter ν, and $p = 1, 2, 3, \ldots$

Well-known special cases of (1.7) include Chaundy's result ([6], p. 62, (25)), where $M(x) = 1$, $Q(x) = -x$, $p = 1$, and the relatively recent formula due to Brafman (see [3], p. 187, (55)), where $M(x) = 1$, $Q(x) \doteq (-p)^{-p}\cdot x$, which, in turn, reduces to equation (24), p. 947 of [2] when $p = 2$.

Formula (1.3) is obviously a very special case of Chaundy's result already referred to (see also [17], p. 24, (4.15)).

2. Further results. The form of formula (7), p. 432 in [8] suggests the existence of the more general result

(2.1)

$$\sum_{n=0}^{\infty}\binom{\lambda+n}{n}\binom{n-\alpha}{n}\binom{n-\alpha-\beta}{n}^{-1}\frac{[(a)]_n}{[(b)]_n}\,{}_{C+2}F_{D+1}\left[\begin{matrix}-n,\alpha+\beta-n,(c);\\ \alpha-n,(d);\end{matrix}\;x\right]t^n$$
$$=F\left[\begin{matrix}\lambda+1,(a): & 1-\alpha; & (c);\\ (b): & 1-\alpha-\beta; & (d);\end{matrix}\;t,-xt\right],$$

where the notation for the double hypergeometric function is due to Burchnall and Chaundy ([5], p. 112) in preference, for the sake of brevity and elegance, to an earlier one introduced by Kampé de Fériet ([1], p. 150).

A proof of (2.1) by the method of finite mathematical induction would require the Laplace and the inverse Laplace transform techniques. Indeed the formula holds for $A=B=0$ and $C=D=1$ by virtue of result (7), p. 432 in [8]. Assuming, therefore, that it remains true for some values of A, B, C, and D, let us replace x by xz, multiply both sides by

$$z^{c_{C+1}-1},$$

and take their Laplace transforms with respect to z. Then using the known formula (3.2.15), p. 43 in [11] we find that C is replaced by $C+1$, thus completing the induction on C. In order to effect the induction on D, we replace x by x/z, multiply both sides by

$$z^{-d_{D+1}},$$

and then take their inverse Laplace transforms with the aid of the known result (3.2.38), p. 45 in [11]. The inductions on A and B can be effected fairly easily by similarly considering the variable t instead of x.

Similar are the proofs of the following extensions of formulas (9) and (11) in [8]:

$$(2.2)\quad \sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}\,{}_{A+2}F_B\left[\begin{matrix}-m,-k,(a);\\ (b);\end{matrix}\;x\right]{}_{C+2}F_D\left[\begin{matrix}-n,\lambda+k,(c);\\ (d);\end{matrix}\;y\right]t^k$$
$$=(1-t)^{-\lambda}F\left[\begin{matrix}\lambda: & -m,(a); & -n,(c);\\ & (b); & (d);\end{matrix}\;-\frac{xt}{1-t},\frac{y}{1-t}\right],$$

and

$$(2.3)\quad \sum_{n=0}^{\infty}\binom{n-\alpha}{n}\binom{n-\alpha-\beta}{n}^{-1}\binom{n-\gamma}{n}\frac{[(a)]_n}{[(b)]_n}\times$$
$$\times\,{}_{A+2}F_{B+1}\left[\begin{matrix}-n,\alpha+\beta-n,(c);\\ \alpha-n,(d);\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}-m-n,\gamma+\delta-n;\\ \gamma-n;\end{matrix}\;y\right]t^n$$

$$= \frac{[\gamma+\delta]_m}{[\gamma]_m}(-y)^{\delta}(1-y)^{m-\delta}\times$$

$$\times F^{(3)}\left[\begin{matrix} 1-\gamma-\delta :: (a);\ -;\ -:\ 1-a;\quad (c);\ -\delta; \\ -:: (b);\ -;\ -: 1-a-\beta;\ (d);\ \gamma-\delta-m-1; \end{matrix}\ (1-y)t,\ -x(1-y)t,\ y^{-1}\right],$$

where $F^{(3)}[x, y, z]$ denotes a general triple hypergeometric series introduced by us in the form ([13], p. 428)

