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# Relations between extrema of parametrical integrals and ordinary integrals

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**Abstract.** In the paper we introduce the concept of a run and an equivalence relation in the set of runs; the elements of the quotient space are called *curves*. Then some sets of runs and some sets of curves are provided with topologies and a functional is defined on each topological spaces.

Let us suppose that two of the functionals are investigated. Assume that one of them has a local extremum for a certain element of its domain and consider the corresponding element of the domain of the second functional; the function establishing the correspondence of the elements is either the identity function or a canonical transformation. The main purpose of the paper is whether that element is a local extremum of the second functional. The solution of this problem is given in Theorems 2 and 6.

The examples adduced in the paper seem to be interesting. They show that a curve which is a strong local extremum of a functional in the class of normally situated curves may not be a strong local extremum of this functional in the class of all curves.

## INTRODUCTION

K. Weierstrass indicated for the first time the analytical relations between "ordinary" and "parametric" problems of the calculus of variations (see [6], p. 93, 239). These relations were later inwestigated by O. Bolza (see [2], p. 198–201) and J. Hadamard (see [4], p. 78).

The mentioned authors dropped the topological side of those relations.

Weierstrass defines the idea of "nearness" of curves very imprecisely (see [6], p. 77, 177). Moreover, he thinks that the parametric problem is more general and in this connection one cannot be interested in the ordinary problem (see [6], p. 84). Weierstrass does not pay any attention to the existence of the wide class of physical problems which lead to the extremal-problem for ordinary integrals by means of distinguishing of the time-axis. It is obvious that one can apply to these problems parametrical integrals corresponding to the initial ones; however, it should be noted that the functionals obtained in this way have different domains than the functionals determined by the initial integrals.

An interesting attempt of exhibition of the topological aspect of the calculus of variations is taken up by C. Carathéodory (see [3], p. 227). Unfortunately he does not go in for the topological relations between ordinary and parametric problems.

These problems are treated interchangeably in the local study of the authors of the later works. Such treatment seems to be unjustified since the problems must be examined in topological spaces with different elements. Moreover, it cannot be supposed a priori that the topologies of these spaces correspond to each other. This assumption is not generally true even for the "natural" topologies of each of the spaces. In order to check this it is sufficient to note that the solution curve of the parametric problem may not be normally situated in relation to $Ox$-axis. The considerations of Section 2 also show that such an assumption cannot be made.

In this paper we introduce the concept of a run as a function whose values belong to a normed linear space; it is worth noting that this space is not necessarily complete.

We also introduce an equivalence relation in the set of runs. The elements of the quotient space are called *curves*.

Then some sets of runs and some sets of curves are provided with topologies and a functional is defined on each of these topological spaces. Suppose that two of the functionals are investigated. Assume that one of them reaches a local extremum for a certain element of its domain and consider the corresponding element of the domain of the second functional. The main purpose of this paper is whether that element is a local extremum of the second functional. The solution of this problem is given in Theorems 2 and 6.

The functionals considered in the paper are of more general form than those expressible as integrals. However, the title of the paper seems to be appropriate since it indicates the connections between the results of the paper and the classical theory.

The mark ■ denotes the end of the proof. If it appears immediately after the formulation of a theorem then it means that the proof of this theorem is obvious.

Finally I would like to express my thanks to Professor K. Tatarkiewicz for drawing my attention to the problem of the paper (see also [5], p. 182).

## 1. TOPOLOGIES ON THE SET OF RUNS AND ON THE SET OF CURVES

**1.1. Runs and curves.** Let $W$ and $Y$ be real normed linear spaces, let $R$ be the reals, $W = R \times Y$, and

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2} \quad \text{for every } (x, y) \in W.$$

Denote by $\boldsymbol{E}$ the set of all functions $\boldsymbol{w}$: $\boldsymbol{Dw} \to \boldsymbol{W}$ such that $\boldsymbol{Dw}$ is a closed interval of $\boldsymbol{R}$, $\boldsymbol{w}$ is continuously differentiable and $\|\boldsymbol{w}(t)\| > 0$ for $t \in \boldsymbol{Dw}$. The elements of $\boldsymbol{E}$ are said to be *runs*.

