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A partition property of cardinal numbers

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CONTENTS

Introduction

The partition property

(1) $$\varkappa \to (\eta_0, \eta_1, \ldots)^n$$

has been extensively studied by Erdös and his collaborators (see [1] and [2] for further references). The present paper deals with a generalization of this property, a generalization obtained by considering not partitions of the n-element subsets of $\varkappa$, as is the case with property (1), but rather partitions of certain finite sequences of n-element subsets. This is in fact a special case of the polarized partition relation defined in [2, p. 100].

The appropriate definitions appear in § 1, along with notation and some simple observations. A brief discussion of when certain partition properties may fail appears in § 2. In § 3, the Ramification Lemma of [2] is recalled, and certain applications of it made. This is in fact the major tool used in obtaining most of the results in this paper. The main theorem is presented in § 4, along with several corollaries to it. Finally, in § 5, a particular polarized relation for cardinals cofinal with $\aleph_0$, which is proved in [2] for the case of a partition into two parts, will be shown to hold for partitions into any finite number of parts.

§ 1. Notation and definitions

Standard notation from set theory will be used throughout. In particular, sequences are written $\langle x_\alpha; \alpha \in A\rangle$, except that for finite n an ordered n-tuple is usually written $\langle x_1, \ldots, x_n\rangle$. The restriction of A to B is denoted $A \upharpoonright B$. The set of all functions with domain x and range contained in y is denoted ${}^x y$. The cartesian product of a family $\{x_i; i \in I\}$ is written $\prod \{x_i; i \in I\}$. The powerset of x is $\mathscr{P}x$, and $x - y$ denotes set-difference.

Ordinals are defined in some standard manner (e.g. [5]) so that if α is an ordinal, $\alpha = \{\beta; \beta \textit{ is an ordinal and } \beta < \alpha\}$. The initial ordinals, i.e. ordinals having power larger than that of any of their members, are identified with the cardinals. The sequence of infinite cardinals is $\aleph_0, \aleph_1, \aleph_2, \ldots$ Ordinal sum and ordinal product are denoted by $+$ and $\cdot$ respectively. Cardinal sum is $+_C$ and cardinal product is indicated by

$\times_C$. Unless the contrary is stated, $\varkappa^\lambda$ means cardinal exponentiation and $\sum$ means the infinite sum of cardinal numbers. If β is a limit ordinal (in particular, a cardinal) the cofinality of β, $\mathrm{Cf}(\beta)$, is defined to be the smallest cardinal $\varkappa$ such that β is the union of $\varkappa$ smaller ordinals. For $\varkappa$ a cardinal, $\varkappa^+$ is the cardinal next after $\varkappa$, and $\varkappa^-$ is the cardinal immediately before $\varkappa$ if there is such a cardinal, whereas otherwise $\varkappa^- = \varkappa$. A cardinal $\varkappa$ is called regular if $\mathrm{Cf}(\varkappa) = \varkappa$, and singular if not regular. A cardinal $\varkappa$ is termed inaccessible if $\varkappa$ is regular and $\lambda < \varkappa \Rightarrow 2^\lambda < \varkappa$.

The cardinality of a set x is denoted $|x|$. The set $\{y \in \mathscr{P}x;\ |y| = \varkappa\}$ of exactly $\varkappa$-element subsets of x (for $\varkappa$ a cardinal) is denoted $[x]^\varkappa$. Similarly, $[x]^{<\varkappa}$, $[x]^{\leqslant\varkappa}$ and $[x]^{\geqslant\varkappa}$ stand for the sets of subsets of x which have cardinality less than $\varkappa$, at most $\varkappa$ and at least $\varkappa$. In particular, $[x]^{<\omega}$ is the set of finite subsets of x.

A partition $\Delta = \{\Delta_l;\ l < \nu\}$ of a set A into ν parts is a decomposition $A = \bigcup \{\Delta_l;\ l < \nu\}$, where no Δ_l is empty. Given such a partition, if $a, b \in A$, then $a \equiv b (\mathrm{mod}\, \Delta)$ indicates that there is some class Δ_l which has both a and b as members. The partition is called disjoint if the Δ_l are pairwise disjoint.

Finally, a word on the conventions concerning variables. Unless the contrary is stated or implied, ι, $\varkappa$, λ denote infinite cardinals. Other small Greek letters denote ordinals, as also frequently do i, j, k, l. Natural numbers, usually non-zero, are represented by m, n. In addition, p, q, r, s also represent arbitrary natural numbers. For partitions of some set, Δ, Γ, Λ will be used. Capital Roman letters and the remaining small Roman letters denote arbitrary sets.

The class of all functions with domain σ and taking ordinal values will be denoted by SEQ_σ. If $X \in \mathrm{SEQ}_\sigma$ for a non-limit ordinal σ, say $\sigma = \tau + 1$, then $X \restriction \tau$ will be abbreviated to $\bar{X}$. Thus $X = \bar{X} \cup \{\langle \tau, X(\tau) \rangle\}$.

The Generalized Continuum Hypothesis will be assumed wherever it leads to a simplication in the results. Theorems reached with its aid will be marked (*).

The partition property to be studied can now be defined.

1.1. Definition. *$\varkappa \to {}^m(\eta_l;\ l < \nu)^n$ if for all partitions $\Delta = \{\Delta_l;\ l < \nu\}$ of ${}^m([\varkappa]^n)$ into ν parts, there are $l < \nu$ and a sequence $H_1, \ldots, H_m$ (where each $H_i \subseteq \varkappa$ and has order type η_l), which is homogeneous for Δ, in the sense that $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_l$.*

Property (1) above is thus seen to be that case of Definition 1.1 for which $m = 1$. The special case of 1.1 in which $\eta_l = \eta$ for all $l < \nu$ will be written $\varkappa \to {}^m(\eta)^n_\nu$. The property $\varkappa \to {}^m(\eta_l; l < \nu,\ \theta_k;\ k < \mu)^n$ has its obvious meaning.

Property 1.1 itself is a special case of the general polarized partition property defined in [2]. This may be expressed as follows.

1.2. DEFINITION.

$$\begin{pmatrix}\varkappa_1\\ \vdots\\ \varkappa_m\end{pmatrix} \to \begin{pmatrix}\eta_{1l}\\ \vdots\\ \eta_{ml}\end{pmatrix}^{n_1,\dots,n_m}_{l<\nu}$$

if for all partitions $\Delta = \{\Delta_l;\ l < \nu\}$ of $[\varkappa_1]^{n_1}\times \dots \times [\varkappa_m]^{n_m}$ into ν parts, there are $l < \nu$ and a sequence $H_1, \dots, H_m$ (where for each i, $H_i \subseteq \varkappa_i$ and has order type η_{il}) such that $[H_1]^{n_1}\times \dots \times [H_m]^{n_m} \subseteq \Delta_l$.

Thus property 1.1 is that case of 1.2 in which $n_1 = \dots = n_m = n$, $\varkappa_1 = \dots = \varkappa_m = \varkappa$ and $\eta_{1l} = \dots = \eta_{ml} = \eta_l$ for each $l < \nu$.

A symbol similar to that used in either of 1.1 or 1.2, but with the $\to$ replaced by $\nrightarrow$, indicates that the appropriate property fails to hold.

Before a more detailed discussion of the various partition relations, a few simple remarks are in order (see also [3]).

The use of the ordinal ν to index the classes of the partition Δ is not essential. Any set with the same power as ν will serve equally well, and the truth or falsity of the relation will remain unchanged. In particular, the ordinals η_l for $l < \nu$ in the relation $\varkappa \to {}^m(\eta_l;\ l < \nu)^n$ may be permuted without affecting the relation.

If $\varkappa$ has a partition property and λ is any cardinal at least as big as $\varkappa$, then λ has that same property.

If $\varkappa$ enjoys a partition property with ν classes and μ is any ordinal smaller than ν, then the corresponding property with μ classes also holds for $\varkappa$, since a partition with a small number of classes can be extended to a partition with a larger number of classes by adding superfluous parts (of, for example, one element).

If $\varkappa$ has a partition property in which the homogeneous sets have order type η_l, then for any ordinals $\theta_l \leqslant \eta_l$ the corresponding property with η_l replaced by θ_l also holds for $\varkappa$.

If $\varkappa$ has a partition property involving sequences of length m, and if $m' \leqslant m$, then $\varkappa$ has the corresponding property for sequences of length m'. (Given any partition involving sequences of length m', choose a partition of m-length sequences for which membership of any partition class depends on only the first m' places of the sequences involved.)

In the case that all the η_l of 1.1 are cardinal numbers (and similarly for the η_{il} of 1.2), rather than requiring that the homogeneous sets have order type exactly η_l, it suffices to specify that their power be η_l. Thus, when the η_l are cardinals, $\varkappa \to (\eta_l;\ l < \nu)^n$ if and only if for all partitions ${}^m([\varkappa]^n) = \bigcup\{\Delta_l;\ l < \nu\}$ there are $l < \nu$ and a sequence $H_1, \dots, H_m$ from $[\varkappa]^{\eta_l}$ such that $[H_1]^n\times \dots \times [H_m]^n \subseteq \Delta_l$. Frequent use of this equivalence will be made, without further comment.

Definitions 1.1 and 1.2 are expressed as properties of cardinal numbers. They could equally well be expressed as properties of arbitrary sets of the appropriate powers. For example, $\varkappa \to {}^m(\eta)^n_{l<\nu}$ is equivalent to the following:

For any sequence of sets $S_1, \ldots, S_m$, where each S_i has power $\varkappa$ and is well ordered by a relation $<_i$, let there be given any partition $\Delta = \{\Delta_l;\ l < \nu\}$ of $[S_1]^n \times \ldots \times [S_m]^n$. Then there is a sequence $H_1, \ldots, H_m$, where each $H_i \subseteq S_i$ and has order type η in $<_i$, such that $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_l$.

