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MULTIPLE STOCHASTIC POINT PROCESSES

1. Introduction. Recently many stochastic point processes have
been studied with particular reference to multiplicative processes. Here
we encounter the statistical problem of defining the distribution of a
discrete number of entities distributed in a continuous parameter
space. The continuous parameter which we denote by x might refer to
the energy of a cosmic ray particle in a cascade process or the age of
an individual in an evolutionary process or the lateral displacement of
a particle moving in a scattering medium. The difficulty in defining a prop-
er probability frequency function has been noticed by mathematicians,
physicists and statisticians (see references [1]-[6]) who have developed
suitable methods to study problems relating to such point processes
as mentioned above[1]. One of the methods consists in defining suitable
densities of various orders (called cumulant functions by Kendall and
product densities by Ramakrishnan) in the parameter space and connect-
ing the moments with the densities. The theory of product densities
and cumulant functions is based on the assumption that the probability
that an "entity" has a value in an infinitesimal interval Δ of parametric
space is $O(\Delta)$, the probability of more than one entity, say n, having
a parametric value within Δ being $O(\Delta^n)$. Thus the above method seems
to be unsuitable for the description of processes in which the probability
that more than one entity have a parametric value within Δ is of the
same order of magnitude as that of one. This is particularly so in the
case of lateral distribution of particles in cosmic ray cascades or the age
distribution of members in a population in which twins and higher mul-
tiplets exist. Hence it is worthwhile to examine how far the density
techniques can be extended to meet such situations. An attempt is made
in this direction by Ramakrishnan and the author (1958) in the particular
problem of age distribution in population growth. We shall here present
a general mathematical formulation of the densities involving multiple
points.

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[1] For an extensive review on the subject see references [7] and [11].

2. Product densities of multiple points. Let $M(a, b; t)$ be the random variable representing the number of entities with parametric values between a and b. t here represents the parameter with respect to which the process evolves. We shall assume that the maximum number of entities having parametric values in the infinitesimal range dx is n. In counting the number of entities, an i-tuplet will be counted as i entities since the members of a multiplet evolve individually with respect to the parameter. Our starting point is the density function introduced by Janossy (1950) suitably modified to accommodate multiplets.

We shall consider

$$J_{m_1 m_2 \dots m_n}(x_1^1, x_2^1, \dots, x_{m_1}^1, \dots, x_1^n, x_2^n, \dots, x_{m_n}^n)$$

where

$$J_{m_1 m_2 \dots m_n} \prod_{ij_i} dx_{j_i}^i \qquad (j_i = 1, 2, \dots, m_i, \, i = 1, 2, \dots, n)$$

represents the joint probability that there exist m_i i-tuples and that a typical i-tuple has a parametric value between $x_{j_i}^i$ and $x_{j_i}^i + dx_{j_i}^i$. To obtain information regarding the moments of the total number of entities having some parametric value in the range (a, b), we consider an assembly of entities $(m_1, m_2, \dots, m_n)$, where m_i is the number of i-tuples, the i-tuples having parametric values in the intervals $(x_1^i, x_1^i + dx_1^i)$, $(x_2^i, x_2^i + dx_2^i)$, $\dots$, $(x_{m_i}^i, x_{m_i}^i + dx_{m_i}^i)$. Then the random variable $M(a, b; t)$ takes the value

$$\sum_{i=1}^{n} \sum_{j_i=1}^{m_i} iH(x_{j_i}^i - a)H(b - x_{j_i}^i)$$

with probability $J_{m_1 m_2 \dots m_n} \prod_{ij_i} dx_{j_i}^i$. $H(x)$ is the Heaviside unit function defined as

$$(1) \qquad H(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The mean of the total number of entities having parametric values in the range (a, b) is given by

$$(2) \quad E\{M(a, b; t)\}$$

$$= \sum_{i=1}^{n} \sum_{m_i} \sum_{j_i=1}^{m_i} \int dx_1^1 \int dx_2^1 \dots \int dx_{m_1}^1 \dots \int dx_1^n \int dx_2^n \dots \int dx_{m_n}^n \times$$