Denote by $\boldsymbol{F}$ the set of all real continuously differentiable functions defined and having positive first derivatives on closed intervals of $\boldsymbol{R}$.

Introduce in $\boldsymbol{E}$ the equivalence relation $\sim$, where $\boldsymbol{w} \sim \boldsymbol{v}$ if and only if there exists $\boldsymbol{h} \in \boldsymbol{F}$ such that $\boldsymbol{w} = \boldsymbol{v} \circ \boldsymbol{h}$, $\boldsymbol{h}(a) = c$, $\boldsymbol{h}(b) = d$, where $\langle a, b\rangle = \boldsymbol{Dw}$, $\langle c, d\rangle = \boldsymbol{Dv}$.

The quotient space will be denoted by $\mathscr{K}$. The elements of $\mathscr{K}$ are called *curves*. It will be convenient at times to regard a curve as a subset of the set of runs, however, we shall make no distinction in notation.

**1.2. Metrics on the sets of fixed-domain runs.** Let $\langle r, s\rangle$ be a fixed interval. Denote by $\boldsymbol{E}\langle r, s\rangle$ the set of all runs having this interval as their domain.

If $\boldsymbol{C}$ is a curve, define $\boldsymbol{C}\langle r, s\rangle$ by $\boldsymbol{C} \cap \boldsymbol{E}\langle r, s\rangle$.

For $k \in \{0, 1\}$ define the metric $\zeta_k^{(r,s)}$ on the set $\boldsymbol{E}\langle r, s\rangle$ by

$$\zeta_k^{(r,s)}(\boldsymbol{v}, \boldsymbol{w}) = \max_{j \in \{0,k\}}\Big(\max_{t \in \langle r,s\rangle} \|\boldsymbol{v}^{(j)}(t) - \boldsymbol{w}^{(j)}(t)\|\Big), \tag{1}$$

where $\boldsymbol{v}^{(j)}$ and $\boldsymbol{w}^{(j)}$ are the derivatives of $j$-th order of $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively.

The topology generated by $\zeta_k^{(r,s)}$ will be denoted by the same symbol.

**1.3. Topologies on the set of curves.** If $\boldsymbol{e} \in \boldsymbol{E}\langle 0, d\rangle$ and $\|\dot{\boldsymbol{e}}(t)\| = 1$ for every $t \in \boldsymbol{De}$, then $\boldsymbol{e}$ is said to be a *natural run.*

For each $\boldsymbol{w}$ in $\boldsymbol{E}$ there is $d > 0$ and a unique natural run $\boldsymbol{e} \in \boldsymbol{E}\langle 0, d\rangle$ with $\boldsymbol{w} \sim \boldsymbol{e}$. If $\boldsymbol{C}$ denotes the equivalence class of $\boldsymbol{w}$ we refer to $\boldsymbol{e}$ as the natural representation of $\boldsymbol{C}$.

For $k \in \{0, 1\}$ define the function $\delta_k$: $\mathscr{K} \times \mathscr{K} \to \boldsymbol{R}$ by

$$\delta_k(\boldsymbol{C}_1, \boldsymbol{C}_2) = \max\{\inf_{\boldsymbol{w}_2 \in \boldsymbol{C}_2\langle 0, d_1\rangle} \zeta_k^{(0,d_1)}(\boldsymbol{e}_1, \boldsymbol{w}_2) \inf_{\boldsymbol{w}_1 \in \boldsymbol{C}_1\langle 0, d_2\rangle} \zeta_k^{(0,d_2)}(\boldsymbol{w}_1, \boldsymbol{e}_2)\}, \tag{2}$$

where $\boldsymbol{e}_i$ is the natural representation of $\boldsymbol{C}_i$ $(i = 1, 2)$.