There is an obvious extension of relation 1.1 to the case that n is an infinite cardinal. A well-known example of Sierpiński [6] shows that any such relation with infinite n is false. Similarly, there is an obvious extension of 1.1 to the case that m is an infinite ordinal. Again, any such relation is known to be false (see, for example, [7]).

I conclude this section by mentioning a simple method of generating relations of the kind in Definition 1.1 from known relations of type (1) above.

1.3. THEOREM. *Let ν be a cardinal and for $l < \nu$ let η_l be ordinals. Put $\eta = \sup\{\eta_l;\ l < \nu\}$. Suppose $\varkappa$ is a cardinal such that $\varkappa \to (\eta\cdot m)^{mn}_\nu$. Then $\varkappa \to {}^m(\eta_l;\ l < \nu)^n$.*

Proof. Let $\Delta = \{\Delta_l; l < \nu\}$ be any partition of ${}^m([\varkappa]^n)$. Choose any partition $\Gamma = \{\Gamma_l;\ l < \nu\}$ of $[\varkappa]^{mn}$ which satisfies: if $a_1 < a_2 < \ldots < a_{mn} < \varkappa$, then for all $l < \nu$,

$$\{a_1, a_2, \ldots, a_{mn}\} \in \Gamma_l \Leftrightarrow \langle\{a_1, \ldots, a_n\}, \ldots, \{a_{mn-n+1}, \ldots, a_{mn}\}\rangle \in \Delta_l.$$

Since $\varkappa \to (\eta\cdot m)^{mn}_\nu$, there is $H \subseteq \varkappa$ having order type $\eta\cdot m$ such that $[H]^{mn} \subseteq \Gamma_l$ for some $l < \nu$. Divide H into m pieces, $H_1, \ldots, H_m$, each having order type η, such that $\sup(H_i) \leqslant \min(H_{i+1})$ for each i. However, then $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_l$, and since $\eta_l \leqslant \eta$ it follows that $\varkappa \to {}^m(\eta_l;\ l < \nu)^n$.

In view of Theorem B of Ramsey [4] and Theorem 39 (iii) of [1], we obtain the following corollaries:

1.4. COROLLARY. *For all $l, m, n, r \in \aleph_0$ there is $k = k(l, m, n, r) \in \aleph_0$ such that $k \to {}^m(l)^n_r$.*

1.5. COROLLARY (*). *If $a \geqslant 0$ and $\eta_l < \aleph_{a+1}$ for $l < \aleph_a$, then*

$$\aleph_{a+mn} \to {}^m(\eta_l;\ l < \aleph_a)^n.$$

In particular, $\aleph_{a+mn} \to {}^m(\aleph_a)^n_{\aleph_a}$.

The result of Corollary 1.5 is not the best possible. A stronger result will appear in § 4.

§ 2. Negative relations

In this section, some cases where a partition property fails to hold will be discussed.

Cardinals with a property of the form $\varkappa \to {}^m(\varkappa)^n_\gamma$, where $m = 1$ have been considered by various authors. There certainly appears to be no reason to exclude their existence. However, when $m > 1$, this is not the case. The following theorem shows that even the weakest such property fails to hold.

2.1. THEOREM. *For all* $\varkappa$, $\varkappa \nrightarrow {}^2(\varkappa)^1_2$.

Proof. Take any cardinal $\varkappa$, and define a partition $\Delta = \{\Delta_0, \Delta_1\}$ of $[\varkappa]^1 \times [\varkappa]^1$ as follows:

$$\langle\{a\}, \{\beta\}\rangle \in \Delta_0 \Leftrightarrow a \leqslant \beta; \quad \langle\{a\}, \{\beta\}\rangle \in \Delta_1 \Leftrightarrow \beta \leqslant a.$$

Let H_0, H_1 be any pair homogeneous for Δ. Suppose, say, that $[H_0]^1 \times \times [H_1]^1 \subseteq \Delta_0$. If $\beta = \min(H_1)$ and $a \in H_0$, then $a \leqslant \beta$. But $\beta \in \varkappa$, and so $|H_0| < \varkappa$. Similarly, if $[H_0]^1 \times [H_1]^1 \subseteq \Delta_1$, then $|H_1| < \varkappa$. Thus no pair homogeneous for Δ can have $|H_0| = |H_1| = \varkappa$. This proves the theorem.

In fact, even more may fail:

2.2. THEOREM (*). *Provided* $\varkappa > \aleph_0$, *then* $\varkappa^+ \nrightarrow {}^2(\varkappa^+, \varkappa)^1$.

This follows from Theorem 43 of [2]. The proof will not be given here.

2.3. THEOREM. *Suppose* $\lambda < \varkappa \Rightarrow 2^\lambda < \varkappa$. *Then*

$$\begin{pmatrix} 2^\varkappa \\ 2^\varkappa \end{pmatrix} \nrightarrow \begin{pmatrix} \varkappa & 2 \\ 2 & \varkappa \end{pmatrix}^{2,2}.$$

Proof. For distinct elements x and y in ${}^\varkappa 2$, define $\delta(x, y)$ to be the least $a < \varkappa$ for which $x(a) \neq y(a)$. Define a disjoint partition $[{}^\varkappa 2]^2 \times \times [{}^\varkappa 2]^2 = \Delta_0 \cup \Delta_1$ by

$$\langle\{x_0, x_1\}, \{y_0, y_1\}\rangle \in \Delta_0 \Leftrightarrow \delta(x_0, x_1) \leqslant \delta(y_0, y_1).$$

Take any $y_0, y_1 \in {}^\varkappa 2$ with $y_0 \neq y_1$. Then $\delta(y_0, y_1) < \varkappa$. However, if $x_0, x_1 \in {}^\varkappa 2$ are such that $\delta(x_0, x_1) \leqslant \delta(y_0, y_1)$, then x_0 and x_1 differ no later than at $\delta(y_0, y_1)$. Hence if $H \subseteq {}^\varkappa 2$ has the property

$$\{x_0, x_1\} \in [H]^2 \Rightarrow \delta(x_0, x_1) \leqslant \delta(y_0, y_1),$$

then $|H| \leqslant 2^{|\delta(y_0, y_1)+1|} < \varkappa$. Thus there are no $H_0 \in [{}^\varkappa 2]^\varkappa, H_1 \in [{}^\varkappa 2]^2$ such that $[H_0]^2 \times H_1 \subseteq \Delta_0$. Similarly, there are no $H_0 \in [{}^\varkappa 2]^2, H_1 \in [{}^\varkappa 2]^\varkappa$ such that $H_0 \times [H_1]^2 \subseteq \Delta_1$. This proves Theorem 2.3.

2.4. COROLLARY (*). *If* $\varkappa^- = \varkappa$, *then* $\varkappa^+ \nrightarrow {}^2(\varkappa)^2_2$.

A similar construction may be used to show:

2.5. THEOREM. *Suppose* $\lambda < \varkappa \Rightarrow 2^\lambda < \varkappa$. *Then*

$$\binom{2^\varkappa}{\varkappa} \nrightarrow \begin{pmatrix} \varkappa & 2 \\ 1 & \varkappa \end{pmatrix}^{2,1}.$$

The following theorem gives a method of stepping up a negative result for a cardinal $\varkappa$ to a negative result for $2^\varkappa$. The proof is an extension of the method of proof of Lemma 5A of [2], and will not be given here.

2.6. THEOREM. *Let* $n \geqslant 3$. *Let* η_l *for* $l < \nu$ *be cardinals such that* $\eta_0, \eta_1 \geqslant \aleph_0$ *and* η_0 *is regular. If* $\varkappa$ *is a cardinal such that* $\varkappa \nrightarrow {}^m(\eta_l;\, l < \nu)^{n-1}$, *then* $2^\varkappa \nrightarrow {}^m(\eta_l;\, l < \nu)^n$.

Theorem 2.1 yields that $\varkappa \nrightarrow {}^2(\varkappa)^1_2$, and so certainly $\varkappa \nrightarrow {}^2(\varkappa)^2_2$. Thus by Theorem 2.6, $\varkappa^+ \nrightarrow {}^2(\varkappa)^3_2$ for regular $\varkappa$. If $\varkappa$ is inaccessible, applying Theorem 2.6 to Corollary 2.4 (*) leads to the stronger result that $\varkappa^{++} \nrightarrow {}^2(\varkappa)^3_2$. Likewise, Theorem 2.2 gives that $\varkappa^+ \nrightarrow {}^2(\varkappa^+, \varkappa)^1$ for $\varkappa > \aleph_0$, so that $\varkappa^+ \nrightarrow {}^2(\varkappa^+, \varkappa)^2$. Then 2.6 yields that $\varkappa^{++} \nrightarrow {}^2(\varkappa^+, \varkappa)^3$.

§ 3. The Ramification Lemma

Many of the positive results obtained in [2] depend on the following lemma, the Ramification Lemma. Likewise, it is the main lemma on which rest most of the proofs of the relations established in §§ 4 and 5.

3.1. LEMMA. *Let* $\varrho > 0$ *be a limit ordinal. For all sequences* $X \in \mathrm{SEQ}_\sigma$ *and* $Y \in \mathrm{SEQ}_{\sigma+1}$, *where* $\sigma < \varrho$, *let there be given sets* $S(Y), F(X)$ *and an ordinal* $n(X)$. *Let a set* $S = S(\emptyset)$ *be given. Put*

$$N = \{X \in \mathrm{SEQ}_\sigma;\ \sigma \leqslant \varrho \text{ and } \forall \tau < \sigma(X(\tau) < n(X \upharpoonright \tau))\},$$

and for $X \in \mathrm{SEQ}_\sigma$ *define*

$$S'(X) = S \cap \bigcap \{S(X \upharpoonright \tau+1);\ \tau < \sigma\}.$$

Suppose that whenever $\sigma < \varrho$ *and* $X \in N \cap \mathrm{SEQ}_\sigma$ *then*

(a) $S'(X) = F(X) \cup \bigcup \{S(Y);\ Y \in \mathrm{SEQ}_{\sigma+1} \text{ and } \overline{Y} = X \text{ and } Y(\sigma) < n(X)\}$,

(b) $F(X) \cap \bigcup \{S(Y);\ Y \in \mathrm{SEQ}_{\sigma+1} \text{ and } \overline{Y} = X \text{ and } Y(\sigma) < n(X)\} = \emptyset$.