$$\times J_{m_1 m_2 \dots m_n} \frac{iH(x_{j_i}^i - a)H(b - x_{j_i}^i)}{m_1! m_2! \dots (m_i - 1)! \dots m_n!} =$$

$$= \sum_{i=1}^{n} i \int_{a}^{b} dx \left[\sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{m_1! \, m_2! \, \ldots \, (m_i-1)! \, \ldots \, m_n!} \times \right.$$

$$\times \int_{0}^{\infty} dx_1^1 \int_{0}^{\infty} dx_2^1 \ldots \int_{0}^{\infty} dx_{m_1}^1 \ldots \int_{0}^{\infty} dx_1^i \int_{0}^{\infty} dx_2^i \ldots \int_{0}^{\infty} dx_{m_i-1}^i \int_{0}^{\infty} dx_1^{i+1} \ldots \int_{0}^{\infty} dx_{m_n}^n \times$$

$$\left. \times J_{m_1 m_2 \ldots m_n}(x_1^1, x_2^1, \ldots, x_{m_1}^1, \ldots, x_1^i, x_2^i, \ldots, x_{m_i-1}^i, x_1^{i+1}, \ldots, x_1^n, x_2^n, \ldots, x_{m_n}^n) \right].$$

We immediately recognise that the quantity within the square bracket denotes the probability that there is an i-tuple having a parametric value between x and $x+dx$ irrespective of the number elsewhere. Writing

$$(3) \qquad E\{M(a, b; t)\} = \sum_{i=1}^{n} i \int_{a}^{b} f_1^i(x, t)\,dx = \int_{a}^{b} f_1(x, t)\,dx$$

we can identify $f_1(x, t)$ in the notation of Ramakrishnan as the product density of degree one. Thus we can call $f_1^i(x, t)$ as the product density of degree one of i-tuples. It is well recognised that it is easy to derive differential equations for $f_1^i(x, t)$ in any particular process.

We next proceed to obtain the ν-th moment to identify the product density of degree ν and the manner in which contributions from the product densities of lower order arise.

Writing

$$(4) \qquad g(x_{j_i}^i) = H(x_{j_i}^i - a) H(b - x_{j_i}^i)$$

we observe that

$$(5) \qquad \left[\sum_{i=1}^{n} i \sum_{j_i=1}^{m_i} g(x_{j_i}^i) \right]^{\nu}$$

$$= \sum_{s=1}^{\nu} \sum_{i_1=1}^{n} \cdots \sum_{i_s=1}^{n} \sum_{\substack{h_1 \, h_2 \quad h_s \\ h_1+h_2+\ldots+h_s=\nu}} \sum \cdots \sum \sum_{l_1=1}^{h_1} \sum_{l_2=1}^{h_2} \cdots \sum_{l_s=1}^{h_s} C_{l_1}^{h_1} C_{l_2}^{h_2} \ldots C_{l_s}^{h_s} \times$$

$$\times i_1^{h_1} i_2^{h_2} \ldots i_s^{h_s} g(x_1^{i_1}) g(x_2^{i_1}) \ldots g(x_{l_1}^{i_1}) \ldots g(x_1^{i_s}) g(x_2^{i_s}) \ldots g(x_{l_s}^{i_s}),$$

where C_s^{ν} denotes the $\nu-s$ fold degeneracy in a product of ν terms and has been tabulated for various values by Stevens (see Ramakrishnan (1950)). Using (5) and observing that

$$\sideset{}{'}\sum_{m_1 m_2 \ldots m_n} \frac{1}{m_1! \, m_2! \, \ldots \, m_{i_1-1}! \, (m_{i_1}-l_1)! \, \ldots \, (m_{i_s}-l_s)! \, \ldots \, m_n!} \int dx_1^1 \int dx_2^1 \ldots \int dx_{m_1}^1 \ldots$$

