ON INDUCTIVE LIMITS OF TOPOLOGICAL ALGEBRAS

BY

JERZY KAKOL (POZNAŃ)

Let $X$ be an algebra. A linear space topology $\sigma$ on $X$ is called multiplicative if the map $(x, y) \mapsto xy$ of $X \times X$ into $X$ is (jointly) continuous. $X$ equipped with such a topology is called a topological algebra. Clearly, $\sigma$ is multiplicative iff for every neighbourhood $U$ of zero there exists a neighbourhood $V$ of zero such that $VV \subset U$. If the absolutely convex neighbourhoods $U$ of zero such that $UU \subset U$ form a base of neighbourhoods of zero, then $X$ is called locally multiplicatively-convex (locally $m$-convex) [8]. We say that a subset $S$ of a topological algebra is $m$-bounded if for each neighbourhood $U$ of zero there exists a neighbourhood $V$ of zero such that $SV \cup VS \subset U$ or, equivalently, if the maps $x \mapsto xy_0$ and $y \mapsto x_0y$, where $x_0, y_0 \in S$, are equicontinuous at zero. If $S$ is a bounded subset of $X$, then $S$ is $m$-bounded (cf. the proof of Corollary 1). The converse holds when $X$ has a unit.

In his fundamental work [8] on locally $m$-convex algebras Michael gave some sufficient conditions for the local $m$-convexity of the algebra $X$ equipped with the linear inductive limit topology associated with an increasing sequence $(X_n, \sigma_n)$ of locally $m$-convex subalgebras of $X$. This study was continued by Warner [13] who gave some other conditions to this effect with many applications.

The present paper deals with a similar problem in the context of the so-called generalized inductive limits of topological algebras.

The notion of generalized inductive limit of locally convex spaces was introduced first by Garling [5] who was inspired by some ideas contained in the earlier work of Wiweger [14]; a careful study of an important particular case was carried out by Roeleke [9]. Extensions of this notion to arbitrary topological linear spaces are due to Turpin (1971) and Adasch and Ernst (1974) (see [10], [11], and [1] for an account of their investigations). In the sequel we shall essentially follow Turpin [11].

Let $X$ be a linear space over the field $K$ of real or complex scalars and let $D = \{a \in K: |a| \leq 1\}$. By a balanced topological space we mean a balanced subset $S$ (of $X$) equipped with a topology $\sigma$ such that the map
(a, x) \mapsto ax of D \times S into S is continuous. By an *inductive system* (of balanced topological spaces) on \(X\) we shall understand a sequence

\[ \Gamma = (S_n, \sigma_n : n \in \mathbb{N}) \]

of balanced topological subspaces of \(X\) such that

(I) \(X = \bigcup_{n=1}^{\infty} S_n;\)

(I) \(S_n + S_n \subseteq S_{n+1}\) and the map \((x, y) \mapsto x + y\) of \(S_n \times S_n\) into \(S_{n+1}\) is continuous at zero for all \(n \in \mathbb{N} = \{1, 2, \ldots\}.\)

It follows from (I) that \(S_n \subseteq S_{n+1}\) and the inclusion map is continuous, i.e., the topology induced by \(\sigma_{n+1}\) on \(S_n\) is weaker than \(\sigma_n\) (in symbols: \(\sigma_{n+1} \leq \sigma_n\)).

Let \(\Gamma = (S_n, \sigma_n : n \in \mathbb{N})\) be an inductive system on \(X\). We denote by \(\sigma_r\) the finest linear topology on \(X\) such that \(\sigma_r|S_n \leq \sigma_n\) for all \(n \in \mathbb{N}\).

\[ \mathcal{B}_n = \mathcal{B}(\sigma_n)\]

be a base of balanced neighbourhoods of zero in \((S_n, \sigma_n)\).

Then the family of all sets

\[ U = \bigcup_{n=1}^{\infty} U_n := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} U_k, \]

where \(U_n \in \mathcal{B}_n\) (\(n = 1, 2, \ldots\)), is a neighbourhood base of zero for \(\sigma_r\).