Under these conditions,

(i) $\tau < \sigma < \varrho$ *and* $X \in N \cap \mathrm{SEQ}_\sigma \Rightarrow F(X) \cap F(X \upharpoonright \tau) = \emptyset$;

(ii) $S = \bigcup\{F(X);\ \exists \sigma < \varrho(X \in N \cap \mathrm{SEQ}_\sigma)\} \cup \bigcup\{S'(X);\ X \in N \cap \mathrm{SEQ}_\varrho\}$;

(iii) *Suppose* $\aleph_0 \leqslant \varkappa \leqslant |S|$, $|\varrho| < \mathrm{Cf}(\varkappa)$ *and* $|F(X)| < \varkappa$ *whenever* $X \in N$.

For $\sigma < \varrho$ *let there be given cardinals* λ_σ *such that* $\lambda_\sigma^{|\sigma|} < \mathrm{Cf}(\varkappa)$, *and suppose that*

$$\tau < \sigma < \varrho \quad \textit{and} \quad X \in N \cap \mathrm{SEQ}_\sigma \Rightarrow |n(X \upharpoonright \tau)| \leqslant \lambda_\sigma.$$

Then there is $X \in N \cap \mathrm{SEQ}_\varrho$ *for which* $S'(X)$ *is non-empty.*

(iv) *Suppose* $\varkappa$ *is strongly inaccessible. Let* $|S| \geqslant \varkappa$, $|\varrho| < \varkappa$ *and assume that for* $\sigma < \varrho$,

$$X \in N \cap \mathrm{SEQ}_\sigma \Rightarrow |F(X)| < \varkappa \quad \textit{and} \quad |n(X)| < \varkappa.$$

Then there is $X \in N \cap \mathrm{SEQ}_\varrho$ *for which* $S'(X)$ *is non-empty.*

The proof is in [2], pp. 103-105. A diagram of a ramification system may be found on p. 105 of [2].

The system $\mathscr{R}$ of sets $N, F(X)$ and $S(X)$ is called a ramification system on S of length ϱ. In all references to the Ramification Lemma, the symbols $\varrho, N, S(X), \ldots$ are to have the significance ascribed to them in that lemma. In applications, the ramification system $\mathscr{R}$ will be constructed inductively. To do this, it is sufficient to assume for any $\sigma < \varrho$ that $S'(X)$ has already been defined for some fixed $X \in \mathrm{SEQ}_\sigma$, and to define $n(X), F(X)$ and each $S(Y)$ for $Y \in \mathrm{SEQ}_{\sigma+1}$ such that $\overline{Y} = X$ and $Y(\sigma) < n(X)$.

As a first application of this principle, the following theorem will be established.

3.2. THEOREM. *Let* $\lambda_1 \geqslant \aleph_0$ *and* λ_2 *be cardinals such that* $\lambda_1 \to {}^1(\eta_l \cdot 2; l < \nu)^1$ *and* $\lambda_2 \to {}^2(\theta_k; k < \mu)^1$. *Suppose* $\varkappa$ *is a regular cardinal such that* $\lambda_1^+, \lambda_2 \leqslant \varkappa$, *and* $\iota < \lambda_1 \Rightarrow (|\nu| \times_C \lambda_2^-)^\iota < \varkappa$. *Then* $\varkappa \to {}^2(\theta_k; k < \mu, \eta_l; l < \nu)^1$.

Proof. Consider first the case where the η_l are all cardinals. Suppose $\varkappa \times \varkappa$ is partitioned,

$$\varkappa \times \varkappa = \bigcup \{\Delta_k; k < \mu\} \cup \bigcup \{\Gamma_l; l < \nu\}.$$

It may be assumed for all $H_0, H_1 \subseteq \varkappa$ with $|H_0|, |H_1| \geqslant \lambda_2$ that $H_0 \times \times H_1 \nsubseteq \bigcup \{\Delta_k; k < \mu\}$, since otherwise the theorem follows from the property $\lambda_2 \to {}^2(\theta_k; k < \mu)^1$. Thus it suffices to find $l < \nu$ and $H_0, H_1 \subseteq \varkappa$ such that H_0, H_1 have power η_l and $H_0 \times H_1 \subseteq \Gamma_l$.

Define a ramification system $\mathscr{R}$ on $\varkappa$ of length $\varrho = \lambda_1$ as follows. Take $\sigma < \varrho$ and $X \in \mathrm{SEQ}_\sigma$. Suppose $S'(X)$ has already been defined, and consider two cases.

Case 1. σ even.

If for no $x \in S'(X)$ is it the case that $\langle x, x \rangle \in \bigcup \{\Delta_k; k < \mu\}$, choose $x \in S'(X)$ and put $F(X) = \{x\}$. Otherwise, choose $F(X) \subseteq S'(X)$ maximal with the property $F(X) \times F(X) \subseteq \bigcup \{\Delta_k; k < \mu\}$. In either case, $|F(X)| < \lambda_2$. Choose $Q(X) \subseteq S'(X)$ maximal such that $F(X) \subseteq Q(X)$ and $Q(X) \times \times F(X) \subseteq \bigcup \{\Delta_k; k < \mu\}$, except that if there is no $x \in S'(X)$ with $\langle x, x \rangle \in \bigcup \{\Delta_k; k < \mu\}$, then the requirement $F(X) \subseteq Q(X)$ is to be ignored.

Then if $y \in S'(X) - Q(X)$, there is $x \in F(X)$ such that $\langle y, x\rangle \notin \bigcup\{\Delta_k; k < \mu\}$, by the maximality of $Q(X)$. Thus there is a decomposition

$$S'(X) - \big(Q(X) \cup F(X)\big) = \bigcup\{S(Y);\ \bar{Y} = X \textit{ and } Y(\sigma) < n'(X)\},$$

where given $Y \in \mathrm{SEQ}_{\sigma+1}$ such that $\bar{Y} = X$, for some $x(Y) \in F(X)$ and $l(Y) < \nu$,

$$y \in S(Y) \Leftrightarrow \langle y, x(Y)\rangle \in \Gamma_{l(Y)}.$$

Further, $|n'(X)| \leqslant |\nu| \times_C |F(X)| \leqslant |\nu| \times_C \lambda_2^-$. Finally, if

$$\bigcap\{S(Y);\ \bar{Y} = X \text{ and } Y(\sigma) < n'(X)\} - F(X) = S'(X) - F(X),$$

put $n(X) = n'(X)$; otherwise $n(X) = n'(X)+1$ and $S\big(X \cup \{\langle\sigma, n'(X)\rangle\}\big) = Q(X) - F(X)$.

Case 2. σ odd, say $\sigma = \zeta + 1$.

If $X(\zeta) < n'(\bar{X})$, put $F(X) = \emptyset$, $n(X) = 1$ and $S(X \cup \{\langle\sigma, 0\rangle\}) = S'(X)$. If $X(\zeta) = n'(\bar{X})$, still put $F(X) = \emptyset$. However, in this case $S'(X) \subseteq Q(\bar{X}) - F(\bar{X})$. Hence by the maximality of $F(\bar{X})$, for any $y \in S'(X)$ either $\langle y, y\rangle \notin \bigcup\{\Delta_k; k < \mu\}$ or there is $x \in F(\bar{X})$ such that $\langle x, y\rangle \notin \bigcup\{\Delta_k; k < \mu\}$. Thus there is a decomposition

$$S'(X) = \bigcup\{S(Y);\ \bar{Y} = X \text{ and } Y(\sigma) < n'(X)\},$$

where given Y such that $\bar{Y} = X$, for some $x(Y) \in F(x)$ and $l(Y) < \nu$,

$$y \in S(Y) \Leftrightarrow \langle x(Y), y\rangle \in \Gamma_{l(Y)} \quad \text{or} \quad \langle y, y\rangle \in \Gamma_{l(Y)}.$$

Again, $|n'(X)| \leqslant |\nu| \times_C |F(X)| \leqslant |\nu| \times_C \lambda_2^-$. Put $n(X) = n'(X)$.

This completes the definition of the ramification system $\mathscr{R}$. Further, Lemma 3.1 (iii) applies to $\mathscr{R}$. Hence there is a sequence $X \in N \cap \mathrm{SEQ}_\varrho$ for which $S'(X) \neq \emptyset$. Choose such a sequence X. For each $\sigma < \varrho$ put $x_\sigma = x(X \restriction \sigma+1)$ whenever $x(X \restriction \sigma+1)$ is defined. By Lemma 3.1 (i), if $\tau + 1 < \sigma$, then $x_\tau \neq x_\sigma$. Now it follows

(1) $\quad \tau < \sigma < \varrho \quad$ and $\quad X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau) \Rightarrow \langle x_{2\cdot\sigma}, x_{2\cdot\tau}\rangle \in \Delta_{l(X \restriction 2\cdot\tau+1)}$,

for $x_{2\cdot\sigma} \in F(X \restriction 2\cdot\sigma) \subseteq S'(X \restriction 2\cdot\sigma) \subseteq S(X \restriction 2\cdot\tau+1)$. Also

(2) $\quad \tau < \sigma < \varrho$ and $X(2\cdot\tau) = n'(X \restriction 2\cdot\tau) \Rightarrow \langle x_{2\cdot\tau+1}, x_{2\cdot\sigma+1}\rangle \in \Delta_{l(X \restriction 2\cdot\tau+2)}$,

for $x_{2\cdot\sigma+1} \in F(X \restriction 2\cdot\sigma) \subseteq S'(X \restriction 2\cdot\sigma) \subseteq S(X \restriction 2\cdot\tau+2)$.