$$\ldots \int dx_1^n \int dx_2^n \ldots \int dx_{m_n}^n J_{m_1 m_2 \ldots m_n} g(x_1^{i_1}) g(x_2^{i_1}) \ldots g(x_{l_1}^{i_1}) \ldots$$

$$\ldots g(x_1^{i_s}) g(x_2^{i_s}) \ldots g(x_{l_s}^{i_s})$$

could be written as

$$\int\limits_a^b dx_1 \int\limits_a^b dx_2 \ldots \int\limits_a^b dx_{k_s} f_{k_s}^{l_1, l_2, \ldots, l_s}(x_1, x_2, \ldots, x_{k_s}, t) \quad (2)$$

we can express the ν-th moment of the variable $M(a, b; t)$ as '

(6) $\quad E\{[M(a, b; t)]^\nu\}$

$$= \sum_{i_1=1}^n \sum_{i_2=1}^n \ldots \sum_{i_s=1}^n \sum_{h_1=1}^\nu \sum_{h_2=1}^\nu \ldots \sum_{\substack{h_s=1 \\ h_1+h_2+\ldots+h_s=\nu}}^\nu \sum_{l_1=1}^{h_1} \sum_{l_2=1}^{h_2} \ldots \sum_{l_s=1}^{h_s}$$

$$\int dx_1 \int dx_2 \ldots \int dx_{k_s} C_{l_1}^{h_1} C_{l_2}^{h_2} \ldots C_{l_s}^{h_s} f_{k_s}^{l_1, l_2, \ldots, l_s}(x_1, x_2, \ldots, x_{k_s}, t),$$

where $f_{k_s}^{l_1, l_2, \ldots, l_s}(x_1, x_2, \ldots, x_{k_s}, t)\, dx_1 dx_2 \ldots dx_{k_s}$ represents the joint probability that there are l_1 i_1-tuples, l_2 i_2-tuples, $\ldots$, l_s i_s-tuples, the parametric values of these entities lying respectively in the ranges (x_1, x_1+dx_1), $(x_2, x_2+dx_2), \ldots, (x_{k_s}, x_{k_s}+dx_{k_s})$ irrespective of the distribution of the entities elsewhere.

For any process, we can deal with $f_{k_s}^{l_1, l_2, \ldots, l_s}(x_1, x_2, \ldots, x_{k_s}, t)$ and obtain differential or integral equations by studing the variation with respect to t. In the next section, we shall illustrate the use of density techniques by some examples.

3. Applications of multiple point densities

1. Multiple scattering in multiplicative processes. When a charged particle passes through matter it suffers collisions with the nucleii of the medium and during a collision, the particle is scattered or new particles are produced. We shall assume that the "parent" and the "offsprings" continue to move in the same direction as that of the parent before collision. If we assume that in a single collision the particle is scattered through an angle a_i, $\theta = \sum_i a_i$ being small and that the particle pursues an almost straight path except for slight lateral displacements, then the lateral displacement at depth t is given by

(7) $$x(t) = \int\limits_0^t \theta(\tau)\, d\tau.$$

We shall consider a simple model of electromagnetic cascade. An electron undergoes multiple scattering as explained above, its lateral displacement being given by (7). We take into account the processes of pair

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(²) For convenience of notation we denote $l_1+l_2+\ldots+l_s$ by k_s, k_s having no other significance.

creation by photons and bremmstrahlung by electrons, the cross sections for these processes being assumed to be λ and μ per unit thickness of matter which are independent of energy[3]. Our object is to determine information regarding the mean and mean square of number of electrons that emerge at t with lateral displacement lying between a and b, given that there is an electron at $t = 0$ with $x = 0$. Since only pair creation by photons is taken into consideration, we encounter only multiple points up to second order [4].

