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MIROSŁAWA SITEK (Poznań)

APPLICATION OF THE SELECTION PROCEDURE R TO UNEQUAL OBSERVATION NUMBERS

1. Introduction. In agricultural and other investigations we often meet the problem of selecting the best population or a subset containing the best population. As *best population* we understand the population having the highest value of a determined location parameter which will generally be the mean. For instance, we have some varieties of lupine and we have to choose the possibly smallest set containing the best, i.e., the highest, yielding variety. In another case we have to compare some remedies with a standard one, and are interested in the selection of a subset containing remedies being either better or as good as the standard one. The remaining remedies will be of lower quality than the standard one. In all these cases we want that the probability of selecting the subset containing the best population or the subset containing all populations not worse than a standard one be equal to a predetermined value P^*. The value P^* may be chosen equal 0.95, 0.99, etc. This problem was analysed by Gupta and Sobel [4] who presented the selection procedure R to equal observation numbers. The problem of comparison with a standard was investigated by Dunnett [1]. In this paper we give a method of applying procedure R to unequal observation numbers. At the end of this paper are given tables of the upper $5^0/_0$ and $1^0/_0$ critical values of the multivatiate t-distribution and tables of corrections permitting the practical application of the proposed method.

2. Selection procedure R. We denote the random variables by capital letters and their values by small ones. Let $X_1, X_2, \ldots, X_k$ be k independent random variables representing the populations $\Pi_1, \Pi_2, \ldots, \Pi_k$ with location parameters $\theta_1, \theta_2, \ldots, \theta_k$, respectively. Let the best population be that one which has the highest value of the parameter θ. The R principle says the following:

Choose the i-th population if and only if

$$x_i \geqslant x_{\max} - d,$$

where x_i is the observed value of the i-th random variable X_i and $x_{\max}$ is the highest value of $x_1, x_2, \ldots, x_k$.

The value d is chosen in a manner to fulfil the probability of correct selection. Let CS denote the *correct selection*, i.e., the selection of the subset containing the best population. The ordered parameters θ will be presented as $\theta_{[1]} \leqslant \theta_{[2]} \leqslant \ldots \leqslant \theta_{[k]}$.

The probability of correct selection will then be

(1) $$\mathrm{P}\{CS|R\} = \mathrm{P}\{X_k \geqslant X_{\max} - d\}.$$

In formula (1) there is one unknown variable connected with θ_k. We do not know a priori the correct pairing of $\{X_i\}$ and the ordered θ_j. If the random variable X_i has the density function $f(x-\theta_i)$ and the distribution function $F(x-\theta_i)$, $i = 1, 2, \ldots, k$, then

(2) $$\mathrm{P}\{CS|R\} = \int_{-\infty}^{\infty} \Big[\prod_{j=1}^{k-1} F(t+d+\theta_{[k]}-\theta_{[j]})\Big] f(t)\,dt.$$

If $\theta_{[k]} = \theta_{[j]}$ for $j = 1, 2, \ldots, k-1$, we obtain $\inf\limits_{\Omega} \mathrm{P}\{CS|R\}$; then

$$\inf_{\Omega} \mathrm{P}\{CS|R\} = \int_{-\infty}^{\infty} F^{k-1}(t+d) f(t)\,dt,$$

where Ω is the space of the parameters $\theta_1, \theta_2, \ldots, \theta_k$.

If we choose some d satisfying the equality

$$\int_{-\infty}^{\infty} F^{k-1}(t+d) f(t)\,dt = P^*,$$

then we have determined the smallest value of d for which

$$\inf_{\Omega} \mathrm{P}\{CS|R\} = P^*.$$

In the procedure R the size of the chosen subset is a random variable which can take the values $1, 2, \ldots, k$. If S denotes the size of the chosen subset, its expected value equals

$$E(S) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} \Big[\prod_{\substack{j=1 \\ j\neq i}}^{k} F(t+d+\theta_{[i]}-\theta_{[j]})\Big] f(t)\,dt$$

and $\max E(S|R) = k \cdot P^*$.

The procedure R has two properties: that of monotonity and that of optimality.

1. If $\theta_{[i]}$ is greater than or equals to $\theta_{[j]}$, the probability that the subset contains a population of $\theta_{[i]}$ is not lower than the probability that the set contains a population of $\theta_{[j]}$.

2. The procedure R chooses a set of size S (unknown a priori) for which the probability of containing the best population is maximal.

Suppose we have besides the populations $\Pi_1, \Pi_2, \ldots, \Pi_k$ one standard population Π_0 with known or unknown location parameter θ_0. We aim, with a probability of at least P^*, to choose a subset containing all populations with $\theta \geqslant \theta_0$. If θ_0 is unknown, the procedure $\bar{R}$ is as follows:

Choose Π_i if and only if

$$x_i \geqslant x_0 - \bar{d},$$

where x_0 is the observed value of the random variable X_0 corresponding to the population Π_0 and where $\bar{d}$ is chosen in such a way that the probability the selected subset will contain all populations better or as good as the standard one will equal P^*.

Therefore, if we do not know how many populations are better or as good as the standard one, the number $\bar{d}$ must fulfil the equation

$$\int_{-\infty}^{\infty} [1 - F(x - \bar{d})]^k f(x)\, dx = P^*. \tag{3}$$

If it is known that some number g ($\leqslant k$) of populations is not worse than the standard one, we replace k in formula (3) by g. Both the $\bar{R}$ and R principles have the same properties. More accurate considerations concerning the R and $\bar{R}$ principles, as well as the arguments for formulas given in this section, may be found in [3].

3. Application of the R and $\bar{R}$ principles for normal populations.

3.1. *Experiments with equal number of observations.* Given are k random variables $X_1, X_2, \ldots, X_k$, uncorrelated or identically correlated, representing the populations $\Pi_1, \Pi_2, \ldots, \Pi_k$ with normal distributions having mean values $\mu_1, \mu_2, \ldots, \mu_k$ and a common variance σ^2.

Suppose the parameters $\mu_1, \mu_2, \ldots, \mu_k$ and σ^2 are unknown. Given are n observations x_{ij} ($j = 1, 2, \ldots, n$) of each of the populations Π_i ($i = 1, 2, \ldots, k$). Let

$$\bar{x}_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij}$$

be the mean value of a sample taken from the population Π_i and let s_ν^2 be the common evaluation of σ^2 based on ν degrees of freedom. Let $\bar{x}_{\max}$ denote the maximum of the observed values of $\bar{x}_i$ ($i = 1, 2, \ldots, k$). The R principle is stated in this case as follows:

Select as the best populations those for which

$$\bar{x}_i \geqslant \bar{x}_{\max} - q s_\nu \sqrt{2/n}. \tag{4}$$

The value of q must be taken from tables of the multivariate t-distribution for the given $P^* = 1-a$, ν and $p = k-1$. The number q satisfies the equation $\inf \mathrm{P}\{CS|R\} = P^*$, therefore

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} \Phi^{k-1}(u+q\sqrt{2}y)\varphi(u)q_{\nu}(y)\,du\,dy = P^*, \tag{5}$$

where $q_{\nu}(y)$ is the density function of $\chi_{\nu}/\sqrt{\nu}$, and Φ and φ indicate, respectively, the cummulative distribution function and the density function of the standard normal distribution.

