ON INTERSECTIONS OF SETS
OF POSITIVE LEBESGUE MEASURE

BY

J. CICHÓN, A. SZYMAŃSKI AND B. WĘGŁORZ (WROCŁAW)

During the VIII Winter School on Abstract Analysis at Moravská
Bouda (February 1980), Peter Simon asked if one can prove in ZFC that:

(*) for every family \{A_\alpha : \alpha < \omega_1\} of sets of positive Lebesgue measure
on the real line \(\mathbb{R}\) there exists a set \(T \subseteq \omega_1\) of cardinality \(\omega_1\) such that
\(\{A_\alpha : \alpha \in T\}\) has the finite intersection property.

In the present note we show the following

THEOREM. The following three statements are equivalent:

(1) For every family \(\{A_\alpha : \alpha < \omega_1\}\) of sets of positive Lebesgue measure on
\(\mathbb{R}\) there is a set \(T \subseteq \omega_1\) of cardinality \(\omega_1\) such that the family \(\{A_\alpha : \alpha \in T\}\) has
the finite intersection property.

(2) For every family \(\{A_\alpha : \alpha < \omega_1\}\) of sets of positive Lebesgue measure on
\(\mathbb{R}\) there is a set \(T \subseteq \omega_1\) of cardinality \(\omega_1\) such that \(\bigcap \{A_\alpha : \alpha \in T\} \neq \emptyset\).

(3) There is no partition of \(\mathbb{R}\) into \(\omega_1\) sets of Lebesgue measure zero.

Proof. (1) \implies (3). Suppose \(\neg (3)\). Let \(\{B_\alpha : \alpha < \omega_1\}\) be a family of
pairwise disjoint sets of Lebesgue measure zero such that \(\bigcup \{B_\alpha : \alpha < \omega_1\} = \mathbb{R}\). Notice that for each \(\alpha < \omega_1\) the set \(C_\alpha = \bigcup \{B_\beta : \alpha \leq \beta < \omega_1\}\) is of full
measure. For each \(\alpha < \omega_1\) choose a compact subset \(F_\alpha \subseteq C_\alpha\) of positive
Lebesgue measure. Then for each \(T \subseteq \omega_1\), of cardinality \(\omega_1\), we have
\[\bigcap \{F_\alpha : \alpha \in T\} \subseteq \bigcap \{C_\alpha : \alpha \in T\} = \emptyset.\]
Thus, by compactness, the family \(\{F_\alpha : \alpha < \omega_1\}\) is a counterexample for (1).

(3) \implies (2). Suppose \(\neg (2)\). Let \(\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}\) be a counterexample for
(2). We may assume that the union of any subfamily of \(\mathcal{A}\) is measurable (if not, then we can remove from each \(\mathcal{A}_\alpha\) a set of measure zero to get a family
with this property). Let \(\mathcal{F}\) be a maximal family of pairwise disjoint sets of
positive Lebesgue measure, each of which intersects countably many sets
from \(\mathcal{A}\) only. Clearly, \(\mathcal{F}\) is countable and the set \(X = \mathbb{R} - \bigcup \mathcal{F}\) is of positive
measure. For each \(\alpha < \omega_1\) put
\[Z_\alpha = X - \bigcup \{A_\beta : \alpha \leq \beta < \omega_1\}.\]
Notice that each $Z_\alpha$ is measurable and intersects at most countably many sets from $\mathcal{A}$. Hence each $Z_\alpha$ is of measure zero. Finally, notice that $X = \bigcup \{Z_\alpha : \alpha < \omega_1\}$. Thus some measurable set of positive measure can be split into $\omega_1$ sets of measure zero. Consequently, there is a partition of $R$ into $\omega_1$ sets of measure zero, which contradicts (3).

(2) $\Rightarrow$ (1) is obvious.

Remark. C. Ryll-Nardzewski has remarked (oral communication) that each of the statements of the Theorem is equivalent to the following one:

(4) For each separable Boolean measure algebra $(\mathcal{B}, \mu)$ and for every family $\{a_\alpha : \alpha < \omega_1\}$ of elements of positive measure from $\mathcal{B}$ there is a set $T \subseteq \omega_1$ of cardinality $\omega_1$ such that the family $\{a_\alpha : \alpha \in T\}$ has the finite intersection property.

We do not know if the assumption of separability can be omitted from (4).

The following corollary gives an answer to the question of P. Simon mentioned at the beginning.

**Corollary.** (i) ZFC + CH $\vdash \neg (*)$.
(ii) Con(ZFC) $\Rightarrow$ Con(ZFC + $2^\omega = \omega_2 + (\ast)$).
(iii) Con(ZFC) $\Rightarrow$ Con(ZFC + $2^\omega = \omega_2 + \neg (\ast)$).

**Proof**. Obviously, ZFC + CH $\vdash \neg (3)$. Thus (i) follows from (1) $\Leftrightarrow$ (3). If we add $\omega_2$ Cohen reals to a model for ZFC + CH, we get a model for ZFC + $2^\omega = \omega_2 + \neg (3)$. Thus (iii) also follows from (1) $\Leftrightarrow$ (3).

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INSTSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

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