Let $f_1^1(x, t)$ and $f_1^2(x, t)$ be the product densities of degree one of electrons in a shower initiated by an electron, the primary having a value $x = 0$ at $t = 0$. If $g_1^1(x, t)$ and $g_1^2(x, t)$ and $h_1^1(x, t)$ and $h_1^2(x, t)$ are the functions corresponding a photon primary and electron-positron pair, then by the regeneration point method or the so-called backward differential equation method (see reference [2]) we obtain

$$(8) \qquad \frac{\partial f_1^1(x, t)}{\partial t} = -\nu f_1^1(x, t) + \tfrac{1}{2}\nu\{f_1^1(x - at, t) + f_1^1(x + at, t)\} + \lambda g_1^1(x, t),$$

$$(9) \qquad \frac{\partial g_1^1(x, t)}{\partial t} = -\mu g_1^1(x, t) + \mu h_1^1(x, t),$$

$$(10) \qquad \frac{\partial h_1^1(x, t)}{\partial t} = -2\nu h_1^1(x, t) + \nu\{f_1^1(x - at, t) + f_1^1(x + at, t)\} + 2\lambda g_1^1(x, t)$$

with the initial conditions

$$(11) \qquad f_1^1(x, 0) = \delta(x), \qquad g_1^1(x, 0) = h_1^1(x, 0) = 0,$$

a is the small lateral displacement resulting from scattering in an infinitesimal thickness of matter the displacement to the left and right being equally probable and ν is the probability of multiple scattering per unit thickness of matter (see for example Ramakrishnan and Mathews (1956)). $g_1^1(x, t)$ satisfies a simple equation in view of the fact that it does not undergo scattering and is merely drifted, the drift being due to the lateral displacement of the parent electron. We next observe that $f_1^2(x, t)$, $g_1^2(x, t)$ and $h_1^2(x, t)$ satisfy exactly the same equations as $f_1^1(x, t)$, $g_1^1(x, t)$ and $h_1^1(x, t)$ respectively with the initial conditions

$$(12) \qquad f_1^2(x, 0) = g_1^2(x, 0) = 0, \qquad h_1^2(x, 0) = \delta(x).$$

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[3] This is of course unphysical and as our object is merely to illustrate by an application, we wish to avoid complicated integral equations.

[4] In fact introduction of multiple processes like tridents give rise to triple points.

It may be a little surprising to find that the equations for $f_1^1(x, t)$ and $f_1^2(x, t)$ become disjoint. This is essentially due to the mathematical technique of Bellman and Harris (1948) which we have used to overcome the difficulty arising from the non-Markovian nature of the process. We cannot obtain forward differential equations for the densities without introducing additional random variables to render the process Markovian. This has been discussed in detail by Ramakrishnan (1955) in connection with phenomenological interpretation of integrals of random functions.

Equations (8) to (10) are menable to Fourier transform technique provided we make use of the fact that a is small. For small a numerical solution of the equations is not very difficult. Of course analytic solution of these equations is indeed difficult. In a similar manner we could write equations satisfied by second order densities and this we shall not discuss in detail since our model is not a realistic one.

2. A simple multiplicative model with ionisation loss. In the previous problem we have used the regeneration point method to write down differential equations satisfied by product densities. Now we shall deal with a simple multiplicative model and since the process is Markovian, we are able to write forward differential equations which are not disjoint as contrasted with the equations in the previous section.

In this model, we assume that:

(i) a particle during a collision creates another particle, the probability of collision per unit thickness of matter being λ,

(ii) the secondary and the primary share the energy equally,

(iii) each particle having an energy E while traversing matter loses energy at a rate $\beta(E)$ per unit thickness of matter where

$$(13) \qquad \beta(E) = \begin{cases} \text{a constant } \beta & \text{if } E > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence of assumption (ii) we encounter double points in E-space. Let $f_1^1(E, t)$ and $f_1^2(E, t)$ be the product densities of singles and doublets of degree one. The two particles constituting the doublet evolve individually after they are produced. Using the Markovian nature of the process we obtain by the last collision method

$$(14) \qquad \frac{\partial f_1^1(E, t)}{\partial t} = -\lambda f_1^1(E, t) + 2\lambda f_1^2(E, t) + \frac{\partial}{\partial E} [\beta(E) f_1^1(E, t)],$$

$$(15) \qquad \frac{\partial f_1^2(E, t)}{\partial t} = -2\lambda f_1^2(E, t) + 4\lambda f_1^2(2E, t) +$$