For any $\boldsymbol{C}_0 \in \mathscr{K}$ define the following family of sets

$$\mathfrak{B}_k(\boldsymbol{C}_0) = \big\{\{\boldsymbol{C} \in \mathscr{K};\ \delta_k(\boldsymbol{C}_0, \boldsymbol{C}) < a\},\ 0 < a < 1\big\}. \tag{3}$$

THEOREM 1. *The class of families*

$$\{\mathfrak{B}_k(\boldsymbol{C});\ \boldsymbol{C} \in \mathscr{K}\} \tag{4}$$

*forms the system of neighbourhoods for a topological space.*

Proof. If $\boldsymbol{C}_1, \boldsymbol{C}_2 \in \mathscr{K}$, $\boldsymbol{w}_1 \in \boldsymbol{C}_1\langle r, s\rangle$, then

$$\delta_0(\boldsymbol{C}_1, \boldsymbol{C}_2) = \inf_{\boldsymbol{w}_2 \in \boldsymbol{C}_2\langle r,s\rangle} \zeta_0^{(r,s)}(\boldsymbol{w}_1, \boldsymbol{w}_2) = \inf_{p,q} \inf_{\boldsymbol{v}_i \in \boldsymbol{C}_i\langle p,q\rangle} \zeta_0^{(p,q)}(\boldsymbol{v}_1, \boldsymbol{v}_2). \tag{5}$$

It follows from (5) that $\delta_0$ is a pseudometric on $\mathscr{K}$.

The analogue of formula (5) is not true for $\delta_1$. In order to prove the theorem in this case we show that the conditions $C_0 \in \mathscr{K}$, $\mathscr{U} \in \mathfrak{B}_1(C_0)$, and $C_1 \in \mathscr{U}$ imply the existence of $\mathscr{V} \in \mathfrak{B}_1(C_1)$ such that $\mathscr{V} \in \mathscr{U}$.

Indeed, suppose that the above conditions are fulfilled. Let $\mathscr{U} = \{C;\ \delta_1(C_0, C) < a\}$ and let $b > 0$ be chosen such that $\delta_1(C_0, C_1) < a-b$. Then the set $\mathscr{V} = \{C;\ \delta_1(C_1, C) < r\}$, where $r = \dfrac{b}{1+a-b}$, has the desired property.

To prove the last statement, choose $C_2 \in \mathscr{V}$ and denote by $e_i$ the natural representation of $C_i$ $(i = 0, 1, 2)$ and by $\langle 0, d_i\rangle$ the domain of $e_i$. Choose $v_0 \in C_0$ and $w_1 \in C_1$ such that $\zeta_1^{(0,d_1)}(e_1, v_0) < a-b$ and $\zeta_1^{(0,d_2)}(w_1, e_2) < r$. Let $h \in F$ be such that $w_1 = e_1 \circ h$. Put $w_0 = v_0 \circ h$. Since $\dot{h}(t) - 1 \leqslant \|\dot{w}_1(t) - \dot{e}_2(t)\| < r$, hence

$$\zeta_1^{(0,d_2)}(w_0, w_1) = \max\Big(\zeta_0^{(0,d_1)}(v_0, e_1) \max_{t \in \langle 0, d_2\rangle} \dot{h}(t)\big\|\dot{v}_0\big(h(t)\big) - 1\big\|\Big) < (1+r)(a-b).$$

Therefore

$$\inf_{w \in C_0\langle 0, d_2\rangle} \zeta_1^{(0,d_2)}(w, e_2) \leqslant \zeta_1^{(0,d_2)}(w_0, e_2) \leqslant \zeta_1^{(0,d_2)}(w_0, w_1) + \zeta_1^{(0,d_2)}(w_1, e_2) < a. \tag{6}$$

Similarly, we show that

$$\inf_{w \in C_2\langle 0, d_0\rangle} \zeta_1^{(0,d_0)}(e_0, w) < a. \tag{7}$$

The theorem now follows from (6) and (7). ∎

The topology of the above theorem will be denoted by $\delta_k$.

**1.4. Remarks.** It can be proved that $C\langle r, s\rangle$ is a closed subset of the space $(E\langle r, s\rangle, \zeta_1^{(r,s)})$. Thus the condition $\delta_1(C_1, C_2) = 0$ implies $C_1 = C_2$. Therefore $(\mathscr{K}, \delta_1)$ is a $T_1$-space.

The topologies $\delta_k$ remain unchanged when the norm of $W$ is replaced by equivalent one.

If $W = R^n$, then the assumptions of Section 1.1 guarantee that the functions $\zeta_k^{(r,s)}$, $\delta_i$ are invariant under rotations and translations of coordinate system in the space $W$.