$$F^{(3)}\left[\begin{matrix}(a):: (b); (b'); (b''): (c); (c'); (c''); \\ (e):: (f); (f'); (f''): (g); (g'); (g''); \end{matrix}\ x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{[(a)]_{m+n+p}[(b)]_{m+n}[(b')]_{n+p}[(b'')]_{p+m}[(c)]_m[(c')]_n[(c'')]_p}{[(e)]_{m+n+p}[(f)]_{m+n}[(f')]_{n+p}[(f'')]_{p+m}[(g)]_m[(g')]_n[(g'')]_p}\times$$

$$\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

The special cases when $\beta\to 0$ in (2.1) and when $y\to 0$ in (2.2) lead us once again to Chaundy's formula (25), p. 62 in [6], while the last result (2.3) reduces to (2.1) for $\delta\to 0$.

We should like to conclude with the remark that it is not difficult to apply the method illustrated here in order to derive a generalization of our summation formula (1.5), p. 680 in [16], viz.

$$(2.4)\quad \sum_{n=0}^{\infty}\binom{\nu+n}{n}F_2[\lambda, -n, -n; a, \beta; x, y]z^n$$

$$= (1-z)^{-\nu-1}\sum_{n=0}^{\infty}\binom{\nu+n}{n}\frac{[\lambda]_{2n}}{[a]_n[\beta]_n}\left[\frac{xyz}{(1-z)^2}\right]^n\times$$

$$\times F_2\left[\lambda+2n, \nu+n+1, \nu+n+1; a+n, \beta+n; \frac{xz}{z-1}, \frac{yz}{z-1}\right],$$

associated with the Appell function (see, e.g., [12], p. 211)

$$F_2[a, \beta, \beta'; \gamma, \gamma'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_{m+n}[\beta]_m[\beta']_n}{[\gamma]_m[\gamma']_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

in the elegant form(1)

$$(2.5)\quad \sum_{n=0}^{\infty}\binom{\nu+n}{n}F\left[\begin{matrix}(a): -n, (b); -n, (b'); \\ (c):\quad (d);\quad (d'); \end{matrix}\ x, y\right]z^n$$

(1) Formula (2.5) occurred to the author during his recent discussions about the results in [16] with Professor L. Carlitz at Duke University.

$$= (1-z)^{-\nu-1} \sum_{n=0}^{\infty} \binom{\nu+n}{n} \frac{[(a)]_{2n}[(b)]_n[(b')]_n}{[(c)]_{2n}[(d)]_n[(d')]_n} \left[\frac{xyz}{(1-z)^2}\right]^n \times$$

$$\times F \left[\begin{matrix} (a)+2n\colon (b)+n, \nu+n+1; (b')+n, \nu+n+1; \\ (c)+2n\colon \quad (d)+n; \quad (d')+n; \end{matrix} \; \frac{xz}{z-1}, \frac{yz}{z-1}\right],$$

provided $|z| < 1$; nor does it seem out of place to mention that in the course of an attempt elsewhere (see [15]) to give extensions of the well-known Hille-Hardy formula ([7], p. 189, (20)) we have invoked these techniques to obtain several new and distinct bilinear generating functions for certain classes of generalized hypergeometric polynomials, one such formula evidently contained in (2.5) being

$$(2.6) \quad \sum_{n=0}^{\infty} \frac{[\lambda]_n}{n!} {}_{A+1}F_B \left[\begin{matrix} -n, (a); \\ (b); \end{matrix} \; x\right] {}_{C+1}F_D \left[\begin{matrix} -n, (c); \\ (d); \end{matrix} \; y\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{[\lambda]_n [(a)]_n [(c)]_n}{n! \quad [(b)]_n [(d)]_n} {}_{A+1}F_B \left[\begin{matrix} \lambda+n, (a)+n; \\ (b)+n; \end{matrix} \; \frac{xz}{z-1}\right] \times$$

$$\times {}_{C+1}F_D \left[\begin{matrix} \lambda+n, (c)+n; \\ (d)+n; \end{matrix} \; \frac{yz}{z-1}\right] \left[\frac{xyz}{(1-z)^2}\right]^n,$$

which generalizes the earlier results of Brafman ([4], p. 1320, (4)), Chaundy ([6], p. 62, (25)), Meixner [9], Weisner ([18], p. 1037, (4.3), (4.6)), and many others.

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