$$+ 2\lambda f_1^1(2E, t) + \frac{\partial}{\partial E} [\beta(E) f_1^2(E, t)],$$

with the initial conditions

(16)
$$f_1^1(E, 0) = \delta(E - E_0),$$
$$f_1^2(E, 0) = 0$$

corresponding to a single primary of energy E_0. Defining $p_1^1(s, t)$ and $p_1^2(s, t)$ as the Laplace transform of $f_1^1(E, t)$ and $f_1^2(E, t)$ with respect to E, we obtain

(17)
$$\frac{\partial p_1^1(s, t)}{\partial t} = -\lambda p_1^1(s, t) + 2\lambda p_1^2(s, t) + \beta s p_1^1(s, t),$$

(18)
$$\frac{\partial p_1^2(s, t)}{\partial t} = -\lambda p_1^2(s, t) + 2\lambda p_1^2\left(\frac{s}{2}, t\right) + \lambda p_1^1\left(\frac{s}{2}, t\right) + \beta s p_1^2(s, t),$$

where terms of the form $[\beta(E)f_1^1(E, t)e^{-sE}]_{E=0}$ vanish in view of (13). It is to be noted that the peculiar feature of the solution lies in (13); otherwise the extra term introduces complications which render solution difficult. A further Laplace transformation with respect to t taken with the initial conditions (16) yield

(19)
$$\varrho_1^1(s, \xi)(\xi + \lambda - \beta s) = e^{-sE_0} + 2\lambda \varrho_1^2(s, \xi),$$

(20)
$$\varrho_1^2(s, \xi)(\xi + 2\lambda - \beta s) = 2\lambda \varrho_1^2\left(\frac{s}{2}, \xi\right) + \lambda \varrho_1^1\left(\frac{s}{2}, \xi\right),$$

$\varrho_1^1(s, \xi)$ and $\varrho_1^2(s, \xi)$ could be explicitly solved by iteration and inverted. The complete solution for $f_1^1(E, t)$ is given by

(21)
$$f_1^1(E, t) + 2f_1^2(E, t) = f_1(E, t),$$

(22) $\quad f_1(E, t) = e^{-\lambda t}\delta(E - E_0 + \beta t) +$

$$+ e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \left(\frac{-\lambda}{\beta}\right)^n \frac{A_i^n}{(n-1)!} (E - E_0 q^n + \beta t q^i)^{n-1} H(E - E_0 q^n + \beta t q^i),$$

where

(23)
$$A_i^n = \prod_{\substack{j=0 \\ j \neq i}}^{n} (q^j - q^i)^{-1}, \quad q = \frac{1}{2}.$$

The mean of the total number of particles above energy E_c obtained by integrating the right hand side of (22) over the range (E_c, E_0). To obtain the fluctuation about the mean, we must proceed to obtain the second order densities. If $f_2^{1,1}(E_1, E_2, t)$, $f_2^{2,1}(E_1, E_2, t)$, $f_2^{1,2}(E_1, E_2, t)$

and $f_2^{2,2}(E_1, E_2, t)$ are the second order product densities, then using the last collision method we obtain,

(24) $\quad \dfrac{\partial f_2^{1,1}(E_1, E_2, t)}{\partial t} = -2\lambda f_2^{1,1}(E_1, E_2, t) + 2\lambda f_2^{2,1}(E_1, E_2, t) +$

$$+ 2\lambda f_2^{1,2}(E_1, E_2, t) + \beta \left(\dfrac{\partial}{\partial E_1} + \dfrac{\partial}{\partial E_2} \right) f_2^{1,1}(E_1, E_2, t),$$