**Special cases.** We shall call \(\Gamma\)

(i) *locally convex* if, for each \(n \in \mathbb{N}, S_n\) is absolutely convex and \(\sigma_n\) has a base of zero consisting of absolutely convex sets (in this case \(\sigma_r\) is obviously locally convex);

(ii) *strict* if \(\sigma_{n+1} \leq \sigma_n\) for all \(n \in \mathbb{N}\) (in this case \(\sigma_r \leq \sigma_n\); [11], p. 41);  

(iii) *bounded* if each \(S_n\) is a bounded subset of \(S_{n+1}\) (i.e., \(S_n\) is absorbed by every neighbourhood of zero in \(S_{n+1}\));

(iv) *simple* if \(X\) is equipped with a linear topology \(\sigma\) and \(\sigma_n = \sigma|S_n\) for all \(n \in \mathbb{N}\);

(v) *bornivorous* if \(\Gamma\) is simple and every \(\sigma\)-bounded subset of \(X\) is contained in some \(S_m\);

(vi) *usual* if each \(S_n\) is a linear subspace of \(X\) and \((S_n, \sigma_n)\) is a topological linear space.

If \(\Gamma_1\) and \(\Gamma_2\) are two inductive systems on \(X\), then we call them *equivalent* and write \(\Gamma_1 \sim \Gamma_2\) if \(\sigma_{r_1} = \sigma_{r_2}\). It is easily seen that if \(\Gamma = (S_k, \sigma_k : k \in \mathbb{N})\) and \((k_n : n \in \mathbb{N}), (m_n : n \in \mathbb{N})\) are strictly increasing sequences in \(\mathbb{N}\) such that \(k_n \leq m_n\) (\(n \in \mathbb{N}\)), then \(\Gamma' = (S_{k_n}, \sigma_{m_n}|S_{k_n} : n \in \mathbb{N})\) is an inductive system on \(X\) and \(\Gamma \sim \Gamma'\).

Now we suppose \(X\) is a (linear) algebra and let \(\Gamma = (S_n, \sigma_n : n \in \mathbb{N})\) be an inductive system on \(X\). It is easily seen that if, for every \(n \in \mathbb{N},\)
$S_nS_n \subseteq S_{n+1}$ and the map $(x, y) \mapsto xy$ of $S_n \times S_n$ into $S_{n+1}$ is separately continuous, then the multiplication on $X$ is also separately continuous under $\sigma_r$. As we would like $(X, \sigma_r)$ to be a topological algebra, it is natural to impose somewhat stronger conditions on $\Gamma$. We shall therefore say that the system $\Gamma$ is algebraic if

(I$_3$) $S_nS_n \subseteq S_{n+1}$ and the map $(x, y) \mapsto xy$ from $S_n \times S_n$ into $S_{n+1}$ is continuous at zero for all $n \in N$.

The system $\Gamma$ is said to be $m$-bounded if

(m) for every $n \in N$ and every $U \in \mathcal{A}_{n+1}$ there exists $V \in \mathcal{A}_n$ such that $VS_n \cup S_nV \subseteq U$.

Note that if $S_nS_n \subseteq S_{n+1}$ for all $n \in N$, then (m) implies (I$_3$).

Our main result is Theorem 1 which shows that (m) suffices for the multiplicativity of $\sigma_r$; for a simple and bounded inductive system $\Gamma$ it is also necessary (Corollary 1). In corollaries to Theorem 1 we indicate also a number of cases where the initial algebraic system $\Gamma$ does not satisfy (m) but for which an equivalent $m$-bounded system $\Gamma'$ can be found.

**Theorem 1.** If $\Gamma = (S_n, \sigma_n; n \in N)$ is an $m$-bounded inductive system on the algebra $X$, then $(X, \sigma_r)$ is a topological algebra.

**Proof.** Let $p : N \times N \to N$ be an injective map such that $p(1, 1) = 2$ and $p(i, j) \geq i+j$ for all $i, j \in N$. Let

$$U = \sum_{n=1}^{\infty} U_n,$$

where $U_n \in \mathcal{A}_n$ for all $n \in N$. We shall find sets $V_n \in \mathcal{A}_n$ such that

(1) $V_nV_m \subseteq U_{p(n,m)}$ for all $n, m \in N$.

Hence it will follow that the $\sigma_r$-neighbourhood of zero

$$V = \sum_{n=1}^{\infty} V_n$$

satisfies $VV \subseteq U$.