The R principle for comparison with the control is stated similarly: Select for as good as the control one those random variables for which

$$\bar{x}_i \geqslant \bar{x}_0 - qs_{\nu}\sqrt{2/n}, \tag{6}$$

where $\bar{x}_0$ is the mean value of the sample control population.

Here also the number q satisfies the equation $\inf \mathrm{P}\{CS|\dot{R}\} = P^*$, therefore

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} \Phi^{k}(u+q\sqrt{2}y)\varphi(u)q_{\nu}(y)\,du\,dy = P^*. \tag{7}$$

The value of q must be taken from tables of the multivariate t-distribution for the given P^*, ν and $p = k$.

3.2. *Experiments with unequal observation numbers.* Given are k independent random variables representing populations with normal distributions having equal variances. For each random variable X_i given are n_i observations x_{ij} $(j = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, k)$.

I propose the use of the following R procedure:

Select as the best those random variables for which

$$\bar{x}_i \geqslant \bar{x}_{\max} - qs_{\nu}\sqrt{1/n_i+1/n}, \tag{8}$$

where n is the observation number of the random variable having the maximum sample mean $\bar{x}_{\max}$. We find the value of q by using the tables of the multivariate t-distribution and the appropriate correction talbes. The method of using these tables is given in chapter 5.

In case of comparison with the control, we select as the best or as good as the control those random variables for which

$$\bar{x}_i \geqslant \bar{x}_0 - qs_{\nu}\sqrt{1/n_0+1/n_i}, \tag{9}$$

where n_0 is the observation number of the control population.

The arguments for formulas (8) and (9), i.e., the principles $\bar{R}$ and R for unequal observation numbers, are given in the following chapter.

4. The multivariate t-distribution and argumentation for the proposed method. In case of unequal observation numbers, the variance of the difference $\bar{x}_i - \bar{x}_j$ equals $\sigma^2\sqrt{1/n_i+1/n_j}$ and the evaluation of this variance equals $s_\nu^2\sqrt{1/n_i+1/n_j}$. Therefore, in formula (8), we find $\sqrt{1/n_i+1/n}$ instead of $\sqrt{2/n}$, and, in formula (9), $\sqrt{1/n_0+1/n_i}$ instead of $\sqrt{2/n}$. We remark that for $n_i = n$ or $n = n_0$ formulas (8) and (9) are the same as formulas (4) and (6), respectively.

Now we show that $\inf \mathrm{P}\{CS|R\}$ is the cummulative distribution function of the multivariate t-distribution. The probability of a correct selection for procedure R is equal to

$$\mathrm{P}\{CS|R\} = \mathrm{P}\{\bar{X}_k \geqslant \bar{X}_{\max} - qs_\nu\sqrt{1/n_i+1/n}\}.$$

In case of $\theta_1 = \theta_2 = \ldots = \theta_k$, the probability determined on the right-hand side of this formula equals

$$\mathrm{P}\left\{\frac{X_{\mathrm{pmax}}-X}{s'} \leqslant q\sqrt{2}\right\} = \mathrm{P}\{Y \leqslant q\sqrt{2}\},$$

where $Y = (X_{\mathrm{pmax}} - X)/s'$, X_{pmax} is the maximum of p independent random variables with a normal distribution with the same mean value and variance σ^2, and $s_\nu'^2$ is the evaluation of the common variance of $p+1 = k$ random variables ($s_\nu'^2$ has the distribution $\sigma^2\chi_\nu^2/\nu$ with ν degrees of freedom). Therefore, we may write $\inf\mathrm{P}\{CS|R\} = \mathrm{P}\{Y \leqslant q\sqrt{2}\}$.

Now we denote by Z_i ($i = 1, 2, \ldots, p$) the random variables of the distribution $N(0, \sigma^2)$ with correlations ϱ_{ij}; in this case we have the relation

$$\begin{aligned}\mathrm{P}\{Y \leqslant q\sqrt{2}\} &= \mathrm{P}\{Y/\sqrt{2} \leqslant q\} \\ &= \mathrm{P}\{Z_1/s_\nu' \leqslant q, \ldots, Z_p/s_\nu' \leqslant q;\ \varrho_{ij} \text{ for } i, j = 1, \ldots, p\} \\ &= \mathrm{P}\{t_i \leqslant q,\ i = 1, \ldots, p,\ \varrho_{ij}\},\end{aligned}$$

where $t_i = Z_i/s_\nu'$ is the Student t-statistics. Therefore $\inf\mathrm{P}\{CS|R\}$ is determined by the cummulative distribution function with correlation matrix (ϱ_{ij}). It can be proved that in our case

$$\varrho_{ij} = \frac{1}{\sqrt{(1+n/n_i)(1+n/n_j)}}; \tag{10}$$

if $n_i = n$ for $i = 1, 2, \ldots, k$, then $\varrho_{ij} = 1/2$. Therefore, for an equal observation number the integral on the left-hand side of formulas (5) and (7),

(11)

$$\int_0^\infty \int_{-\infty}^\infty \Phi^p(u+q\sqrt{2}y)\varphi(u)q_\nu(y)\,du\,dy = \mathrm{P}\{t_i \leqslant q,\ i = 1, 2, \ldots, p,\ \varrho_{ij} = \tfrac{1}{2}\}$$

is the cummulative distribution function of the multivariate t-distribution with identical correlation coefficient 1/2. Gupta and Sobel in [4] have analysed methods for the calculation of integral (11) and presented a table of the values of $q\sqrt{2}$ for $k = 2, 5, 10(1)16, 18, 20(5)40, 50$, $\nu = 15(1)20, 24, 30, 36, 40, 48, 60, 80, 100, 120, 360, \infty$, $P^* = 0.75$, $0.95, 0.975, 0.99$. Dunnett in [1] has also given tables of the q values for $p = 1(1)9$, selected values of ν and $\alpha = 1-P^* = 0.05, 0.01$. The critical values of t for $\varrho_{ij} = \varrho = 0.0(0.1)0.9$, $p = 1(1)10$, $\nu = 5(1)35$ and the method of their calculation are given by Krishnaiah in [6].

At the end of this paper are given tables of critical values of the multivariate t-distribution for $\alpha = 1-P^* = 0.05$ and 0.01, $p = 1(1)10$ and $\nu = 5(1)35, 40, 60, 120, \infty$. Values of t for $p = 1(1)9$ and $\nu = 5(1)35$ are taken from [6], for $p = 1(1)9$ and $\nu = 40, 60, 120, \infty$ from [1], and for $p = 10$ and $\nu = 40, 60, 120, \infty$ from [4].