(25) $\quad \dfrac{\partial f_2^{2,1}(E_1, E_2, t)}{\partial t} = -3\lambda f_2^{2,1}(E_1, E_2, t) + 2\lambda f_2^{2,2}(E_1, E_2, t) +$

$$+ 2\lambda f_2^{1,1}(2E_1, E_2, t) + 4\lambda f_2^{2,1}(2E_1, E_2, t) +$$

$$+ \beta \left(\dfrac{\partial}{\partial E_1} + \dfrac{\partial}{\partial E_2} \right) f_2^{2,2}(E_1, E_2, t) + 2\lambda f_1^2(E_2, t)\,\delta(2E_1 - E_2),$$

(26) $\quad \dfrac{\partial f_2^{1,2}(E_1, E_2, t)}{\partial t} = -3\lambda f_2^{1,2}(E_1, E_2, t) + 2\lambda f_2^{2,2}(E_1, E_2, t) +$

$$+ 2\lambda f_2^{1,1}(E_1, 2E_2, t) + 4\lambda f_2^{1,2}(E_1, 2E_2, t) + 4\lambda f_2^{1,2}(E_1, 2E_2, t) +$$

$$+ \beta \left(\dfrac{\partial}{\partial E_1} + \dfrac{\partial}{\partial E_2} \right) f_2^{1,2}(E_1, E_2, t) + 2\lambda f_1^2(E_1, t)\,\delta(E_1 - 2E_2),$$

(27) $\quad \dfrac{\partial f_2^{2,2}(E_1, E_2, t)}{\partial t} = -4\lambda f_2^{2,2}(E_1, E_2, t) + 2\lambda f_2^{2,1}(E_1, 2E_2, t) +$

$$+ 2\lambda f_2^{1,2}(2E_1, E_2, t) + 4\lambda f_2^{2,2}(E_1, 2E_2, t) +$$

$$+ 4\lambda f_2^{2,2}(2E_1, E_2, t) + \beta \left(\dfrac{\partial}{\partial E_1} + \dfrac{\partial}{\partial E_2} \right) f_2^{2,2}(E_1, E_2, t).$$

The initial conditions are

(28) $\quad f_2^{1,1}(E_1, E_2, 0) = f_2^{1,2}(E_1, E_2, 0) = f_2^{2,1}(E_1, E_2, 0) = f_2^{2,2}(E_1, E_2, 0) = 0.$

Of course, as usual, we can obtain a non-zero solution in view of the correlation term arising from first order densities in equations (25) and (26). We observe that the above equations are not easy to handle. Since we are interested in the moments of the total number of particles above E_c, from (6) it is obvious we need consider what we may call tha overall product density of degree two defined by

(29) $\quad f_2(E_1, E_2, t) = f_2^{1,1}(E_1, E_2, t) + 2f_2^{1,2}(E_1, E_2, t) +$

$$+ 2f_2^{2,1}(E_1, E_2, t) + 4f_2^{2,2}(E_1, E_2, t).$$

From equations (25) to (28), we obtain

$$(30) \quad \frac{\partial f_2(E_1, E_2, t)}{\partial t} = -2\lambda f_2(E_1, E_2, t) + 2\lambda f_2(2E_1, E_2, t) +$$

$$+ 2\lambda f_2(E_1, 2E_2, t) + 4\lambda f_1^2(E_2, t)\,\delta(E_2 - 2E_1) +$$

$$+ 4\lambda f_1^2(E_1, t)\,\delta(E_1 - 2E_2) +$$

$$+ \beta\left(\frac{\partial}{\partial E_1} + \frac{\partial}{\partial E_2}\right) f_2(E_1, E_2, t).$$

If we define $p_2(s_1, s_2, t)$ as the Laplace transform of $f_2(E_1, E_2, t)$, $p_2(s_1, s_2, t)$ satisfied the equation

$$(31) \quad \frac{\partial p_2(s_1, s_2, t)}{\partial t} = -\{(s_1 + s_2)\beta - 2\lambda\} p_2(s_1, s_2, t) +$$

$$+ 2\lambda \left\{ p_2\left(s_1, \frac{s_2}{2}, t\right) + p_2\left(\frac{s_1}{2}, s_2, t\right) \right\} +$$

$$+ 4\lambda \left\{ p_1^2\left(s_2 + \frac{s_1}{2}, t\right) + p_1^2\left(s_1 + \frac{s_2}{2}, t\right) \right\}.$$

(31) may be handled in exactly the same way as (17) and (18) and a series solution for $p_2(s_1, s_2, t)$ can be obtained by iteration without much difficulty. The solution is in inverse powers of β and since β has to be small in any model close to reality, this offers serious difficulty in numerical computation. However an important feature of the model lies in our being able to obtain an explicit solution as contrasted with the electromagnetic cascades, where explicit analytical solution is not possible.