For $\tau < \varrho$, either $X(2\cdot\tau) = n'(X \restriction 2\cdot\tau)$ or $X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau)$. Hence there is $H \subseteq \varrho$ with $|H| = \varrho$ such that either

(3) $$\tau \in H \Rightarrow X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau),$$

or

(4) $$\tau \in H \Rightarrow X(2\cdot\tau) = n'(X \restriction 2\cdot\tau).$$

Let H be partitioned, $H = \bigcup\{\Lambda_l;\ l < \nu\}$, where

$$\Lambda_l = \{\tau \in H;\ l(X \upharpoonright 2\cdot\tau+1) = l\}$$

if (3) holds, or if (4) holds

$$\Lambda_l = \{\tau \in H;\ l(X \upharpoonright 2\cdot\tau+2) = l\}.$$

Since by assumption $\lambda_1 \to {}^1(\eta_l\cdot 2;\ l < \nu)^1$ and $|H| = \varrho = \lambda_1$, there are $I \subseteq H$ and $l < \nu$ such that I has order type $\eta_l\cdot 2$ and $I \subseteq \Lambda_l$. Enumerate I as $\{\tau(\alpha);\ \alpha < \eta_l\cdot 2\}$, in increasing order. If (3) holds, put

$$H_0 = \{x_{2\cdot\tau(\alpha)};\ \eta_l \leqslant \alpha < \eta_l\cdot 2\}, \qquad H_1 = \{x_{2\cdot\tau(\alpha)};\ \alpha < \eta_l\},$$

whereas if (4) holds, put

$$H_0 = \{x_{2\cdot\tau(\alpha)+1};\ \alpha < \eta_l\}, \qquad H_1 = \{x_{2\cdot\tau(\alpha)+1};\ \eta_l \leqslant \alpha < \eta_l\cdot 2\}.$$

Then H_0, H_1 both have power η_l, and it follows from (1) and (2) that $H_0 \times H_1 \subseteq \Gamma_l$.

This proves the theorem when the η_l are all cardinals. A slight modification of the argument above allows the x_σ to be chosen so that $\tau+1 < \sigma \Rightarrow x_\tau < x_\sigma$. The general case then follows.

3.3. Corollary (*). *Let* $\alpha \geqslant 0$, $\nu < \mathrm{Cf}(\aleph_\alpha)$ *and* $\eta_l < \mathrm{Cf}(\aleph_\alpha)$ *for* $l < \nu$. *Then* $\aleph_{\alpha+1} \to {}^2(\aleph_{\alpha+1}, \eta_l;\ l < \nu)^1$.

Proof. Put $\lambda_1 = \mathrm{Cf}(\aleph_\alpha)$ and $\lambda_2 = \aleph_{\alpha+1}$. Then λ_1 is regular, so that $\lambda_1 \to {}^1(\lambda_1)^1_\nu$ and hence, in particular, $\lambda_1 \to {}^1(\eta_l\cdot 2;\ l < \nu)^1$. Since $\lambda_1 \leqslant \aleph_\alpha$, if $\iota < \lambda_1$, then

$$(|\nu| \times_C \lambda_2^-)' \leqslant (\mathrm{Cf}(\aleph_\alpha) \times_C \aleph_\alpha)' = (\aleph_\alpha)' = \aleph_\alpha < \aleph_{\alpha+1}.$$

Further $\lambda_2 \to {}^2(\lambda_2)^1_1$. Thus Theorem 3.2 applies with $\varkappa = \aleph_{\alpha+1}$. This yields the result.

A similar result can be reached for inaccessible cardinals.

3.4. Theorem (*). *Let* $\varkappa$ *be strongly inaccessible. Take* $\nu < \varkappa$ *and for each* $l < \nu$ *let* $\eta_l < \varkappa$. *Then* $\varkappa \to {}^2(\varkappa, \eta_l;\ l < \nu)^1$.

Proof. The proof is very similar to that of Theorem 3.2. Put $\lambda_2 = \varkappa$, so $\lambda_2 \to {}^2(\varkappa)^1_1$. Put $\lambda_1 = \left(\sum\{|\eta_l|^+;\ l < \nu\}\right)^+$, so $\lambda_1 < \varkappa$. Then $\lambda_1 \to {}^1(|\eta_l|^+;\ l < \nu)^1$, so certainly $\lambda_1 \to {}^1(\eta_l\cdot 2;\ l < \nu)^1$.

Suppose $\varkappa \times \varkappa$ is partitioned, $\varkappa \times \varkappa = \Delta \cup \bigcup\{\Gamma_l;\ l < \nu\}$. The proof now becomes almost identical to that of Theorem 3.2, except that the application of Lemma 3.1 (iii) is replaced by an appeal to Lemma 3.1 (iv).

§ 4. The main theorem

The method of proof of the Stepping-up Lemma of [2, p. 107] is used to prove the following two lemmas. Together, these yield the main theorem, Theorem 4.3.

4.1. LEMMA. *Let* $m, n_1, \ldots, n_m \geqslant 1$. *Let* λ *be a cardinal such that*

$$\begin{pmatrix}\lambda\\ \vdots\\ \lambda\end{pmatrix} \to \begin{pmatrix}\eta_{1l}\\ \vdots\\ \eta_{ml}\end{pmatrix}^{n_1,\ldots,n_m}_{l<\nu},$$

where the η_{il} *are infinite cardinals. Suppose* $\varkappa$ *is such that* $\lambda < \mathrm{Cf}(\varkappa)$ *and* $\sigma < \lambda \Rightarrow |\nu|^{|\sigma|} < \mathrm{Cf}(\varkappa)$. *For each* $l < \nu$, *suppose that* $\mathrm{Cf}(\eta_{1l}) > \eta_{2l}, \ldots, \eta_{ml}$. *Then*

$$\begin{pmatrix}\varkappa\\ \vdots\\ \varkappa\end{pmatrix} \to \begin{pmatrix}\eta_{1l}\\ \vdots\\ \eta_{ml}\end{pmatrix}^{n_1+1,n_2,\ldots,n_m}_{l<\nu}.$$

Proof. Suppose the hypotheses of the lemma hold. Let any disjoint partition $\Delta = \{\Delta_l;\ l < \nu\}$ of $[\varkappa]^{n_1+1}\times[\varkappa]^{n_2}\times\ldots\times[\varkappa]^{n_m}$ be given. Define inductively a ramification system $\mathscr{R}$ on $\varkappa$ of length $\varrho = \lambda$, together with elements $x(X)$ for certain $X \in \mathrm{SEQ}_\sigma$ with $\sigma < \varrho$. Let $\sigma < \varrho$ and suppose $S'(X)$ has already been defined for some $X \in \mathrm{SEQ}_\sigma$. If $S'(X) = \emptyset$, put $F(X) = \emptyset$ and $n(X) = 0$. Otherwise, choose $x(X) \in S'(X)$ and put $F(X) = \{x(X)\}$. Place $G(X) = \{x(X \restriction \tau);\ \tau \leqslant \sigma\}$. Define a partition $\Gamma(X)$ of $S'(X) - F(X)$ as follows

$$y \equiv z\,(\mathrm{mod}\,\Gamma(X)) \Leftrightarrow \forall a_1 \in [G(X)]^{n_1}\forall a_2 \in [G(X)]^{n_2}\ldots\forall a_m \in [G(X)]^{n_m} \big(\langle a_1 \cup \{y\}, a_2, \ldots, a_m\rangle \equiv \langle a_1 \cup \{z\}, a_2, \ldots, a_m\rangle\,(\mathrm{mod}\,\Delta)\big).$$

Put $n(X) = |\Gamma(X)|$, and for $Y \in \mathrm{SEQ}_{\sigma+1}$ such that $\overline{Y} = X$ and $Y(\sigma) < n(X)$, let $S(Y)$ range over the classes of $\Gamma(X)$. This defines $\mathscr{R}$.

For $\sigma < \varrho$, put $\lambda_\sigma = |\nu|^{\mu}$, where $\mu = |\sigma|^{n_1\cdots n_m}$. Then $|n(X\restriction\tau)| \leqslant \lambda_\sigma$ whenever $X \in \mathrm{SEQ}_\sigma$ and $\tau < \sigma < \varrho$. Moreover, if $\sigma < \varrho$, then $\lambda_\sigma^{|\sigma|} < \mathrm{Cf}(\varkappa)$. Hence Lemma 3.1 (iii) applies to $\mathscr{R}$, and so there is $X \in N \cap \mathrm{SEQ}_\varrho$ for which $S'(X) \neq \emptyset$. Choose such a sequence X. Then $x(X\restriction\sigma)$ is defined for each $\sigma < \varrho$. Put $x_\sigma = x(X\restriction\sigma)$ for $\sigma < \varrho$, and choose $x_\varrho \in S'(X)$. Thus for any $\sigma < \varrho$, if $y, z \in S'(X\restriction\sigma)$, then

$$y \equiv z\,(\mathrm{mod}\,\Gamma(X\restriction\sigma)) \Leftrightarrow \forall a_1 \in [\{x_\alpha;\ \alpha \leqslant \sigma\}]^{n_1}\ldots\forall a_m \in [\{x_\alpha;\ \alpha \leqslant \sigma\}]^{n_m} \big(\langle a_1 \cup \{y\}, a_2, \ldots, a_m\rangle \equiv \langle a_1 \cup \{z\}, a_2, \ldots, a_m\rangle\,(\mathrm{mod}\,\Delta)\big).$$

Now if $\sigma < \tau \leqslant \varrho$, then $x_\tau \in F(X\restriction\tau) \subseteq S'(X\restriction\tau) \subseteq S(X\restriction\sigma+1)$, and so if $\tau < \varrho$ and $a_i \in [\{x_\alpha;\ \alpha < \tau\}]^{n_i}$ for $i = 1, \ldots, m$, then

$$\langle a_1 \cup \{x_\tau\}, a_2, \ldots, a_m\rangle \equiv \langle a_1 \cup \{x_\varrho\}, a_2, \ldots, a_m\rangle\,(\mathrm{mod}\,\Delta).$$

Further, if $\sigma < \tau < \varrho$, then by Lemma 3.1 (i), $F(X\restriction\sigma) \cap F(X\restriction\tau) = \emptyset$, and so $x_\sigma \neq x_\tau$.