Dunnett stated in [2] that, for $0.125 < \varrho < 0.50$, the values of q are approximately linearly dependent on the reciprocal of $1-\varrho$ or, more accurately, on the value $(1-2\varrho)/(1-\varrho)$. Using this relation, I have calculated corrections permitting to find the values of q for $\varrho < 0.05$, if one has only tables of the critical values of t for $\varrho = 0.5$. Of course, having tables of the critical values of the multivariate t-distribution, the use of the correction tables is superfluous. The correction tables replace tables (one for every value of ϱ) of the critical values of the multivariate t-distribution.

5. Correction tables. Calculation difficulties did not permit to build tables of the critical values of the multivariate t-distribution for various values of ϱ_{ij}. Therefore, there exist only tables for identical $\varrho_{ij} = \varrho$. Generally, the condition of equality of the correlation coefficients is not satisfied (see formula (10)). Therefore, one must select a common ϱ such that the inequality $\inf \mathrm{P}\{CS|R\} \geqslant P^*$ is fulfilled. The shape of the cummulation distribution function of the multivariate t-distribution, given by John in [5], shows that the critical values decrease with increasing ϱ. We have the inequality

$$\mathrm{P}\{t_i \leqslant q,\ i = 1, \ldots, p;\ \varrho_{ij}\} \geqslant \mathrm{P}\{t_i \leqslant q,\ i = 1, \ldots, p;\ \min \varrho_{ij}\}.$$

If we equal the right-hand side of this inequality with P^*, we obtain the required condition

$$\inf \mathrm{P}\{CS|R\} = \mathrm{P}\{t_i \leqslant q,\ i = 1, \ldots, p;\ \varrho_{ij}\} \geqslant P^*.$$

Thus, it is better to choose a common value ϱ. Therefore the minimal value $\min \varrho_{ij} = \varrho$. From formula (10) we obtain

$$\varrho = \min \varrho_{ij} = \frac{1}{n/\min\limits_i n_i + 1} \quad \text{and} \quad \frac{1-2\varrho}{1-\varrho} = 1 - \min_i \frac{n_i}{n}.$$

Because the critical values of the multivariate t-distribution are only approximately linearly dependent on the values $(1-2\varrho)/(1-\varrho)$, three corrections, for given ν and p and for $\varrho = 0.0, 0.2, 0.4$, are given to obtain more accurate critical values corresponding to $\min_i n_i/n = 0, \frac{1}{4}, \frac{2}{3}$. As a result of the linear dependence of t and $(1-2\varrho)/(1-\varrho)$, we obtain a formula for the calculation of the critical value, for a given ϱ (we denote it by t_ϱ),

$$t_\varrho = \left(\frac{1-2\varrho}{1-\varrho}\cdot\frac{x}{100}+1\right)\cdot t_{\varrho=0.5},$$

where x is the correction given in percents. The following formula will be more practical:

$$t_\varrho = \left[\left(1-\frac{\min n_i}{n}\right)\frac{x}{100}+1\right]\cdot t_{\varrho=0.5}. \tag{12}$$

For example: we have $k = 5$ populations, $\nu = 16$ and $\min n_i/n = 0.27$. We look in the tables for corrections for $p = k-1 = 4$, $\nu = 16$, $P^* = 0.95$ and $\min n_i/n = 1/4$. The correction value is $x = 4.6\%$ and in the table of the critical values of the multivariate t-distribution is $t_{\varrho=0.5} = 2.34$. The required critical value will be

$$t_\varrho = (0.046\cdot 0.73+1)\cdot 2.34 = 2.42.$$

6. Examples. In the Chair for Fancy Plants of the School of Agriculture in Poznań there were carried out some comparative investigations on 10 varieties of roses. The data of these investigations concern the length of the flower branches and are the following:

$[i]$	1	2	3	4	5	6	7	8	9	10
i	B	D	G	F	J	I	C	H	A	E
$\bar{x}_i$	35.5	35.6	36.3	36.4	37.5	40.5	41.0	41.3	43.4	56.8
n_i	19	21	28	25	25	29	28	22	20	20

The maximal mean was observed on the variety E, $\bar{x}_{\max} = 56.8$. The observation number for this variety was $n = 20$, the minimum number of observations $\min n_i = 19$, the number of degrees of freedom $\nu = 227$, and $s_\nu = 6.87$. Since the tables of the critical values of the multivariate t-distirbution do not contain values for $\nu = 227$, we find the necessary value by inverse interpolation. We find, for $p = k-1 = 9$, $P^* = 1-\alpha = 0.95$, $t_{\nu=120} = 2.45$ and $t_{\nu=\infty} = 2.42$; we calculate $t_{\nu=227}$ using the formula

$$t_{\nu=120}-t_{\nu=227} = \frac{1/120\ -\ 1/227}{1/120\ -\ 1/\infty}(t_{\nu=120}-t_{\nu=\infty}) \tag{13}$$

and obtain $t_{\nu=227} = 2.44$. The minimum value $n_i/n = 19/20$, therefore we find the correction for $\min n_i/n = 2/3$ and $\nu = 120$ obtaining $x = 4.9\%$. It can be stated that $\min n_i/n$ is near 1 and ϱ is near 1/2, therefore there is no necessity to use a correction, but we will do it for demonstration purposes. We calculate the value of q using formula (12),

$$q = \left(\frac{4.9}{20\cdot 100}+1\right)\cdot 2.44 = 2.45 .$$

We calculate $q\ s_\nu\sqrt{1/n_i+1/n} = 16.8\cdot 16.8\cdot\sqrt{1/20+1/n_i}$ for the varieties $A, H, C, \ldots$ They are $5.3, 5.2, 4.9, \ldots$, respectively, and the values of $\bar{x}_{\max} - qs_\nu\sqrt{1/n_i+1/n}$ are $51.5, 51.6, \ldots$, respectively. We see that only for the variety E inequality (8) is fulfilled, therefore we select it as the best one.

Now we shall find the next group containing the best varieties. The next maximal mean value among the remaining 9 varieties is the mean value for variety A. Now we have $\bar{x}_{\max} = 43.4$. We proceed in a way similar, as before, for $p = 8$, $P^* = 1-a = 0.95$ and find $t_{\nu=120} = 2.41$ and $t_{\nu=\infty} = 2.38$. Using formula (13) we calculate $t_{\nu=227} = 2.40$, the minimum value $n_i/n = 19/20$, the correction $x = 5.0\%$. We calculate the value of q using formula (12) and obtain $q = 2.41$. The values $qs_\nu\sqrt{1/n_i+1/n}$ for the varieties H, C, I, J, F equal $5.1, 4.9, 4.8, 5.0, 5.0$, respectively, and the values $\bar{x}_{\max} - qs\ \sqrt{1/n_i+1/n}$ are $38.3, 38.5, 38.6, 38.4, 38.4, \ldots$, respectively. We see that only the varieties I, C, H, A fulfil inequality (8), therefore they form a group containing the best varieties among the 9 varieties in question.