For $n = m = 1$ choose $V_1 \in \mathcal{A}_1$ such that $S_1V_1 \cup V_1S_1 \subseteq U_2 = U_{p(1,1)}$; then (1) is satisfied for $n = m = 1$. Suppose we have already found sets $V_i \in \mathcal{A}_i$, $i = 1, 2, \ldots, n$, for some $n \geq 1$, so that (1) is satisfied for $1 \leq m \leq n$. If $1 \leq k \leq n+1$, then $p(n+1, k) \geq n+2$, and hence by (m) we may find $W^{(k)} \in \mathcal{A}_{n+1}$ such that

$$W^{(k)}S_{n+1} \subseteq U_{p(n+1,k)} \quad \text{and} \quad S_{n+1}W^{(k)} \subseteq U_{p(k,n+1)}.$$

Let $V_{n+1} \in \mathcal{A}_{n+1}$ be such that

$$V_{n+1} \subseteq W^{(1)} \cap W^{(2)} \cap \ldots \cap W^{(n+1)}.$$
Now, if $1 \leq m \leq n+1$, then

$$V_m V_{n+1} \subseteq S_{n+1} W^{(m)} \subseteq U_{p(m,n+1)}, \quad V_{n+1} V_m \subseteq W^{(m)} S_{n+1} \subseteq U_{p(n+1,m)}.$$ 

This completes the proof.

**Corollary 1.** If $\Gamma$ is a bounded algebraic inductive system on the algebra $X$, then $(X, \sigma_\Gamma)$ is a topological algebra.

**Proof.** If $U \in \mathcal{A}_{n+2}$, then by (I3) there exists $V \in \mathcal{A}_{n+1}$ such that $VV \subseteq U$. Since $S_n$ is bounded in $S_{n+1}$, there exists $a \in (0, 1)$ such that $aS_n \subseteq V$. Choose $W \in \mathcal{A}_n$ so that $W \subseteq aV$. Then

$$S_n W \subseteq S_n (aV) = (aS_n) V \subseteq VV \subseteq U$$

and, similarly, $WS_n \subseteq U$. It follows that the inductive system

$$\Gamma_1 = (S_{2n-1}, \sigma_{2n-1} : n \in \mathbb{N})$$

is m-bounded. Evidently, $\Gamma \sim \Gamma_1$, and so we may apply Theorem 1, which completes the proof.

**Corollary 2.** Let $\Gamma$ be a bounded simple inductive system on the algebra $X$. Then $\sigma_\Gamma$ is multiplicative iff $\Gamma$ is m-bounded.

**Corollary 3.** Let $(X, \sigma)$ be a topological algebra with a fundamental sequence of bounded sets and let $\tau^*$ be another multiplicative topology on $X$ such that $\tau^* \leq \sigma$. Then the finest linear topology $\gamma$ on $X$ agreeing with $\tau^*$ on all $\sigma$-bounded sets is multiplicative.

**Proof.** From the assumption on $(X, \sigma)$ it follows that it has a fundamental sequence $(S_n : n \in \mathbb{N})$ of bounded balanced sets such that $(S_n + S_n) \cup (S_n S_n) \subseteq S_{n+1}$ for all $n \in \mathbb{N}$. Then $\gamma = \tau^*_\sigma$, where $\Gamma = (S_n, \tau^* : S_n : n \in \mathbb{N})$, and so it is enough to apply Corollary 1.

**Example.** Let $C(S)$ be the topological algebra of all bounded and continuous (real- or complex-valued) functions on a locally compact Hausdorff space $S$ equipped with the sup-norm topology $\sigma$. From Corollary 3 it follows immediately that the strict topology $\beta$ on $C(S)$ (cf. [3]), i.e., the finest locally convex topology on $C(S)$ agreeing with the compact-open topology on all $\sigma$-bounded sets, is multiplicative. For another proof see [3], p. 152.

Let $(Y, \partial)$ be a topological linear space. Then $\mathcal{A}(\partial)$ will denote a (fixed) base of balanced $\partial$-neighbourhoods of zero and $\text{Bd}(\partial)$ the class of all $\partial$-bounded subsets of $Y$.