## 2. EXTREMA OF FUNCTIONALS DEFINED ON THE SET OF RUNS AND ON THE SET OF CURVES

**2.1. Definitions.** We shall consider a functional $\mathbf{I}\colon E \to R$ having constant values on curves regarded as subsets of $E$ and the functional $\mathbf{J}\colon \mathscr{K} \to R$ generated by $\mathbf{I}$.

If $p, q \in W$ denote by $E(p, q)$ $(\mathscr{K}(p, q))$ the set of all runs (curves) beginning at $p$ and ending at $q$.

The run of the shape $(\boldsymbol{id}, \boldsymbol{y})$ is said to be *super-normal run*. A curve $\boldsymbol{C}$ with $(\boldsymbol{id}, \boldsymbol{y}) \in \boldsymbol{C}$ for some $\boldsymbol{y}$ is called a *normal curve*. The set of all super-normal runs is denoted by $\boldsymbol{E}(x)$. The set of all normal curves is denoted by $\mathscr{K}(x)$. We adopt the following notation: $\boldsymbol{E}(p, q;\, r, s) = \boldsymbol{E}(p, q) \cap \cap \boldsymbol{E}\langle r, s\rangle$; $(\boldsymbol{E}(p, q);\, x) = \boldsymbol{E}(p, q) \cap \boldsymbol{E}(x)$; $\mathscr{K}(p, q; x) = \mathscr{K}(p, q) \cap \mathscr{K}(x)$.

For each $\boldsymbol{C} \in \mathscr{K}(x)$ there is a unique super-normal run in the equivalence class $\boldsymbol{C}$ which we call the super-normal representation of $\boldsymbol{C}$.

**2.2. The difference between strong extrema of I and J.** The following two examples show that the investigation of strong extrema of the functionals $\mathbf{I}/\boldsymbol{E}\langle r, s\rangle$ and $\mathbf{I}/\boldsymbol{E}(x)$ cannot be treated interchangeably.

EXAMPLE 1. Take $W = R^2$ and define $\mathbf{I}\colon \boldsymbol{E} \to \boldsymbol{R}$,

$$\mathbf{I}(\boldsymbol{x}, \boldsymbol{y}) = \int_r^s \boldsymbol{x}\,\|(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}})\|\,dt,$$

where $D(\boldsymbol{x}, \boldsymbol{y}) = \langle r, s\rangle$.

Put $p = (0, 0)$, $q = (1, 0)$. The run $\boldsymbol{w}_0 = (\boldsymbol{id}, \boldsymbol{0})$ is an absolute minimum of $\mathbf{I}/\boldsymbol{E}(p, q; x)$.

Suppose that $0 < a < 1$ and define the functions $\boldsymbol{x}$ and $\boldsymbol{y}$ by

$$\boldsymbol{x}(t) = \begin{cases} -\dfrac{a}{2}\sin\left(\pi - \dfrac{10\pi}{a}\cdot t\right)\cos\left(\dfrac{\pi}{2} - \dfrac{5\pi}{a}\cdot t\right) & \text{if } t \in \left\langle 0, \dfrac{a}{10}\right\rangle; \\ \boldsymbol{f}(t) & \text{if } t \in \left\langle \dfrac{a}{10}, \dfrac{a}{5}\right\rangle; \\ t & \text{if } t \in \left\langle \dfrac{a}{5}, 1\right\rangle; \end{cases}$$

$$\boldsymbol{y}(t) = \begin{cases} \dfrac{a}{2}\sin\left(\pi - \dfrac{10\pi}{a}\cdot t\right)\sin\left(\dfrac{\pi}{2} - \dfrac{5\pi}{a}\cdot t\right) & \text{if } t \in \left\langle 0, \dfrac{a}{10}\right\rangle; \\ 0 & \text{if } t \in \left\langle \dfrac{a}{10}; 1\right\rangle, \end{cases}$$

where $\boldsymbol{f}$ is such that $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{E}(p, q; 0, 1)$ and $\zeta_0^{(0,1)}(\boldsymbol{w}, \boldsymbol{w}_0) < a$.