Now we have still 5 varieties. We shall find among them the group containing the best varieties in a similar way as before. Now $\bar{x}_{\max} = 37.5$, $p = 4$, $\min n_i/n = 19/25$, the correction $x = 4.1$, $t_{\nu=120} = 2.18$, and $t_{\nu=\infty} = 2.16$. Using formula (13) we obtain $t_{\nu=227} = 2.17$ and, using formula (12), $q = 2.19$. The values $\bar{x}_{\max} - qs_\nu\sqrt{1/n_i+1/n}$ for the varieties F, G, D, B are $33.2, 33.4, 33.0, 32.9$, respectively. Therefore all these varieties form a group containing the best varieties among the 5 varieties in question. We shall apply the procedure $\bar{R}$ to the same data, to carry out a comparison with a control. Suppose the variety H is a control one. Then $\bar{x}_0 = 41.3$, $n_0 = 22$, $p = 9$, $\nu = 227$, $q = 2.45$, $s_\nu = 6.87$. The values $qs_\nu\sqrt{1/n_i+1/n_0} = 16.8\sqrt{1/n_i+1/22}$ for the varieties $E, A, C, I, J, F, G, D, B$ are $5.2, 5.2, 4.8, 4.9, 4.9, 4.8, 5.1, 5.3$, respectively, and the values $\bar{x}_0 - qs_\nu\sqrt{1/n_i+1/n_0}$ are $36.1, 36.1, 36.5, 36.5, 36.4, 36.4, 36.2, 36.2, 36.0$, respectively. Only the varieties E, A, C, I, J, F fulfil inequality (9), therefore they are as good or better as the control variety H. The remaining varieties G, D, B are worse than the control variety H.

7. Complementary notes. The procedure R is a one-side test and is seemingly similar to other well known tests. The differences are involved in the theoretical basis. In the procedure R the error of first kind is constant and equals $\alpha = 1-P^*$. For the Duncan test, being a two-side test, the error of first kind for a particular comparison has the value α and for p comparisons the value $1-(1-\alpha)^p$. Therefore, for the whole experiment, this error has the value $1-(1-\alpha)^{k-1}$ and tends towards 1 when k tends towards infinity. The Duncan test is used for formation of homogeneous groups. Using procedure R, we may form groups containing the best population (see chapter 6) and in each case we know that the error of first kind is always the same (α). If an experimenter wants to find the group of the best varieties, it is better to use the procedure R than the Duncan test since the committed error of first kind is smaller.

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MIROSŁAWA SITEK (Poznań)

ZASTOSOWANIE ZASADY WYBORU R PRZY NIEJEDNAKOWYCH LICZEBNOŚCIACH OBSERWACJI

STRESZCZENIE

W pracy zasada wyboru R, podana przez Guptę [3], została rozszerzona na niejednakową liczbę obserwacji. W tym celu zbudowano tablice poprawek do wartości krytycznych wielowymiarowego rozkładu t-Studenta i podano metodę posługiwania się nimi.

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1. Upper 1% critical values of the multivariate t-distribution for

1	2	3	4	5	6	7	8	9
3.36	3.90	4.21	4.43	4.60	4.73	4.85	4.94	5.03
3.14	3.61	3.88	4.06	4.21	4.32	4.42	4.51	4.58
3.00	3.42	3.66	3.83	3.96	4.06	4.15	4.22	4.29
2.90	3.29	3.51	3.66	3.78	3.88	3.96	4.03	4.09
2.82	3.19	3.40	3.54	3.66	3.75	3.82	3.89	3.94
2.76	3.11	3.31	3.45	3.56	3.64	3.72	3.78	3.83
2.72	3.06	3.25	3.38	3.48	3,56	3.63	3.69	3.74
2.68	3.01	3.19	3.32	3.42	3.50	3.56	3.62	3.67
2.65	2.97	3.15	3.27	3.37	3.44	3.51	3.56	3.61
2.62	2.93	3.11	3.23	3.32	3.40	3.46	3.51	3.56
2.60	2.91	3.08	3.20	3.29	3.36	3.42	3.47	3.52
2.58	2.88	3.05	3.17	3.26	3.33	3.39	3.44	3.48
2.57	2.86	3.03	3.14	3.23	3.30	3.36	3.41	3.45
2.55	2.84	3.01	3.12	3.20	3.27	3.33	3.38	3.42
2.54	2.83	2.99	3.10	3.18	3.25	3.31	3.36	3.40
2.53	2.81	2.97	3.08	3.16	3.23	3.29	3.34	3.38
2.52	2.80	2.96	3.07	3.15	3.21	3.27	3.32	3.36
2.51	2.79	2.94	3.05	3.13	3.20	3.25	3.30	3.34
2.50	2.77	2.93	3.04	3.12	3.18	3.24	3.28	3.32
2.49	2.77	2.92	3.02	3.11	3.17	3.22	3.27	3.31
2.48	2.76	2.91	3.01	3.10	3.16	3.21	3.26	3.30
2.48	2.75	2.90	3.00	3.08	3.15	3.20	3.24	3.29
2.47	2.74	2.89	2.99	3.07	3.14	3.19	3.23	3.27
2.47	2.73	2.88	2.99	3.06	3.13	3.18	3.22	3.26
2.46	2.73	2.88	2.98	3.06	3.12	3.17	3.21	3.25
2.46	2.72	2.87	2.97	3.05	3.11	3.16	3.20	3.24
2.45	2.72	2.86	2.96	3.04	3.10	3.15	3.20	3.23
2.45	2.71	2.86	2.96	3.03	3.10	3.15	3.19	3.23
2.44	2.71	2.85	2.95	3.03	3.09	3.14	3.18	3.22
2.44	2.70	2.85	2.95	3.02	3.08	3.13	3.17	3.21
2.44	2.70	2.84	2.94	3.01	3.08	3.13	3.17	3.21
2.42	2.68	2.82	2.92	2.99	3.05	3.10	3.14	3.18
2.39	2.64	2.78	2.87	2.94	3.00	3.04	3.08	3.12
2.36	2.60	2.73	2.82	2.89	2.94	2.99	3.03	3.06
2.33	2.56	2.68	2.77	2.84	2.89	2.93	2.97	3.00

inal data are taken out of [6], table 36, p. 43; [1], table 1b, p. 1118; and [4], table 1, p. 9