Put $W = \{x_\alpha;\ \alpha < \varrho\}$, so $|W| = \varrho = \lambda$. Define a partition $[W]^{n_1}\times \times\ldots\times[W]^{n_m} = \bigcup\{\Delta'_l;\ l < \nu\}$ by, for $l < \nu$,

$$\langle a_1, \ldots, a_m\rangle \in \Delta'_l \Leftrightarrow \langle a_1 \cup \{x_\varrho\}, a_2, \ldots, a_m\rangle \in \Delta_l.$$

By relation (1), there are $l < \nu$ and a sequence $H_1, \dots, H_m$, where $H_i \in [W]^{\eta_{il}}$ such that $[H_1]^{n_1} \times \dots \times [H_m]^{n_m} \subseteq \Delta_l'$. Since it is assumed that $\mathrm{Cf}(\eta_{1l}) > \eta_{2l}, \dots, \eta_{ml}$ take $H \in [H_1]^{\eta_{1l}}$ such that $x_\tau \in H$ and $x_\sigma \in H_2 \cup \dots \cup H_m \Rightarrow \sigma < \tau$. But then $[H]^{n_1+1} \times [H_2]^{n_2} \times \dots \times [H_m]^{n_m} \subseteq \Delta_l$. This proves 4.1.

4.2. Lemma. *Let $m, n_1, \dots, n_m \geqslant 1$. Let λ be a cardinal such that*

$$(2) \qquad \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} \to \begin{pmatrix} \eta_{1l} \\ \vdots \\ \eta_{ml} \end{pmatrix}^{n_1, \dots, n_m}_{l<\nu},$$

where the η_{il} are infinite cardinals. Suppose $\varkappa$ is such that $\lambda < \mathrm{Cf}(\varkappa)$ and $\sigma < \lambda \Rightarrow |\nu|^{|\sigma|} < \mathrm{Cf}(\varkappa)$. For each $l < \nu$ suppose that $\mathrm{Cf}(\lambda) > \eta_{1l}, \dots, \eta_{ml}$. Then

$$\begin{pmatrix} \varkappa \\ \varkappa \\ \vdots \\ \varkappa \end{pmatrix} \to \begin{pmatrix} \lambda \\ \eta_{1l} \\ \vdots \\ \eta_{ml} \end{pmatrix}^{1, n_1, \dots, n_m}_{l<\nu}$$

Proof. The proof is similar to that of Lemma 4.1, and so will not be given in full detail. Any undefined notation is taken from 4.1.

Define a ramification system similar to that used in the last proof, except that the partition $\Gamma(X)$ of $S'(X) - F(X)$ should satisfy:

$$y \equiv z \big(\mathrm{mod}\,\Gamma(X)\big) \Leftrightarrow \forall a_1 \in [G(X)]^{n_1} \dots \forall a_m \in [G(X)]^{n_m} \big(\langle y, a_1, \dots, a_m\rangle \equiv \langle z, a_1, \dots, a_m\rangle \,(\mathrm{mod}\,\Delta)\big).$$

An application of the Ramification Lemma yields distinct elements x_a for $a < \varrho$ such that if $\tau < \varrho$ and $a_i \in [\{x_a;\ a < \tau\}]^{n_i}$ for $i = 1, \dots, m$, then

$$\langle x_\tau, a_1, \dots, a_m\rangle \equiv \langle x_\varrho, a_1, \dots, a_m\rangle \,(\mathrm{mod}\,\Delta).$$

Put $W = \{x_a;\ a < \varrho\}$, and define a partition $[W]^{n_1} \times \dots \times [W]^{n_m} = \bigcup \{\Delta_l';\ l < \nu\}$ by, for $l < \nu$,

$$\langle a_1, \dots, a_m\rangle \in \Delta_l' \Leftrightarrow \langle x_\varrho, a_1, \dots, a_m\rangle \in \Delta_l.$$

By property (2), there are $l < \nu$ and a sequence $H_1, \dots, H_m$, where each $H_i \subseteq W$ and has power η_{il}, such that $[H_1]^{n_1} \times \dots \times [H_m]^{n_m} \subseteq \Delta_l'$. Put $H = \{x_a \in W;\ \forall \tau < \varrho(x_\tau \in H_1 \cup \dots \cup H_m \Rightarrow \tau < a)\}$. By the property assumed for λ, then H has power λ. Since $H \times [H_1]^{n_1} \times \dots \times [H_m]^{n_m} \subseteq \Delta_l$, the proof of 4.2 is complete.

These lemmas combine to yield:

4.3. Theorem (*). *Let $\nu < \mathrm{Cf}(\aleph_a)$ and let $n_1, \dots, n_m \geqslant 1$ with $n_m > 1$.*

Then

(1)
$$\begin{pmatrix}\aleph_{a+n_1+\ldots+n_m-1}\\ \vdots\\ \aleph_{a+n_1+\ldots+n_m-1}\end{pmatrix} \to \begin{pmatrix}\aleph_{a+n_2+\ldots+n_m-1}\\ \vdots\\ \aleph_{a+n_m-1}\\ \aleph_a\end{pmatrix}_{\nu}^{n_1,\ldots,n_m}.$$

Proof. We shall use the known result

(2)
$$\aleph_{a+n_m-1} \to {}^{1}(\aleph_a)_{\nu}^{n_m}$$

from [1, p. 468]. Note that $\sigma < \aleph_{a+n_m-1} \Rightarrow |\nu|^{|\sigma|} \leqslant \aleph_{a+n_m-1}$, and that $\aleph_{a+n_m-1} < \aleph_{a+n_m} = \mathrm{Cf}(\aleph_{a+n_m})$. Also since $n_m > 1$, it follows that $\aleph_{a+n_m-1} > \aleph_a$. Thus Lemma 4.2 may be applied to relation (2), to deduce

(3)
$$\begin{pmatrix}\aleph_{a+n_m}\\ \aleph_{a+n_m}\end{pmatrix} \to \begin{pmatrix}\aleph_{a+n_m-1}\\ \aleph_a\end{pmatrix}_{\nu}^{1,n_m}.$$

Lemma 4.1 is now applied to relation (3) $n_{m-1}-1$ times in succession, and yields

(4)
$$\begin{pmatrix}\aleph_{a+n_{m-1}+n_m-1}\\ \aleph_{a+n_{m-1}+n_m-1}\end{pmatrix} \to \begin{pmatrix}\aleph_{a+n_m-1}\\ \aleph_a\end{pmatrix}_{\nu}^{n_{m-1},n_m}.$$

Another application of Lemma 4.2, this time to property (4), gives

$$\begin{pmatrix}\aleph_{a+n_{m-1}+n_m}\\ \aleph_{a+n_{m-1}+n_m}\\ \aleph_{a+n_{m-1}+n_m}\end{pmatrix} \to \begin{pmatrix}\aleph_{a+n_{m-1}+n_m-1}\\ \aleph_{a+n_m-1}\\ \aleph_a\end{pmatrix}_{\nu}^{1,n_{m-1},n_m}.$$

By continuing in this manner, relation (1) can be established. This concludes the proof.

The following special cases are worth noting:

4.4. Corollary (*). *If* $\nu < \mathrm{Cf}(\varkappa)$, *then*

$$\begin{pmatrix}\varkappa^{++}\\ \varkappa^{++}\end{pmatrix} \to \begin{pmatrix}\varkappa^{+}\\ \varkappa\end{pmatrix}_{\nu}^{1,2}.$$

This provides the answer to a question posed in [7, p. 85].

4.5. Corollary (*). *If* $\nu < \mathrm{Cf}(\varkappa)$, *then*

$$\begin{pmatrix}\varkappa^{+++}\\ \varkappa^{+++}\end{pmatrix} \to \begin{pmatrix}\varkappa^{+}\\ \varkappa\end{pmatrix}_{\nu}^{2,2}.$$

4.6. Corollary (*). *If* $\nu < \mathrm{Cf}(\aleph_a)$ *and* $n > 1$, *then* $\aleph_{a+mn-1} \to {}^{m}(\aleph_a)_{\nu}^{n}$.

This is the promised improvement to Corollary 1.5. Note that the result excludes the case $n = 1$. In this event, the following theorem appears as Corollary 17 in [2]:

4.7. THEOREM (*). *For all infinite* $\varkappa$, $\varkappa^+ \to {}^2(\varkappa)_2^1$.

For cardinals cofinal with $\aleph_0$, a slightly stronger result will be proved in § 5. By applying Lemma 4.2 to Theorem 4.7, we obtain

4.8. THEOREM (*). *If* $a \geqslant 0$, *then* $\aleph_{a+m-1} \to {}^m(\aleph_a)_2^1$.

In fact, slightly stronger results pertain, e.g.

4.9. THEOREM (*). *For all infinite* $\varkappa$,

$$\begin{pmatrix} \varkappa^{++} \\ \varkappa^{++} \\ \varkappa^{++} \end{pmatrix} \to \begin{pmatrix} \varkappa^{+} \\ \varkappa \\ \varkappa \end{pmatrix}_2^{1,1,1}.$$

In view of the negative results of § 2, some of the most simple unanswered questions are seen to be:

4.10. PROBLEM (*). *Is* $\varkappa^{++} \to {}^2(\varkappa)_2^2$ *true? If* $\varkappa > \varkappa^-$, *is in fact* $\varkappa^+ \to {}^2(\varkappa)_2^2$ *true?*

4.11. PROBLEM (*). *Is* $\varkappa^+ \to {}^3(\varkappa)_2^1$ *true? Is* $\varkappa^{++} \to {}^4(\varkappa)_2^1$ *true?*

Here, Lemma 4.2 shows that a positive answer to the second part of Problem 4.11 is implied by a positive answer to the first.