Now

$$\mathbf{I}(\boldsymbol{w}) = \int_0^{a/10} \boldsymbol{x}\,\|\dot{\boldsymbol{w}}\|\,dt + \int_{a/10}^{1} \boldsymbol{x}\,\|\dot{\boldsymbol{w}}\|\,dt$$

and the first of these integrals is negative while the second is equal to $\mathbf{I}(\boldsymbol{w}_0)$.

Thus $\boldsymbol{w}_0$ is not a strong local minimum of $\mathbf{I}/\boldsymbol{E}(p, q; 0, 1)$.

EXAMPLE 2. Take $r < s$ and denote for any natural $k$:

$$a_k = r + \frac{s-r}{2^{2k-1}}; \quad b_k = \frac{s-r}{2^{2k}};$$

$$Z_k = \{(x, y);\ r + b_k < x < a_k,\ y > 0\}$$
$$\{(x, y);\ \big((x-a_k)^2 + (y-b_k)^2\big)^{1/2} = b_k,\ y \leqslant b_k\};$$
$$Z = R^2 \setminus \big(\bigcup_{k=1}^{\infty} Z_k\big).$$

Put $W = R^2$ and define $\mathbf{I}$: $E \to R$

$$\mathbf{I}(x, y) = \int_m^n \dot{x}\big(g \circ (x, y)\big)dt,$$

where $D(x, y) = \langle m, n\rangle$ and $g(x, y)$ is the distance of $(x, y) \in R^2$ to $Z$.

The run $w_0 = (id, 0)$ is an absolute minimum of $\mathbf{I}/E(x)$.

Define the run $w_k$ such that $w_k(t) = (t, 0)$ if $t \in \langle r, s\rangle \setminus \langle a_k, a_k + 2\pi b_k\rangle$ and $w_k(t)$ traces out the circle $(x-a_k)^2 + (y-b_k)^2 = b_k^2$ if $t$ ranges over $(a_k, a_k + 2\pi b_k)$. Then for $k \geqslant 2$ the domain of $w_k$ is $\langle r, s\rangle$ and for any $a > 0$, $\zeta_0^{(r,s)}(w_0, w_k) < a$ provided that $k$ is sufficiently large. Since $\mathbf{I}(w_k) < 0$ hence $w_0$ is not a strong local minimum of $\mathbf{I}/E\langle r, s\rangle$.

It may be worth noting that the set of values of any function $w_k$ is contained in the set $\{(x, y) \mid r \leqslant x \leqslant s\}$. The function $w$ constructed in Example 1 does not have the analogous property.

The above examples and formula (5) yield the following theorem.

THEOREM 2 (about strong extrema). *Strong local extrema of the functionals $\mathbf{J}/\mathcal{K}(p, q)$ and $\mathbf{I}/E(p, q; r, s)$ correspond and so also do those of the functionals $\mathbf{J}/\mathcal{K}(p, q; x)$ and $\mathbf{I}/E(p, q; x)$. The strong local extrema of the functionals $\mathbf{J}/\mathcal{K}(p, q)$ and $\mathbf{J}/\mathcal{K}(p, q; x)$ as well as $\mathbf{I}/E(p, q; r, s)$ and $\mathbf{I}/E(p, q; x)$ do not necessarily correspond.*

Clearly, the term correspondence in the above theorem is interpreted to mean that if $w$ is a minimum of one functional then $\mathbf{T}w$ is a minimum of the second functional, where $\mathbf{T}$ is the canonical transformation, i.e., $\mathbf{T}$ is either the quotient mapping or identity mapping. A minimum of the second functional might not be in the image under $\mathbf{T}$ of any element in the domain of the first functional.

**2.3. Relationships between topologies on the set of runs and on the set of curves.** In this section $Z_1$ and $Z_2$ are topological spaces which as sets are subsets of $E$ or $\mathcal{K}$. Taking an arbitrary element of $Z_2$ and an open neighbourhood $U$ of it we establish the existence of an open neighbourhood 0 of the corresponding element of $Z_1$ and a set $S$ contained in $U$ such that the image of $S$ under the canonical transformation covers 0.

It should be noted that this property of the canonical transformation is not continuity.