2. Upper 5 % critical values of the multivariate t-distribution for

1	2	3	4	5	6	7	8	9
2.01	2.44	2.68	2.85	2.98	3.08	3.16	3.24	3.30
1.94	2.34	2.56	2.71	2.83	2.92	3.00	3.06	3.12
1.89	2.27	2.48	2.62	2.73	2.81	2.89	2.95	3.00
1.86	2.22	2.42	2.55	2.66	2.74	2.81	2.87	2.92
1.83	2.18	2.37	2.50	2.60	2.68	2.75	2.81	2.86
1.81	2.15	2.34	2.47	2.56	2.64	2.70	2.76	2.81
1.79	2.13	2.31	2.43	2.53	2.60	2.67	2.72	2.77
1.78	2.11	2.29	2.41	2.50	2.58	2.64	2.69	2.73
1.77	2.09	2.27	2.39	2.48	2.55	2.61	2.66	2.71
1.76	2.08	2.25	2.37	2.46	2.53	2.59	2.64	2.69
1.75	2.07	2.24	2.36	2.44	2.51	2.57	2.62	2.67
1.75	2.06	2.23	2.34	2.43	2.50	2.56	2.61	2.65
1.74	2.05	2.22	2.33	2.42	2.49	2.54	2.59	2.63
1.73	2.04	2.21	2.32	2.41	2.48	2.53	2.58	2.62
1.73	2.03	2.20	2.31	2.40	2.46	2.52	2.57	2.61
1.72	2.03	2.19	2.30	2.39	2.46	2.51	2.56	2.60
1.72	2.02	2.18	2.30	2.38	2.45	2.50	2.55	2.59
1.72	2.02	2.18	2.29	2.37	2.44	2.50	2.54	2.58
1.71	2.01	2.17	2.28	2.37	2.43	2.49	2.53	2.57
1.71	2.01	2.17	2.28	2.36	2.43	2.48	2.53	2.57
1.71	2.00	2.16	2.27	2.36	2.42	2.48	2.52	2.56
1.71	2.00	2.16	2.27	2.35	2.42	2.47	2.52	2.56
1.70	2.00	2.16	2.26	2.35	2.41	2.46	2.51	2.55
1.70	1.99	2.15	2.26	2.34	2.41	2.46	2.51	2.54
1.70	1.99	2.15	2.26	2.34	2.40	2.46	2.50	2.54
1.70	1.99	2.15	2.25	2.33	2.40	2.45	2.50	2.54
1.70	1.99	2.14	2.25	2.33	2.39	2.45	2.49	2.53
1.69	1.98	2.14	2.25	2.33	2.39	2.44	2.49	2.53
1.69	1.98	2.14	2.24	2.32	2.39	2.44	2.49	2.52
1.69	1.98	2.14	2.24	2.32	2.38	2.44	2.48	2.52
1.69	1.98	2.13	2.24	2.32	2.38	2.43	2.48	2.52
1.68	1.97	2.13	2.23	2.31	2.37	2.42	2.47	2.51
1.67	1.95	2.10	2.21	2.28	2.35	2.39	2.44	2.48
1.66	1.93	2.08	2.18	2.26	2.32	2.37	2.41	2.45
1.64	1.92	2.06	2.16	2.23	2.29	2.34	2.38	2.42

inal data are taken out of [6], table 16, p. 23; [1], table 1a, p. 1117; and [4], table 1, p. 9

TABLE 3. Correction table for $\alpha = 0.01$

ν	$\frac{\min n_i}{n}$ \ p	2	3	4	5	6	7	8	9	10
5	0	2.6	4.3	5.4	6.5	7.4	8.0	8.9	9.3	9.8
	$\frac{1}{4}$	2.4	4.1	5.1	5.8	6.8	7.4	8.1	8.5	8.6
	$\frac{2}{3}$	2.3	3.6	4.7	5.2	6.3	6.2	7.3	7.2	7.0
6	0	1.9	3.3	4.7	5.5	6.2	7.0	7.3	7.9	8.4
	$\frac{1}{4}$	1.8	3.1	4.6	5.1	5.9	6.3	6.8	7.3	7.8
	$\frac{2}{3}$	1.7	3.1	4.4	4.3	5.6	6.1	6.0	6.5	7.1
7	0	1.7	3.0	3.9	4.8	5.4	6.0	6.6	7.0	7.4
	$\frac{1}{4}$	1.9	2.9	3.8	4.7	5.3	5.8	6.3	6.5	6.7
	$\frac{2}{3}$	1.7	2.5	3.1	3.8	5.2	5.1	5.7	5.6	6.2
8	0	1.5	2.8	3.8	4.5	4.9	5.3	5.7	6.1	6.8
	$\frac{1}{4}$	1.6	2.7	3.6	4.2	4.8	5.0	5.6	5.9	6.4
	$\frac{2}{3}$	0.9	2.6	4.1	4.0	4.6	4.5	5.2	5.1	5.8
9	0	1.6	2.6	3.4	3.8	4.3	5.0	5.1	5.8	6.0
	$\frac{1}{4}$	1.7	2.3	3.4	3.6	4.3	4.9	5.1	5.7	6.0
	$\frac{2}{3}$	0.9	2.6	3.4	3.3	4.0	4.7	4.6	5.3	5.3
10	0	1.6	2.4	3.2	3.6	4.1	4.6	4.8	5.2	5.4
	$\frac{1}{4}$	1.7	2.4	3.1	3.7	4.4	4.3	4.6	5.2	5.5
	$\frac{2}{3}$	1.9	2.7	3.5	3.4	4.1	4.0	4.0	4.7	4.6
11	0	1.3	2.1	3.0	3.4	3.9	4.4	4.6	5.1	5.0
	$\frac{1}{4}$	1.3	2.0	2.8	3.4	4.1	4.4	4.7	5.0	4.9
	$\frac{2}{3}$	1.0	1.8	2.7	3.4	4.2	4.1	4.1	4.8	4.7
12	0	1.3	2.2	2.7	3.2	3.7	4.2	4.4	4.6	5.1
	$\frac{1}{4}$	1.3	2.1	2.8	3.1	3.4	4.1	4.4	4.7	5.0
	$\frac{2}{3}$	1.0	1.9	2.7	3.5	3.4	4.2	4.1	4.1	4.8
13	0	1.0	1.9	2.7	3.0	3.5	3.7	4.2	4.4	4.7
	$\frac{1}{4}$	1.3	2.1	2.8	3.2	3.9	3.8	4.1	4.4	4.7
	$\frac{2}{3}$	1.0	1.9	2.7	2.7	3.5	3.4	4.2	4.2	4.9
14	0	1.4	1.9	2.5	3.0	3.2	3.8	4.0	4.2	4.4
	$\frac{1}{4}$	1.4	2.1	2.9	3.2	3.5	3.8	4.2	4.5	4.4
	$\frac{2}{3}$	2.0	1.9	2.8	3.6	3.5	3.5	4.3	4.2	4.2
15	0	1.0	1.9	2.5	2.7	3.3	3.5	3.7	4.0	4.2
	$\frac{1}{4}$	0.9	1.7	2.5	2.8	3.2	3.5	3.8	4.2	4.5
	$\frac{2}{3}$	1.0	1.9	1.9	2.7	3.6	3.5	4.3	3,4	4.2
16	0	1.4	2.0	2.2	2.8	3.0	3.2	3.5	4.0	4.3
	$\frac{1}{4}$	1.4	2.2	2.1	2.9	3.2	3.5	3.5	4.2	4.2
	$\frac{2}{3}$	1.0	2.0	1.9	2.8	2.7	3.5	3.5	4.3	4.3