The question of extending the relation of Theorem 4.7 to partitions involving more than two classes is also unsolved in general. A special case will appear as Corollary 5.14 of the next sext section.

§ 5. A result for cardinals cofinal with $\aleph_0$

The following relation (1) is proved in [2], Theorem 42(*), for the case $\nu = 2$ and $\mathrm{Cf}(\varkappa) = \aleph_0$:

(1) $$\begin{pmatrix} \varkappa \\ \varkappa^{+} \end{pmatrix} \to \begin{pmatrix} \varkappa \\ \varkappa \end{pmatrix}_\nu^{1,1}.$$

In the present section, this result will be extended to cover the situation $\nu < \aleph_0$ and $\mathrm{Cf}(\varkappa) = \aleph_0$. To the best of my knowledge, when $\mathrm{Cf}(\varkappa) > \aleph_0$ the truth of relation (1) is still open. The following sequence of lemmas is required. The result is finally proved in Theorem 5.13.

5.1. LEMMA. *Let* $p < \aleph_0$, *and suppose*

(1) $$\begin{pmatrix} \varkappa \\ \varkappa^{+} \end{pmatrix} \to \begin{pmatrix} \eta_l \\ \zeta_l \end{pmatrix}_{l<\nu}^{1,1}.$$

Then the following relation holds:

$$\begin{pmatrix} \varkappa \\ \varkappa^{+} \end{pmatrix} \to \begin{pmatrix} \varkappa & \begin{pmatrix} \eta_l \\ \xi_l \end{pmatrix}_{l<\nu} \\ p & \end{pmatrix}^{1,1}.$$

Proof. Use induction on p. If $p = 0$ the lemma is trivial. Assume the result true for some $q < \aleph_0$. Take a disjoint partition

$$\varkappa \times \varkappa^+ = \Delta \cup \bigcup \{\Delta_l;\ l < \nu\},$$

and suppose that for $l < \nu$ there are no sets $H_0 \subseteq \varkappa, H_1 \subseteq \varkappa^+$ having order types η_l and ζ_l respectively, for which $H_0 \times H_1 \subseteq \Delta_l$. By the inductive hypothesis, there must then be sets $S \in [\varkappa]^\varkappa$ and $T \in [\varkappa^+]^q$ such that $S \times T \subseteq \Delta$.

For $\beta < \varkappa^+$, put $Q(\beta) = \{\alpha \in \varkappa;\ \langle \alpha, \beta \rangle \in \Delta\}$. If there is $\beta \in \varkappa^+ - T$ such that $|S \cap Q(\beta)| = \varkappa$, then $(S \cap Q(\beta)) \times (T \cup \{\beta\}) \subseteq \Delta$, and the result follows. So suppose that $|S \cap Q(\beta)| < \varkappa$ for all $\beta \in \varkappa^+ - T$. We shall show that a contradiction results.

For $\varkappa' < \varkappa$, put $T(\varkappa') = \{\beta \in \varkappa^+ - T;\ |S \cap Q(\beta)| = \varkappa'\}$. There must be some $\lambda < \varkappa$ with $|T(\lambda)| = \varkappa^+$. Write S as a disjoint union, $S = \bigcup \{S_\mu;\ \mu < \lambda^+\}$, where each $|S_\mu| = \varkappa$. If $\beta \in T(\lambda)$, then $S \cap Q|(\beta)| = \lambda < \lambda^+$, and hence there is $\mu(\beta) < \lambda^+$ for which $S_{\mu(\beta)} \cap Q(\beta) = \emptyset$. Since $\varkappa^+ \to {}^1(\varkappa^+)^1_\varkappa$, there are $Y \in [T(\lambda)]^{\varkappa^+}$ and $\mu < \lambda^+$ such that $\beta \in Y \Rightarrow \Rightarrow \mu(\beta) = \mu$. Then $S_\mu \cap Q(\beta) = \emptyset$ for all $\beta \in Y$. Hence $S_\mu \times Y \subseteq \bigcup \{\Delta_l;\ l < \nu\}$. Now $|S_\mu| = \varkappa$ and $|Y| = \varkappa^+$. Thus by relation (1), for some $l < \nu$ there are $H_0 \subseteq S_\mu$ and $H_1 \subseteq Y$, having order types η_l and ζ_l respectively, for which $H_0 \times H_1 \subseteq \Delta_l$. This contradicts the assumed property of the partition.

Thus the induction step is completed, and the lemma proved.

5.2. LEMMA (*). *Let $\varkappa$ be a singular cardinal with the property*

$$\begin{pmatrix} \varkappa \\ \varkappa^+ \end{pmatrix} \to \begin{pmatrix} \eta_l \\ \zeta_l \end{pmatrix}^{1,1}_{l<\nu}. \tag{1}$$

For some $\iota \geq \varkappa$ let there be given a partition

$$\iota \times \varkappa^+ = \Delta \cup \bigcup \{\Delta_l;\ l < \nu\}$$

such that for no $l < \nu$ are there sets $H_0 \subseteq \iota$ and $H_1 \subseteq \varkappa^+$, having order types η and ζ_l respectively, for which $H_0 \times H_1 \subseteq \Delta_l$. Let λ be any regular cardinal with $\mathrm{Cf}(\varkappa) < \lambda < \varkappa$. Let $A \in [\iota]^\varkappa$ and $B \subseteq \varkappa^+$. Then there is a set $A^* \subseteq [A]^\lambda$ with $|A^*| \leq \varkappa$ and there is a map f from A^* to the subsets of B such that $a \in A^* \Rightarrow a \times f(a) \subseteq \Delta$, and $|B - \bigcup \{f(a);\ a \in A^*\}| \leq \varkappa$.

Proof. Take $A \in [\iota]^\varkappa$ and $B \subseteq \varkappa^+$. For $\beta < \varkappa^+$ put $Q(\beta) = \{\alpha < \iota;\ \langle \alpha, \beta \rangle \in \Delta\}$. Let $B_1 = \{\beta \in B;\ |A \cap Q(\beta)| < \lambda\}$ and $B_2 = B - B_1$. Lemma 5.1 applied to relation (1) yields in particular

$$\begin{pmatrix} \varkappa \\ \varkappa^+ \end{pmatrix} \to \begin{pmatrix} \varkappa & \begin{pmatrix} \eta_l \\ \zeta_l \end{pmatrix}_{l<\nu} \\ 1 & \end{pmatrix}^{1,1}.$$

Now $A\times B_1 \subseteq \Delta \cup \bigcup\{\Delta_l;\ l<\nu\}$. Thus if $|B_1| = \varkappa^+$, either there is $\beta\in B_1$ such that $|A\cap Q(\beta)| = \varkappa$ (contrary to the definition of B_1), or there are $l<\nu$ and $K_0\subseteq A$, $K_1\subseteq B_1$ such that the order type of K_0 is η_l, the order type of K_1 is ζ_l and $K_0\times K_1\subseteq \Delta_l$ (contrary to the choice of the partition). Hence it must be the case that $|B_1|\leqslant \varkappa$.

Choose cardinals $\varkappa_\sigma<\varkappa$ for $\sigma<\mathrm{Cf}(\varkappa)$ such that $\varkappa = \sum\{\varkappa_\sigma;\ \sigma<\mathrm{Cf}(\varkappa)\}$. There is a disjoint partition $A = \bigcup\{A_\sigma;\ \sigma<\mathrm{Cf}(\varkappa)\}$, where $|A_\sigma| = \varkappa_\sigma$. Put $A^* = \bigcup\{[A_\sigma]^\lambda;\ \sigma<\mathrm{Cf}(\varkappa)\}$, so $|A^*|\leqslant \sum\{\varkappa_\sigma^\lambda;\ \sigma<\mathrm{Cf}(\varkappa)\}\leqslant\varkappa$. For $a\in A^*$, define $f(a) = \{\beta\in B_2;\ a\subseteq A\cap Q(\beta)\}$, so $a\times f(a)\subseteq \Delta$ and f maps A^* into $\mathscr{P}B$. Further, for each $\beta\in B_2$ there is $a\in A^*$ for which $\beta\in f(a)$, since otherwise $|A_\sigma\cap Q(\beta)|<\lambda$ for all $\sigma<\mathrm{Cf}(\varkappa)$, and so by the regularity of λ and the definition of B_2,

$$\lambda\leqslant|A\cap Q(\beta)| = \sum\{|A_\sigma\cap Q(\beta)|;\ \sigma<\mathrm{Cf}(\varkappa)\}<\lambda.$$

Hence $B_2 = \bigcup\{f(a);\ a\in A^*\}$. This leads to the required result.

5.3. Lemma. *Let $\varkappa\geqslant\aleph_0$ and $\mathrm{Cf}(\lambda)>\varkappa$. Let $p\in\aleph_0$ and suppose*

$$(1)\qquad \begin{pmatrix}\varkappa\\ \lambda\end{pmatrix}\to\begin{pmatrix}\eta_l\\ \zeta_l\end{pmatrix}^{1,1}_{l<\nu}.$$

Then it follows that

$$(2)\qquad \begin{pmatrix}\varkappa\\ \lambda\end{pmatrix}\to\begin{pmatrix}p & \begin{pmatrix}\eta_l\\ \zeta_l\end{pmatrix}_{l<\nu}\\ \lambda & \end{pmatrix}^{1,1}.$$

Proof. By induction on p. The case $p=0$ is trivial. Assume (2) holds with $p=q$. Take a disjoint partition

$$\varkappa\times\lambda = \Delta\cup\bigcup\{\Delta_l;\ l<\nu\},$$

and suppose that for $l<\nu$ there are no sets $H_0\subseteq\varkappa$, $H_1\subseteq\lambda$ having order types η_l, ζ_l respectively, for which $H_0\times H_1\subseteq\Delta_l$. By the inductive hypothesis, there must then be sets $S\in[\varkappa]^q$ and $T\in[\lambda]^\lambda$ such that $S\times T\subseteq\Delta$.