Finally, we observe that it might be worthwhile to attempt integral equations for $\Pi^j(m_1, m_2, \ldots, m_n, x, x_0; t)$, where Π^j represents the probability that m_i i-tuples have parametric values above x given that at $t = 0$, there is a j-tuple primary with parametric value x_0. It is in this spirit that the problem originated from Janossy (1950) and Bellman and Harris (1948). However the author (1960) has shown in connection with the new approach to cascade theory that this method is more suited to describe fluctuations, equations for the moments being amenable to numerical computations. Of course when collision loss is included, even in such a simple model as the one we have proposed in this paper, the integral equation for the number distribution turns out to be complicated.

We hope that the extension of the density method proposed in the paper will find applications in physical problems. As this particular aspect of stochastic processes is in a nascent state, it has been considered worthwhile to present results relating to multiple points and illustrate them

by simple models. In conclusion, the author wishes to record his indebtedness to Professor Alladi Ramakrishnan for the formulation of the simple multiplicative model with ionisation loss.

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PROCESY STOCHASTYCZNE O PUNKTACH WIELOKROTNYCH

STRESZCZENIE

Rozważamy proces stochastyczny odnoszący się do dyskretnej liczby „cząstek" rozmieszczonych w ciągłej przestrzeni parametru. Parametr ciągły, który będziemy oznaczali symbolem x, może oznaczać energię cząstki promieniowania kosmicznego, wiek osobnika w procesie ewolucyjnym, lub przesunięcie cząsteczki poruszającej się w rozpraszającym ośrodku. Zakładamy, że prawdopodobieństwo, że dla pewnej liczby,

powiedzmy n, „cząstek" wartości parametru leżą w nieskończenie małym przedziale Δ przestrzeni parametru wynosi $O(\Delta)$, a prawdopodobieństwo, że dla więcej niż n „cząstek" wartości parametru leżą w tym przedziale wynosi $o(\Delta)$. Przypadek $n = 1$ był badany przez Ramkrishnana i innych, którzy wprowadzili odpowiednie funkcje gęstości i wyrazili za pomocą nich momenty rozkładu liczby „cząstek". W obecnej pracy uogólniamy tę metodę na przypadek punktów wielokrotnych w przestrzeni parametru, i otrzymujemy jawne wyrażenia na r-ty moment za pomocą różnych funkcji gęstości rzędu nie większego niż r. Dla zilustrowania tej metody bada się prosty model kaskady elektromagnetycznej w związku z zagadnieniem rozpraszania. W tym modelu tworzeniu się par cząstek odpowiadają punkty podwójne w przestrzeni parametru, który równy jest przestrzennemu odchyleniu cząstki. Innym przykładem jest proces Furry'ego ze zmienną energią wywołaną przez stratę na jonizację. Najważniejszą cechą tego modelu jest to, że można było otrzymać jawne wyrażenia, w przeciwieństwie do bardziej skomplikowanych kaskad elektromagnetycznych, gdzie nie można było otrzymać wyrażeń jawnych.

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МНОГОКРАТНЫЕ ТОЧЕЧНЫЕ СТОХАСТИЧЕСКИЕ ПРОЦЕССЫ

РЕЗЮМЕ

Изучаются точечные процессы, которые содержат кратные точки в параметрическом пространстве, на котором они определены. Показывается, что развитый в последнее время метод плотностей может быть распространен на случай кратных точек. Используя плотности Яноши, найдены явные выражения моментов через произведения плотностей. В качестве примера рассматриваются простые модели ветвящихся процессов.