TABLE 3 (ctd.)

v	$\min n_i / n$	p = 2	3	4	5	6	7	8	9	10
17	0	1.0	1.6	2.2	2.8	3.0	3.3	3.5	3.8	4.0
	$\frac{1}{4}$	1.4	1.8	2.5	2.9	2.8	3.2	3.5	3.9	4.2
	$\frac{2}{3}$	1.0	2.0	2.9	2.8	2.7	2.7	3.5	4.3	4.3
18	0	1.1	1.7	2.2	2.8	3.1	3.3	3.5	3.8	3.8
	$\frac{1}{4}$	1.4	1.8	2.1	2.9	3.3	3.2	3.5	3.9	3.8
	$\frac{2}{3}$	1.1	1.0	1.9	2.8	3.7	3.6	3.5	4.4	4.3
19	0	1.1	1.7	1.9	2.5	2.8	3.0	3.3	3.5	3.5
	$\frac{1}{4}$	0.9	1.8	2.1	2.9	2.9	3.2	3.2	3.5	3.9
	$\frac{2}{3}$	1.1	1.0	1.9	2.8	2.8	2.7	2.7	3.5	3.5
20	0	1.1	1.7	1.9	2.5	2.8	2.7	3.0	3.2	3.8
	$\frac{1}{4}$	0.9	1.8	2.2	2.5	2.9	2.8	3.2	3.5	3.9
	$\frac{2}{3}$	1.1	2.0	1.9	2.8	2.8	2.7	2.7	3.5	4.4
21	0	1.1	1.3	1.9	2.2	2.8	2.7	3.0	3.3	3.5
	$\frac{1}{4}$	0.9	1.3	2.2	2.5	2.9	2.8	3.2	3.6	3.9
	$\frac{2}{3}$	1.1	1.0	1.9	1.9	2.8	2.7	2.7	3.6	4.4
22	0	0.7	1.7	2.0	2.2	2.5	2.8	3.0	3.3	3.2
	$\frac{1}{4}$	1.0	1.8	2.2	2.6	2.5	3.3	3.2	3.6	3.5
	$\frac{2}{3}$	1.1	2.0	2.0	2.9	2.8	3.7	2.7	3.6	3.5
23	0	1.1	1.4	2.0	2.2	2.8	2.8	3.0	3.3	3.3
	$\frac{1}{4}$	1.4	1.8	2.2	2.1	2.9	2.9	3.2	3.6	3.6
	$\frac{2}{3}$	1.1	2.0	2.0	1.9	2.8	2.8	3.7	3.6	3.6
24	0	0.7	1.4	2.0	1.9	2.5	2.8	2.7	3.0	3.6
	$\frac{1}{4}$	1.0	1.4	2.2	2.1	2.5	3.3	3.3	3.2	3.6
	$\frac{2}{3}$	0.0	2.0	3.0	1.9	2.8	3.7	2.7	3.6	3.6
25	0	0.7	1.4	2.0	1.9	2.5	2.8	2.8	3.0	3.3
	$\frac{1}{4}$	1.0	1.4	2.2	2.1	2.5	2.9	2.9	3.2	3.6
	$\frac{2}{3}$	1.1	2.1	2.0	1.9	1.9	2.8	2.8	2.7	3.6
26	0	0.7	1.4	2.0	2.3	2.2	2.8	3.1	2.7	3.0
	$\frac{1}{4}$	1.0	1.4	2.2	2.6	2.5	2.9	3.3	2.8	3.2
	$\frac{2}{3}$	1.1	2.1	2.0	2.9	1.9	2.8	3.7	2.7	3.6
27	0	1.1	1.4	2.0	2.3	2.2	2.5	2.8	3.1	3.0
	$\frac{1}{4}$	1.0	1.8	2.2	2.6	2.5	2.9	3.3	3.3	3.2
	$\frac{2}{3}$	1.1	2.1	2.0	2.9	1.9	2.8	3.7	3.7	2.7
28	0	1.1	1.7	1.7	2.3	2.2	2.5	2.8	3.1	3.0
	$\frac{1}{4}$	1.0	1.8	1.8	2.6	2.6	2.5	3.3	3.3	3.2
	$\frac{2}{3}$	1.1	2.1	2.0	2.9	1.9	2.8	3.7	3.7	2.7

TABLE 3 (ctd.)

ν	p / $\frac{\min n}{n}$	2	3	4	5	6	7	8	9	10
29	0	0.7	1.4	1.7	2.0	2.2	2.5	2.8	3.1	3.0
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.6	2.5	3.3	3.3	3.2
	$\frac{2}{3}$	1.1	1.0	2.0	2.0	1.9	2.8	3.7	3.7	2.7
30	0	1.1	1.4	1.7	2.0	2.2	2.5	2.8	3.1	2.7
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.6	2.5	3.3	3.3	3.2
	$\frac{2}{3}$	1.1	2.1	2.0	2.0	2.9	2.8	3.7	3.7	2.7
31	0	0.7	1.4	2.0	2.0	2.3	2.5	2.5	3.1	2.7
	$\frac{1}{4}$	1.0	1.9	2.2	2.2	2.6	3.0	2.9	3.3	3.3
	$\frac{2}{3}$	0.0	2.1	2.0	2.0	2.9	2.9	2.8	3.7	3.7
32	0	0.7	1.4	1.7	2.3	1.9	2.2	2.5	2.8	3.1
	$\frac{1}{4}$	1.0	1.4	1.8	2.6	2.1	2.5	2.9	2.9	3.3
	$\frac{2}{3}$	1.1	1.0	2.0	3.0	1.9	1.9	2.8	2.8	3.7
33	0	0.7	1.4	1.7	2.0	2.3	2.2	2.8	2.8	3.1
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.2	2.5	2.9	2.9	3.3
	$\frac{2}{3}$	1.1	2.1	2.0	2.0	1.9	2.9	2.8	2.8	3.7
34	0	0.7	1.4	1.4	2.0	2.3	2.6	2.8	2.8	2.8
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.6	2.6	2.9	3.3	2.9
	$\frac{2}{3}$	1.1	1.0	1.0	2.0	2.9	2.9	2.8	3.7	2.8
35	0	0.7	1.4	1.7	2.3	1.9	2.2	2.5	2.8	2.8
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.2	2.6	2.5	2.9	3.3
	$\frac{2}{3}$	1.1	2.1	2.0	3.0	1.9	1.9	2.8	2.8	3.7
40	0	0.7	1.4	1.7	2.0	2.0	2.3	2.5	2.8	2.8
	$\frac{1}{4}$	1.0	1.4	1.8	2.2	2.2	2.6	2.5	2.9	3.3
	$\frac{2}{3}$	1.1	2.1	2.0	3.0	2.9	2.9	2.9	2.8	3.7
60	0	0.8	1.1	1.4	2.0	1.7	2.3	2.6	2.6	2.9
	$\frac{1}{4}$	1.0	1.0	1.9	1.8	2.2	2.6	2.6	2.6	3.4
	$\frac{2}{3}$	1.1	1.1	2.1	3.1	2.0	3.0	2.9	2.9	4.8
120	0	0.8	1.1	1.4	1.7	1.7	1.7	2.0	2.3	2.3
	$\frac{1}{4}$	0.5	1.5	1.4	1.8	2.3	2.2	2.2	2.6	2.6
	$\frac{2}{3}$	1.1	2.2	2.1	2.1	3.1	3.0	2.0	2.9	2.9
∞	0	0.8	1.1	1.1	1.4	1.4	1.7	1.7	2.0	2.0
	$\frac{1}{4}$	0.5	1.5	1.4	1.4	1.8	2.3	2.2	2.2	2.2
	$\frac{2}{3}$	1.2	2.2	2.2	2.1	2.1	3.1	2.0	3.0	3.0