Put $P(a) = \{\beta\in\lambda;\ \langle a,\beta\rangle\in\Delta\}$. Then

$$(\varkappa-S)\times\big(T-\bigcup\{P(a);\ a\in\varkappa-S\}\big)\subseteq\bigcup\{\Delta_l;\ l<\nu\}.$$

Now $|\varkappa-S| = \varkappa$, and so by relation (1) and the choice of the partition, $|T-\bigcup\{P(a);\ a\in\varkappa-S\}|<\lambda$. Hence $|T\cap\bigcup\{P(a);\ a\in\varkappa-S\}| = \lambda$. Since $\mathrm{Cf}(\lambda)>\varkappa$, there must be $a\in\varkappa-S$ for which $|T\cap P(a)| = \lambda$. But

$$(S\cup\{a\})\times\big(T\cap P(a)\big)\subseteq\Delta,$$

and so since $|S\cup\{a\}| = q+1$, the induction step is complete. This establisches Lemma 5.3.

The following two lemmas are quoted without proof. They follow from [2], Lemma 8(*) and Lemma 3A(*), respectively.

5.4. LEMMA (*). *Let $\varkappa$ and λ be infinite cardinals with* $\mathrm{Cf}(\varkappa) \neq \mathrm{Cf}(\lambda)$. *Let* $|A| = \varkappa$, $|B| = \varkappa^+$ *and for each* $b \in B$ *let there be given a set* $A_b \in [A]^{\geqslant \lambda}$. *Then there is* $B' \in [B]^{\varkappa^+}$ *such that* $|\bigcap \{A_b;\ b \in B'\}| \geqslant \lambda$.

5.5. LEMMA (*). *Let $\varkappa$ be singular, and let the cardinals $\varkappa_\sigma$ for* $\sigma < \mathrm{Cf}(\varkappa)$ *be such that* $\sigma < \tau < \mathrm{Cf}(\varkappa) \Rightarrow \varkappa_\sigma < \varkappa_\tau < \varkappa$, *and* $\varkappa = \sum \{\varkappa_\sigma;\ \sigma < \mathrm{Cf}(\varkappa)\}$. *Let* $\nu < \varkappa$ *and suppose a disjoint partition is given,* $\varkappa \times \varkappa = \bigcup \{\Delta_l;\ l < \nu\}$. *Then there are sets* A_σ, B_σ *and ordinals* $h(\sigma, \tau) < \nu$ *such that*

$$A_\sigma \in [\varkappa]^{\varkappa_\sigma}, \quad B_\sigma \in [\varkappa]^{\varkappa_\sigma}, \quad \tau \neq \sigma \Rightarrow A_\varrho \cap A_\tau = B_\sigma \cap B_\tau = \emptyset,$$
$$\sigma, \tau < \mathrm{Cf}(\varkappa) \Rightarrow A_\sigma \times B_\tau \subseteq \Delta_{h(\sigma,\tau)}.$$

5.6. LEMMA (*). *Let $\varkappa$ be infinite, and suppose*

(1) $$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \to \begin{pmatrix}\eta_l \\ \zeta_l\end{pmatrix}^{1,1}_{l<\nu}.$$

Then for any $\lambda < \varkappa$,

(2) $$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \to \begin{pmatrix}\lambda & \begin{pmatrix}\eta_l \\ \zeta_l\end{pmatrix}_{l<\nu} \\ \varkappa^+ & \end{pmatrix}^{1,1}.$$

Proof. The case $\varkappa = \aleph_0$ is immediate from Lemma 5.3, so suppose $\varkappa > \aleph_0$. Define ι as follows: if $\mathrm{Cf}(\varkappa) = \varkappa$, then $\iota = \lambda$; otherwise $\iota = \lambda^+ +_C (\mathrm{Cf}(\varkappa))^+$. In either case, $\lambda \leqslant \iota < \varkappa$ and $\mathrm{Cf}(\iota) \neq \mathrm{Cf}(\varkappa)$.

Take any partition, $\varkappa \times \varkappa^+ = \Delta \cup \bigcup \{\Delta_l;\ l < \nu\}$. To establish (2), we must find sets $H_0 \in [\varkappa]^\lambda$, $H_1 \in [\varkappa^+]^{\varkappa^+}$ such that $H_0 \times H_1 \subseteq \Delta$, or else $l < \nu$ and sets $H_0 \subseteq \varkappa$, $H_1 \subseteq \varkappa^+$, having order types η_l, ζ_l respectively, such that $H_0 \times H_1 \subseteq \Delta_l$.

Put $T = \{\beta \in \varkappa^+;\ |Q(\beta)| \geqslant \iota\}$, where $Q(\beta) = \{\alpha \in \varkappa;\ \langle \alpha, \beta\rangle \in \Delta\}$. Suppose $|T| = \varkappa^+$. Then by Lemma 5.4, there is $T' \in [T]^{\varkappa^+}$ such that $|\bigcap \{Q(\beta);\ \beta \in T'\}| \geqslant \iota$. Since $\bigcap \{Q(\beta);\ \beta \in T'\} \times T' \subseteq \Delta$, there is nothing more to be shown. So consider the case when $|T| < \varkappa^+$. Then $|\varkappa^+ - T| = \varkappa^+$. Since $\iota^+ \leqslant \varkappa$, there is a disjoint partition $\varkappa = \bigcup \{S_\mu;\ \mu < \iota^+\}$, where $|S_\mu| = \varkappa$ for each $\mu < \iota^+$. However, $|Q(\beta)| < \iota$ for $\beta \in \varkappa^+ - T$. Hence for each $\beta \in \varkappa^+ - T$, there is $\mu(\beta) < \iota^+$ for which $S_{\mu(\beta)} \cap Q(\beta) = \emptyset$. Moreover, there must be $\mu < \iota^+$ and $T_1 \in [\varkappa^+ - T]^{\varkappa^+}$ for which $\beta \in T_1 \Rightarrow \mu(\beta) = \mu$. Now $S_\mu \cap Q(\beta) = \emptyset$ for $\beta \in T_1$ and so $S_\mu \times T_1 \subseteq \bigcup \{\Delta_l;\ l < \nu\}$. Since $|S_\mu| = \varkappa$ and $|T_1| = \varkappa^+$, the result follows from relation (1). This concludes the proof.

5.7. THEOREM (*). *If* $\mathrm{Cf}(\varkappa) = \aleph_0$, *then for any* $p < \aleph_0$

$$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \to \begin{pmatrix}\varkappa & \begin{pmatrix}\varkappa \\ \aleph_0\end{pmatrix}_p \\ \varkappa^+ & \end{pmatrix}^{1,1}.$$

Proof. By induction on p. The case $p = 0$ is trivial. Assume the result is true for some $q < \aleph_0$. Take a partition

$$\varkappa \times \varkappa^+ = \Delta \cup \bigcup \{\Delta_l;\ l \leqslant q\},$$

and suppose that there are no sets $H_0 \in [\varkappa]^\varkappa$, $H_1 \in [\varkappa^+]^{\varkappa^+}$ with $H_0 \times H_1 \subseteq \Delta$, nor sets $H_0 \in [\varkappa]^\varkappa$, $H_1 \in [\varkappa^+]^{\aleph_0}$ with $H_0 \times H_1 \subseteq \Delta_l$ for any $l \in \{1, \ldots, q\}$. We must find sets $H_0 \in [\varkappa]^\varkappa$, $H_1 \in [\varkappa^+]^{\aleph_0}$ with $H_0 \times H_1 \subseteq \Delta_0$.

Write $\varkappa = \sum \{\varkappa_r;\ r < \aleph_0\}$, where $\varkappa_r < \varkappa$ for $r < \aleph_0$. Define inductively sets D_r, $B_r \in [\varkappa^+]^{\varkappa^+}$, $A_r \in [\varkappa]^\varkappa$ and $y_r \in B_r$ for $r < \aleph_0$ as follows:

By Lemma 5.6 and the inductive hypothesis, choose

$$D_0 \in [\varkappa]^{\varkappa_0} \quad \text{and} \quad B_0 \in [\varkappa^+]^{\varkappa^+} \quad \text{such that} \quad D_0 \times B_0 \subseteq \Delta_0.$$

By Lemma 5.1 and the inductive hypothesis, choose

$$A_0 \in [\varkappa]^\varkappa \quad \text{and} \quad y_0 \in B_0 \quad \text{such that} \quad A_0 \times \{y_0\} \subseteq \Delta_0.$$

Generally, by Lemma 5.6 and the inductive hypothesis, choose

$$D_r \in [A_{r-1}]^{\varkappa_r} \quad \text{and} \quad B_r \in [B_{r-1} - \{y_{r-1}\}]^{\varkappa^+} \quad \text{such that} \quad D_r \times B_r \subseteq \Delta_0.$$

By Lemma 5.1 and the inductive hypothesis, choose

$$A_r \in [A_{r-1}]^\varkappa \quad \text{and} \quad y_r \in B_r \quad \text{such that} \quad A_r \times \{y_r\} \subseteq \Delta_0.$$

Put $H_0 = \bigcup \{D_r;\ r < \aleph_0\}$ and $H_1 = \{y_r;\ r < \aleph_0\}$. Then $|H_0| = \varkappa$ and $|H_1| = \aleph_0$. Note that if $r \leqslant s$, then $D_r \times \{y_s\} \subseteq D_r \times B_s \subseteq D_r \times B_r \subseteq \Delta_0$. And if $r > s$, then $D_r \times \{y_s\} \subseteq A_s \times \{y_s\} \subseteq \Delta_0$. Hence $H_0 \times H_1 \subseteq \Delta_0$, and the induction step is complete. This proves Theorem 5.7.