**Theorem 2.** Let $\Gamma = (S_n, \sigma_n : n \in \mathbb{N})$ be a usual inductive system of topological algebras on $X$ such that

$$(2) \quad \mathcal{A}(\sigma_n) \cap \text{Bd}(\sigma_{n+1}) \neq \emptyset \quad \text{for each} \ n \in \mathbb{N}.$$ 

Then $(X, \sigma_\Gamma)$ is a topological algebra.
Proof. Let $U_n \in \mathcal{B}(\sigma_n) \cap \text{Bd}(\sigma_{n+1})$ for each $n \in N$. First we shall construct an inductive system $\Gamma_1 = (A_n, \sigma_n | A_n ; n \in N)$ such that $\Gamma \sim \Gamma_1$, where $A_n \in \mathcal{B}(\sigma_n) \cap \text{Bd}(\sigma_{n+1})$ and also $(A_n + A_n) \cup (A_n A_n) \subset A_{n+1}$ for all $n \in N$.

Set $A_1 = U_1$. Suppose we have already defined $A_1, A_2, \ldots, A_n$ in such a way that the desired conditions are satisfied. Since $(A_n + A_n) \cup (A_n A_n)$ is $\sigma_{n+1}$-bounded, it is contained in $aU_{n+1}$ for some $a > 0$. Set $A_{n+1} = aU_{n+1}$. It is obvious that $\Gamma_1$ is an algebraic and bounded inductive system on $X$. We have also $\Gamma \sim \Gamma_1$, as is seen from the following simple fact (cf. [5]): If $a$, $\beta$ are two linear topologies on a linear space and $U$ is a $\beta$-neighbourhood of zero, then $a \leq \beta$ iff $a | U \leq \beta | U$. Finally, by Corollary 1, the topology $\sigma_1$ is multiplicative.

Remark. Condition (2) is clearly satisfied when each $(S_n, \sigma_n)$ is locally bounded or when the inclusion map of $S_n$ into $S_{n+1}$ is compact (or pre-compact) for each $n \in N$.

Corollary 1. The (linear topological) direct sum of a sequence of locally bounded topological algebras is a topological algebra.

Remark. The (linear topological) direct sum of a sequence of locally $m$-convex algebras is a locally $m$-convex algebra (cf. Example 9 of [13]).

A topological linear space $(X, \sigma)$ is called an Ultra-$L$-space (respectively, Ultra-$Lb$-space) if $\sigma = \sigma_r$ for every simple (respectively, bornivorous) inductive system $\Gamma$ on $X$. Every ultrabarrelled space is an Ultra-$L$-space and every quasi-ultrabarrelled space is an Ultra-$Lb$-space. It is easily seen that every simple inductive system on an Ultra-$L$-space is bornivorous. For the basic properties of spaces of this type we refer to [1], [6], and [7].

Theorem 3. Let $\Gamma$ be a usual inductive system on the algebra $X$ consisting of topological algebras $(S_n, \sigma_n)$ each of which is an Ultra-$Lb$-space with a fundamental sequence of bounded sets. Then $(X, \sigma_r)$ is a topological algebra.

Proof. Let $(B_m^n : m \in N)$ be an increasing fundamental sequence of $\sigma_n$-bounded balanced sets in $S_n$. Let $A_1 = B_1^{(1)}$. Suppose for some $n \in N$ we have already chosen sets $B_i \in \text{Bd}(\sigma_i)$ so that $(A_i + A_i) \cup (A_i A_i) \subset A_{i+1}$ for $i = 1, 2, \ldots, n$. Since $A_n$ is $\sigma_{n+1}$-bounded, there exists $p \in N$ such that $(A_n + A_n) \cup (A_n A_n) \subset B_p^{(n+1)}$. Then define $A_{n+1} = B_n^{(1)} + B_{n+1}^{(2)} + \ldots + B_{n+1}^{(n+1)} + B_p^{(n+1)}$.

Let $\Gamma_1 = (A_n, \sigma_n | A_n ; n \in N)$. It is obvious that $\sigma_r \leq \sigma_{r_1}$. Now fix $k \in N$. Then $B_n^{(k)} \subset A_n$ and $\sigma_{r_1} | B_n^{(k)} \leq \sigma_k | B_n^{(k)}$ for all $n \geq k$. Since $(S_k, \sigma_k)$ is an Ultra-$Lb$-space, $\sigma_{r_1} | S_k \leq \sigma_k$ for all $k \in N$. 


Hence $\sigma_{r_1} \leqslant \sigma_r$. Thus $\sigma_{r_1} = \sigma_r$ and it suffices to apply Corollary 1 to Theorem 1.