TABLE 4. Correction table for $\alpha = 0.05$

ν	$\min n_i / n$ \ p	2	3	4	5	6	7	8	9	10
5	0	3.7	6.0	7.4	8.4	9.7	10.8	11.1	11.8	12.2
	$\frac{1}{4}$	3.8	5.5	7.0	7.6	8.7	9.7	9.9	10.5	10.7
	$\frac{2}{3}$	3.7	4.5	6.3	6.0	7.8	8.5	8.3	9.1	8.9
6	0	3.4	5.5	6.6	7.8	8.6	9.3	10.1	10.6	11.4
	$\frac{1}{4}$	2.8	5.2	6.4	7.1	8.2	8.4	9.6	9.8	10.1
	$\frac{2}{3}$	2.6	4.7	5.5	6.4	7.2	7.0	8.8	8.6	9.5
7	0	3.1	4.8	6.1	7.0	8.2	8.6	9.1	10.0	10.2
	$\frac{1}{4}$	2.9	4.3	5.6	6.8	7.6	7.8	8.6	9.3	9.6
	$\frac{2}{3}$	2.6	3.6	4.6	5.5	7.5	7.3	7.1	8.0	7.9
8	0	2.7	4.5	5.9	6.8	7.7	8.2	8.7	9.2	9.8
	$\frac{1}{4}$	3.0	4.4	5.7	6.0	7.3	7.6	7.9	8.7	9.0
	$\frac{2}{3}$	2.7	3.7	5.9	5.6	6.6	6.4	7.3	7.2	8.1
9	0	2.7	4.6	5.6	6.5	7.1	7.6	8.2	8.4	9.0
	$\frac{1}{4}$	3.1	4.5	5.3	6.1	7.0	7.3	7.6	7.9	8.3
	$\frac{2}{3}$	2.7	3.8	4.8	5.8	6.7	6.5	6.4	6.3	7.2
10	0	2.8	4.3	5.3	6.2	6.8	7.4	7.6	8.2	8.8
	$\frac{1}{4}$	2.5	4.0	4.9	6.2	6.6	7.4	7.2	7.6	7.9
	$\frac{2}{3}$	2.8	3.8	3.6	5.9	5.7	6.7	6.5	6.4	7.4
11	0	2.3	4.3	5.3	5.9	6.5	7.1	7.7	7.9	8.2
	$\frac{1}{4}$	2.5	4.0	5.5	5.8	6.7	6.5	7.3	7.2	7.6
	$\frac{2}{3}$	1.4	3.9	4.9	4.7	5.8	5.6	6.6	6.5	6.4
12	0	2.4	3.9	5.0	6.0	6.2	6.8	7.4	7.7	8.3
	$\frac{1}{4}$	2.5	3.5	5.0	5.9	6.2	6.6	6.9	7.8	8.2
	$\frac{2}{3}$	1.4	3.9	5.0	4.8	4.6	5.7	6.7	7.7	7.6
13	0	2.9	4.0	5.0	5.6	6.3	6.9	7.1	7.4	7.6
	$\frac{1}{4}$	2.5	3.5	4.5	5.4	6.3	6.6	7.0	6.9	7.3
	$\frac{2}{3}$	2.9	4.0	3.8	4.8	5.9	5.7	6.8	6.6	6.5
14	0	2.4	4.0	4.6	5.3	5.9	6.6	6.8	7.1	7.7
	$\frac{1}{4}$	2.6	4.1	4.5	5.4	5.8	6.2	6.6	6.9	7.3
	$\frac{2}{3}$	1.4	4.0	3.8	4.9	5.9	6.8	6.8	5.6	6.6
15	0	2.4	3.6	4.7	5.3	6.0	6.6	6.9	7.1	7.4
	$\frac{1}{4}$	2.6	3.6	4.5	5.5	5.8	6.2	6.6	6.5	6.9
	$\frac{2}{3}$	1.4	2.7	3.8	4.9	6.0	5.8	6.9	5.6	5.5
16	0	2.4	3.6	4.7	5.3	5.6	6.2	6.5	6.8	7.1
	$\frac{1}{4}$	1.9	3.6	4.6	4.9	5.9	5.7	6.1	6.5	6.9
	$\frac{2}{3}$	1.5	2.7	3.8	4.9	4.8	4.7	5.7	6.8	5.6

TABLE 4 (ctd.)