THEOREM 3. *For any curve $C_0 \in \mathscr{K}(x)$ and $a > 0$ there exists $b > 0$ such that the conditions $\zeta_1^{(0,d_0)}(e_0, w) < b$, $x(0) = r$, $x(d_0) = s$ imply that $C \in \mathscr{K}(x)$ and $\zeta_1^{(r,s)}(w_0^*, w^*) < a$, where $e_0$ is the natural representation of $C_0$, $w = (x, y) \in C$ and $w_0^*$ and $w^*$ are the super-normal representations of $C_0$ and $C$ respectively.*

Proof. Set $e_0 = (g_0, h_0)$, $w_0^* = (id, y_0^*) w^* = (id, y^*)$, $m = \min_t g_0(t)$ and $M = \max_t \|\dot{h}_0(t)\| + m/2$. Let $c > 0$ be such that $\|\dot{y}_0^*(p) - \dot{y}_0^*(q)\| < a/4$ if $|p-q| < c$ and let $d > 0$ be such that

$$\zeta_0^{(r,s)}(w_0^*, w^*) < a \quad \text{if} \quad \zeta_0^{(0,d_0)}(e_0, w) < d. \tag{8}$$

If $b = \min\left(\frac{m}{2}, c, \frac{am^2}{2(m+M)}, d\right)$, then

$$\dot{x}(t) > \dot{g}_0(t) - b \geqslant m - m/2 > 0,$$

$$\begin{aligned}\left\|\dot{y}^*(x(t)) - \dot{y}_0^*(g_0(t))\right\| &= \left\|\frac{\dot{y}(t)}{\dot{x}(t)} - \frac{\dot{h}_0(t)}{\dot{g}(t)}\right\| \\ &\leqslant \frac{1}{\dot{x}(t)\dot{g}_0(t)}\{\dot{x}(t)\|\dot{y}_0(t) - \dot{y}(t)\| + \|\dot{y}(t)\| \cdot |\dot{x}(t) - \dot{g}_0(t)|\} \\ &\leqslant \frac{b}{\dot{g}_0(t)}\left(1 + \frac{1}{\dot{x}(t)}\|\dot{y}(t)\|\right) < \frac{b}{m}\left(1 + \frac{2M}{m}\right) \leqslant \frac{a}{2},\end{aligned}$$

$$\begin{aligned}&\left\|\dot{y}^*(x(t)) - \dot{y}_0^*(x(t))\right\| \\ &\quad\leqslant \left\|\dot{y}^*(x(t)) - \dot{y}_0^*(g_0(t))\right\| + \left\|\dot{y}_0^*(g_0(t)) - \dot{y}_0^*(x(t))\right\| < a.\end{aligned} \tag{9}$$

The theorem is a consequence of (8) and (9). ■

THEOREM 4. *For any natural run $e_0 \in E$ and $a > 0$ there exists $b > 0$ such that the condition $\zeta_1^{(0,d_0)}(w, e_0) < b$ implies that $\zeta_1^{(0,d_0)}(f, e_0) < a$, where $f(t) = e\left(\frac{d}{d_0} t\right)$ for $t \in \langle 0, d_0 \rangle$; $e$ is the natural representation of $[w]^{\sim}$, $De = \langle 0, d \rangle$.*

Proof. Put $b = \min\left(\frac{a}{1+2d_0}, \frac{c}{4d_0}, \frac{a}{6}, \frac{1}{2}\right)$, where $c$ is a positive number such that $\|\dot{e}_0(r) - \dot{e}_0(s)\| < a/3$ if $|r-s| < c$. Suppose that $f = w \circ h$, where $h \in F$.

Since

$$\begin{aligned}\left|\int_t^{h(t)} \|\dot{w}\| ds\right| \leqslant \left|\int_0^t \|\dot{f}\| ds - t\right| + \left|\int_0^t (\|\dot{w}\| - 1) ds\right| < \left|\frac{d}{d_0} - 1\right| \cdot t + bt \\ < 2bt < 2bd_0,\end{aligned} \tag{10}$$

we have

$$(11)\qquad \|f(t)-e_0(t)\| \leqslant \|e_0(t)-w(t)\| + \|w(t)-f(t)\| < b(1+2d_0) \leqslant a.$$

Since $|1-\|\dot{w}\|| < b \leqslant \frac{1}{2}$ hence it follows form (10) that

$$|h(t)-t| < 4bd_0 \leqslant c, \qquad \text{so} \qquad \|\dot{e}_0(h(t))-\dot{e}_0(t)\| < a/3.$$