Now let $X$ be an algebra with a locally convex topology $\sigma$. Such an algebra $(X, \sigma)$ is called inverse continuous if it has a unit $e$, the multiplicative group $G(X)$ of invertible elements is open, and the map $x \mapsto x^{-1}$ is continuous on $G(X)$. It is known (and easy to see) that if the map $x \mapsto x^{-1}$ is continuous at $e$, then it is continuous on $G(X)$. By a theorem due to Turpin (cf. [12], p. 123), every commutative inverse continuous locally convex topological algebra is locally m-convex.

We shall need the following lemma proved in [2].

**Lemma.** Let $X$ be an algebra with the unit $e$ and let $\Gamma = (S_n, \sigma_n : n \in \mathbb{N})$ be a locally convex m-bounded inductive system on $X$ such that for each $n \in \mathbb{N}$

(a) $S_n S_n \subseteq S_{n+1}$,

(b) $S_n$ is contained in a subalgebra $X_n$ of $X$,

(c) $X_1 \subseteq X_2 \subseteq \ldots, X = \bigcup_{n=1}^{\infty} X_n$,

(d) $\sigma_n = \tau_n|_{S_n}$, where $\tau_n$ is a locally convex topology on $X_n$.

Assume that

(*) for every $n \in \mathbb{N}$ there exist $V \in \mathcal{B}(\tau_n)$ and $m \in \mathbb{N}$ such that

$$e + V \cap S_n \subseteq G(X) \quad \text{and} \quad (e + V \cap S_n)^{-1} \subseteq S_m.$$

Then $(X, \sigma_r)$ is inverse continuous.

**Theorem 4.** Let $X$ be an algebra with the unit $e$ and assume that

$\Gamma = (S_n, \sigma_n : n \in \mathbb{N})$ is the usual inductive system of inverse continuous topological algebras $(S_n, \sigma_n)$ on $X$. Suppose also that

$$\mathcal{B}(\sigma_n) \cap \text{Bd}(\sigma_{n+1}) \neq \emptyset \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $(X, \sigma_r)$ is an inverse continuous topological algebra.

**Proof.** Without loss of generality we may assume that $e \in S_n$ for all $n \in \mathbb{N}$. Hence, by assumption, for each $n \in \mathbb{N}$ and each $U \in \mathcal{B}(\sigma_n)$ there exists $V \in \mathcal{B}(\sigma_n)$ such that

$$e + V \subseteq G(S_n) \subset G(X) \quad \text{and} \quad (e + V)^{-1} \subseteq e + U.$$

As in the proof of Theorem 2 we can construct an algebraic bounded inductive system $\Gamma_1 = (A_n, \sigma_n | A_n : n \in \mathbb{N})$ such that $\Gamma \sim \Gamma_1$, where

$$A_n \in \mathcal{B}(\sigma_n) \cap \text{Bd}(\sigma_{n+1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

On the other hand, $\Gamma_2 = (A_{2n-1}, \sigma_{2n-1} | A_{2n-1} : n \in \mathbb{N})$ is an m-bounded inductive system and $\Gamma_2 \sim \Gamma_1$ (see the proof of Corollary 1). Hence it is enough to prove that $\Gamma_1$ satisfies condition (*) from the Lemma. Fix $n \in \mathbb{N}$. Since $A_n \in \mathcal{B}(\sigma_n)$ and $(S_n, \sigma_n)$ is inverse continuous, there exists
\[ V \in \mathcal{A}(\sigma_n) \text{ such that} \]
\[ e + V \subset G(X) \quad \text{and} \quad (e + V)^{-1} \subset e + A_n. \]

Hence
\[ e + V \cap A_n \subset G(X) \quad \text{and} \quad (e + V \cap A_n)^{-1} \subset A_m \]

for some \( m \in \mathcal{N}. \)

Applying the above-mentioned result of Turpin we now prove the following

**Corollary 1.** If \( \Gamma \) is the usual inductive system of commutative Banach algebras on the algebra \( X \) with a unit, then \( (X, \sigma_{\Gamma}) \) is a locally \( m \)-convex algebra.

**Remark.** In general, the usual inductive limit of locally \( m \)-convex algebras need not be locally \( m \)-convex (cf. Example 6 of [13]).

I would like to thank Professor L. Drewnowski for his help in preparation of this paper.

**REFERENCES**


INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY
POZNAŃ

*Reçu par la Rédaction le 12.12.1978; en version modifiée le 23.7.1980*