v	$\min n_i / n$ \ p	2	3	4	5	6	7	8	9	10
17	0	2.4	3.6	4.7	5.0	5.6	6.3	6.6	6.8	7.1
	$\frac{1}{4}$	2.6	3.6	4.6	5.0	5.4	6.3	6.7	7.1	7.0
	$\frac{2}{3}$	1.5	2.7	3.9	5.0	4.8	5.9	5.8	6.8	6.7
18	0	2.4	3.6	4.3	5.0	5.2	5.9	6.2	6.9	6.8
	$\frac{1}{4}$	2.6	3.6	4.6	5.0	5.4	5.8	6.2	6.6	7.0
	$\frac{2}{3}$	2.9	2.7	3.9	3.7	4.8	5.9	5.8	6.9	6.8
19	0	2.5	3.6	4.3	5.0	5.7	5.9	6.2	6.5	6.8
	$\frac{1}{4}$	2.6	3.6	4.6	5.0	6.0	5.8	6.2	6.6	6.5
	$\frac{2}{3}$	3.0	2.7	3.9	3.7	6.1	5.9	5.8	5.7	5.7
20	0	2.5	3.6	4.3	5.0	5.3	6.0	6.2	6.5	6.4
	$\frac{1}{4}$	2.0	3.6	4.6	5.0	4.9	5.8	6.2	6.7	6.6
	$\frac{2}{3}$	1.5	4.1	3.9	3.8	4.9	6.0	5.9	5.8	5.7
21	0	2.5	3.7	3.9	5.0	5.3	6.0	6.3	6.6	6.5
	$\frac{1}{4}$	2.6	3.7	4.1	5.0	5.4	5.9	6.3	6.2	6.6
	$\frac{2}{3}$	3.0	4.1	3.9	5.0	4.9	6.0	5.9	5.8	5.7
22	0	2.0	3.7	4.4	5.1	5.3	5.6	6.3	6.6	6.5
	$\frac{1}{4}$	2.0	3.7	4.1	5.1	5.5	5.3	6.4	6.7	6.6
	$\frac{2}{3}$	1.5	2.7	3.9	5.1	4.9	4.8	5.9	5.8	5.7
23	0	2.5	3.7	4.4	4.6	5.3	5.6	6.3	6.6	6.5
	$\frac{1}{4}$	2.6	3.7	4.7	4.5	5.5	5.4	6.3	6.7	6.6
	$\frac{2}{3}$	3.0	4.1	3.9	3.8	4.9	4.8	5.9	5.8	5.7
24	0	2.0	3.2	3.9	4.7	4.9	5.6	5.9	6.2	6.5
	$\frac{1}{4}$	2.0	3.1	4.1	5.1	4.9	5.4	5.8	6.2	6.7
	$\frac{2}{3}$	1.5	2.8	3.9	5.1	4.9	4.8	4.7	5.8	6.9
25	0	2.5	3.7	4.4	4.7	5.4	5.2	5.9	6.2	6.1
	$\frac{1}{4}$	2.7	3.7	4.1	4.5	5.5	5.4	5.8	6.2	6.1
	$\frac{2}{3}$	3.0	4.2	4.0	3.8	5.0	4.8	5.9	5.9	5.8
26	0	2.5	3.2	4.0	4.7	5.0	5.3	5.6	5.9	6.2
	$\frac{1}{4}$	2.0	3.1	4.1	4.5	5.0	5.4	5.8	5.7	6.2
	$\frac{2}{3}$	1.5	2.8	4.0	3.8	3.7	4.9	4.8	4.7	5.8
27	0	2.0	3.2	4.4	4.7	5.0	5.7	5.6	5.9	6.6
	$\frac{1}{4}$	2.0	3.1	4.1	4.5	5.0	6.0	5.8	6.3	6.7
	$\frac{2}{3}$	1.5	2.8	5.3	3.8	5.0	6.1	6.0	5.9	7.0
28	0	2.5	3.3	4.0	4.7	5.0	5.3	5.6	6.3	6.2
	$\frac{1}{4}$	2.7	3.7	4.1	4.6	5.0	5.4	5.3	6.3	6.2
	$\frac{2}{3}$	3.0	4.2	4.0	5.1	3.7	4.9	4.8	5.9	5.8

TABLE 4 (ctd.)

ν	$\min n_i / n$	$p=2$	3	4	5	6	7	8	9	10
29	0	2.5	3.3	4.0	4.3	5.0	5.3	5.6	5.9	6.2
	$\frac{1}{4}$	2.0	3.1	4.1	4.6	5.0	4.9	5.9	5.8	6.7
	$\frac{2}{3}$	1.5	2.8	4.0	3.8	5.0	4.9	6.0	5.9	7.0
30	0	2.0	3.3	4.0	4.7	5.0	5.3	5.6	5.9	6.2
	$\frac{1}{4}$	2.0	3.1	4.1	4.6	5.0	5.4	5.3	5.8	6.2
	$\frac{2}{3}$	1.5	2.8	4.0	5.1	5.0	4.9	4.8	4.7	5.8
31	0	2.0	3.3	4.0	4.7	5.0	5.3	5.6	5.9	5.8
	$\frac{1}{4}$	2.0	3.7	4.1	4.6	5.6	5.4	5.9	5.8	6.2
	$\frac{2}{3}$	1.5	4.2	4.0	3.9	5.0	4.9	6.0	5.9	5.8
32	0	2.5	3.3	4.0	4.3	5.0	5.3	5.6	5.9	6.2
	$\frac{1}{4}$	2.7	3.1	4.1	4.6	5.0	5.5	5.3	5.8	6.2
	$\frac{2}{3}$	3.0	2.8	4.0	3.9	5.0	6.1	4.8	4.7	5.9
33	0	2.5	3.3	4.0	4.7	5.0	5.3	5.2	6.0	5.9
	$\frac{1}{4}$	2.0	3.1	4.2	4.6	5.0	5.5	5.4	6.3	6.2
	$\frac{2}{3}$	1.5	2.8	4.0	5.2	5.0	4.9	4.8	5.9	5.9
34	0	2.0	3.3	4.0	4.3	5.0	4.9	5.6	5.9	5.9
	$\frac{1}{4}$	2.0	3.1	4.2	4.6	5.0	4.9	5.9	5.8	5.7
	$\frac{2}{3}$	1.5	2.8	4.0	3.9	5.0	4.9	6.0	5.9	4.7
35	0	2.0	3.3	4.0	4.3	5.0	5.3	5.2	5.6	6.3
	$\frac{1}{4}$	2.0	3.8	4.2	4.6	5.0	5.5	5.4	5.8	6.3
	$\frac{2}{3}$	1.5	4.2	4.0	3.9	5.0	6.2	4.8	4.8	5.9
40	0	2.0	2.8	4.0	4.3	4.6	5.0	4.9	5.2	5.9
	$\frac{1}{4}$	2.0	3.1	4.2	4.6	5.1	5.5	5.4	5.3	5.8
	$\frac{2}{3}$	1.5	2.8	4.0	3.9	5.1	6.2	4.9	4.8	5.9
60	0	2.0	3.3	3.6	4.4	4.2	5.0	4.9	5.2	6.0
	$\frac{1}{4}$	2.7	3.8	3.6	4.7	4.5	5.0	4.9	5.4	6.4
	$\frac{2}{3}$	3.1	4.3	4.1	3.9	3.8	6.3	4.9	4.8	6.0
120	0	2.1	2.9	3.7	4.0	4.3	4.6	5.0	4.9	5.2
	$\frac{1}{4}$	2.8	3.2	4.3	4.1	4.6	4.5	5.0	5.4	5.9
	$\frac{2}{3}$	3.1	2.9	4.1	4.0	5.2	5.1	5.0	4.9	4.8
∞	0	1.6	2.9	3.2	4.0	3.9	4.3	4.6	4.5	4.9
	$\frac{1}{4}$	2.1	3.2	3.7	4.2	4.7	4.6	5.0	5.0	5.4
	$\frac{2}{3}$	1.6	2.9	4.2	4.0	5.2	5.1	5.0	5.0	4.9