Let $w = e\circ \overset{\bullet}{g}, g \in F$. Then

$$\|\dot{f}(t)-\dot{w}(h(t))\| = \left|\frac{d}{d_0} - g(h(t))\right| \leqslant \left|\frac{d}{d_0} - 1\right| + |\dot{g}(h(t))-1| < 2b \leqslant a/3.$$

Therefore

$$|\dot{f}(t)-\dot{e}_0(t)| \leqslant \|\dot{f}(t)-\dot{w}(h(t))\| + \|\dot{w}(h(t))-\dot{e}_0(h(t))\| + \\ + \|\dot{e}_0(h(t))-\dot{e}_0(t)\| < a.$$

The theorem follows from (11) and (12). ∎

Lemma. *For any curve $C_0 \in \mathscr{K}$ and $a > 0$ there exists $b > 0$ such that for $C \in \mathscr{K}$ the condition*

$$\zeta_1^{(0,d_0)}(e_0, f) < b$$

*implies that*

$$\zeta_1^{(0,d)}(f_0, e) < a,$$

*where $e_0$ and $e$ are natural representations of $C_0$ and $C$ respectively,*

$$f_0(t) = e_0\left(\frac{d_0}{d}t\right), \qquad f(t) = e\left(\frac{d}{d_0}t\right).\ ∎$$

Theorem 5. *For any run $w_0 \in E$ and $a > 0$ there exists $b > 0$ such that $\delta_1([w_0]^{\tilde{}}, [w]^{\tilde{}}) < a$ if $\zeta_1^{(r,s)}(w_0, w) < b$.*

Proof. If $w_0$ is a natural run, then the theorem follows from Theorem 4 and the lemma. The general case now follows. ∎

**2.4. The relationship between weak extrema of I and J.**

Theorem 6. (About weak extrema). *The weak local extrema of the functionals $\mathbf{J}/\mathscr{K}(p,q)$, $\mathbf{J}/\mathscr{K}(p,q;x)$, $\mathbf{J}/E(p,q;x)$ and $\mathbf{I}/E(p,q;r,s)$ correspond.*

Proof. The theorem follows from Theorems 3 and 5. ∎

The interpretation of the term correspondence in the above theorem is similar to that in Theorem 2.

**2.5. Final remarks.** Define the function $\gamma_k\colon \mathscr{K}\times\mathscr{K}\to \mathbf{R}$ by

$$\gamma_k(C_0, C) = \inf_{w\in C\langle 0,d_0\rangle} \zeta_k^{(0,d_0)}(e_0, w),$$

where $e_0$ is the natural representation of $C_0$ and $k \in \{0, 1\}$.

Replace $\delta_k$ by $\gamma_k$ in formula (3) and form the topology $\gamma_k$.

From Theorem 5 it follows that $\gamma_k = \delta_k$.

Suppose that $\boldsymbol{A} \subset \boldsymbol{E}\langle r, s\rangle$ and $\mathscr{B} \subset \mathscr{K}$ is the set of curves generated by the elements of $\boldsymbol{A}$. Then formula (5) and Theorem 5 yield the following two theorems.

THEOREM 7. *If $\boldsymbol{A}$ is a compact subset of the space $(\boldsymbol{E}\langle r, s\rangle, \zeta_k^{(r,s)})$, then $\boldsymbol{B}$ is a compact subset of the space $(\mathscr{K}, \delta_k)$, where $k \in \{0, 1\}$.*

THEOREM 8. *If $\boldsymbol{w} \in \boldsymbol{E}\langle r, s\rangle$, then $\mathbf{J}$ is lower (upper) semi-continuous at the point $[\boldsymbol{w}]^{\sim}$ with respect to the topology $\delta_k$ if and only if $\mathbf{I}/\boldsymbol{E}\langle r, s\rangle$ is lower (upper) semi-continuous at the point $\boldsymbol{w}$ with respect to the metric $\zeta_k^{(r,s)}$.*

Theorems 7 and 8 may be used to establish the existence of absolute extrema of the functional $\mathbf{J}/\mathscr{Z}$ when $\mathscr{Z}$ is a compact subset of $\mathscr{K}$.

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