5.8. Corollary (*). *If* $\mathrm{Cf}(\varkappa) = \aleph_0$, *then for any* $p, q < \aleph_0$ *and any* $\lambda < \varkappa$,

$$\binom{\varkappa}{\varkappa^+} \to \left(\binom{\varkappa}{\varkappa^+} \binom{\lambda}{\varkappa^+}_p \binom{\varkappa}{\aleph_0}_q\right)^{1,1}.$$

Perusal of the proofs of the preceding results shows that (*) is not required for the special case $\varkappa = \aleph_0$. Thus we obtain

5.9. Corollary. *For any* $p < \aleph_0$,

$$\binom{\aleph_0}{\aleph_1} \to \left(\binom{\aleph_0}{\aleph_1} \binom{\aleph_0}{\aleph_0}_p\right)^{1,1}.$$

5.10. Lemma (*). *Let* $\varkappa \geqslant \aleph_0$ *and* $p < \aleph_0$. *Take any partition*

$$\varkappa \times \varkappa = \Delta \cup \bigcup \{\Delta_l;\ l < p\}.$$

If there are no sets $H_0, H_1 \in [\varkappa]^\varkappa$ *with* $H_0 \times H_1 \subseteq \Delta$, *then for some* $l < p$ *there are* $H \in [\varkappa]^\varkappa$ *and* $a \in \varkappa$ *such that either* $H \times \{a\} \subseteq \Delta_l$ *or else* $\{a\} \times H \subseteq \Delta_l$.

Proof. By an obvious induction. The case $p = 1$ comes as a special case of Theorem 38(*) of [2].

5.11. THEOREM (*). *Suppose $\varkappa$ is singular; let $\lambda < \varkappa$ and $p < \aleph_0$. Then*

(1) $$\binom{\varkappa}{\varkappa^+} \to \binom{\varkappa}{\lambda}_p^{1,1}.$$

Proof. By induction on p. The case $p = 1$ is trivial, so suppose the result is true for some q with $1 \leqslant q < \aleph_0$.

We may suppose that $\mathrm{Cf}(\varkappa) < \lambda < \varkappa$, and that λ is regular. Put $\varrho = \mathrm{Cf}(\varkappa)$. Choose cardinals $\varkappa_\sigma$ for $\sigma < \varrho$ such that $\varkappa = \sum\{\varkappa_\sigma;\ \sigma < \varrho\}$ and $\lambda < \varkappa_\sigma < \varkappa_\tau < \varkappa$ when $\sigma < \tau < \varrho$. Take any disjoint partition

(2) $$\varkappa \times \varkappa^+ = \bigcup\{\Delta_l;\ l \leqslant q\},$$

and suppose that whenever $1 \leqslant l \leqslant q$ then there are no sets $H_0 \in [\varkappa]^\varkappa$ and $H_1 \in [\varkappa^+]^\lambda$ for which $H_0 \times H_1 \subseteq \Delta_l$. We must find $H_0 \in [\varkappa]^\varkappa$ and $H_1 \in [\varkappa^+]^\lambda$ such that $H_0 \times H_1 \subseteq \Delta_0$.

Suppose that ι is regular with $\varrho < \iota < \varkappa$. Take any $A \in [\varkappa]^\varkappa$ and $B \in [\varkappa^+]^{\varkappa^+}$. Then by Lemma 5.2, there are $A^* \subseteq [A]^\iota$ with $|A^*| \leqslant \varkappa$ and a map f from A^* to the subsets of B such that

(3) $$a \in A^* \Rightarrow a \times f(a) \subseteq \Delta_0, \qquad |B - \bigcup\{f(a);\ a \in A^*\}| \leqslant \varkappa.$$

Define inductively a ramification system $\mathscr{R}$ on $\varkappa^+$ of length ϱ as follows. Take $\sigma < \varrho$ and $X \in \mathrm{SEQ}_\sigma$. Suppose that $S'(X)$ has already been defined. If $|S'(X)| \leqslant \varkappa$, put $F(X) = S'(X)$ and $n(X) = 0$. If $|S'(X)| = \varkappa^+$, put $n(X) = \varkappa$ and choose $R(X) \in [S'(X)]^\varkappa$. Then by applying (3) with $A = \varkappa$, $B = S'(X) - R(X)$ and $\iota = \varkappa_\sigma$, one can find a set $A^*(X) \subseteq [\varkappa]^{\varkappa_\sigma}$ with $|A^*(X)| \leqslant \varkappa$, and a map f_X from $A^*(X)$ to $\mathscr{P}B$, which together satisfy the appropriate form of (3). Write $A^*(X) = \{a(Y);\ \overline{Y} = X$ and $Y(\sigma) < \varkappa\}$. For $Y \in \mathrm{SEQ}_{\sigma+1}$ such that $\overline{Y} = X$ and $Y(\sigma) < \varkappa$, put $S(Y) = f_X(a(Y))$. Define

$$F(X) = R(X) \cup \big(S'(X) - \bigcup\{S(Y);\ \overline{Y} = X \text{ and } Y(\sigma) < \varkappa\}\big).$$

This defines $\mathscr{R}$. Further

$$\begin{aligned} |F| &= |R| +_C |(S' - R) - \bigcup\{S(Y);\ \overline{Y} = X \text{ and } Y(\sigma) < \varkappa\}| \\ &= |R| +_C |B - \bigcup\{f(a);\ a \in A^*\}|, \end{aligned}$$

and so $|F| = \varkappa$, by (3). Hence Lemma 3.1 (iii) applies to $\mathscr{R}$, so choose a sequence $X \in N \cap \mathrm{SEQ}_\varrho$ such that $S'(X) \neq \emptyset$. Then for each $\sigma < \varrho$, it must be that $|S'(X \restriction \sigma)| = \varkappa^+$, and so always $R(X \restriction \sigma)$ is defined. Choose a bijection $g_{X \restriction \sigma}$ from $[\varkappa]^{\varkappa_\sigma}$ onto $[R(X \restriction \sigma)]^{\varkappa_\sigma}$. Put

$$g(\sigma) = g_{X \restriction \sigma}\big(a(X \restriction \sigma + 1)\big);$$

then always $g(\sigma)\in[R(X\restriction\sigma)]^{\varkappa_\sigma}$. Put $A = \bigcup\{a(X\restriction\sigma+1);\ \sigma<\varrho\}$ and $B=\bigcup\{g(\sigma);\ \sigma<\varrho\}$. Then $A\in[\varkappa]^{\varkappa}$ and $B\in[\varkappa^+]^{\varkappa}$. Moreover, if $\sigma<\tau<\varrho$, then

$$g(\tau)\subseteq R(X\restriction\tau)\subseteq S'(X\restriction\tau)\subseteq S(X\restriction\sigma+1).$$

However, $a(X\restriction\sigma+1)\times S(X\restriction\sigma+1)\subseteq\Delta_0$ by (3), and so

(4) $$\sigma<\tau<\varrho\Rightarrow a(X\restriction\sigma+1)\times g(\tau)\subseteq\Delta_0.$$

The partition (2) restricts to a partition of $A\times B$. Hence by Lemma 5.5, there are sets $A_\sigma\in[A]^{\varkappa_\sigma}$, $B_\sigma\in[B]^{\varkappa_\sigma}$ and numbers $h(\sigma,\tau)\leqslant q$ such that if $\sigma<\tau<\varrho$, then $A_\sigma\cap A_\tau=B_\sigma\cap B_\tau=\emptyset$, and also

(5) $$\sigma,\tau<\varrho\Rightarrow A_\sigma\times B_\tau\subseteq\Delta_{h(\sigma,\tau)}.$$

For $l\leqslant q$, put $\Delta'_l=\{\langle\sigma,\tau\rangle\in\varrho\times\varrho;\ h(\sigma,\tau)=l\}$. Thus $\varrho\times\varrho=\bigcup\{\Delta'_l;\ l\leqslant q\}$. Apply Lemma 5.10 to this partition of $\varrho\times\varrho$.

There are three cases to consider.

Case 1. There are $H\in[\varrho]^{\varrho}$, $\tau<\varrho$ and l with $1\leqslant l\leqslant q$ such that $h(\sigma,\tau)=l$ for all $\sigma\in H$. Put $A'=\bigcup\{A_\sigma;\ \sigma\in H\}$; then $|A'|=\varkappa$, and $A'\times B_\tau\subseteq\Delta_l$ by (5). Since $\lambda<\varkappa_\tau$, this contradicts the choice of the partition (2).

Case 2. There are $H\in[\varrho]^{\varrho}$, $\sigma<\varrho$ and l with $1\leqslant l\leqslant q$ such that $h(\sigma,\tau)=l$ for all $\tau\in H$. Put $B'=\bigcup\{B_\tau;\ \tau\in H\}$; then $|B'|=\varkappa$, and $A_\sigma\times B'\subseteq\Delta_l$ by (5). Choose $a\in A_\sigma$. Then $\{a\}\times B'\subseteq\Delta_l$. Further, there is $\sigma_1<\varrho$ such that $a\in a(X\restriction\sigma_1+1)$. Hence it follows from (4) that $B'\subseteq\bigcup\{g(\tau);\ \tau\leqslant\sigma_1\}$. However, this yields the contradiction

$$\varkappa=|B'|\leqslant\sum\{|g(\tau)|;\ \tau\leqslant\sigma_1\}=\sum\{\varkappa_\tau;\ \tau\leqslant\sigma_1\}<\varkappa.$$

Case 3. This case must prevail. There are $K_0,K_1\in[\varrho]^{\varrho}$ such that $h(\sigma,\tau)=0$ for $\sigma\in K_0$ and $\tau\in K_1$. Put $H_0=\bigcup\{A_\sigma;\ \sigma\in K_0\}$ and $H_1=\bigcup\{B_\tau;\ \tau\in K_1\}$. Then $H_0\in[\varkappa]^{\varkappa}$, $H_1\in[\varkappa^+]^{\varkappa}$, and $H_0\times H_1\subseteq\Delta_0$ by (5). Since $\lambda<\varkappa$, the induction step is complete. This proves Theorem 5.11.

5.12. Lemma (*). *Let $\varkappa$ be uncountable with* $\mathrm{Cf}(\varkappa)=\aleph_0$. *Take $p,q<\aleph_0$ with $q\geqslant 1$. Suppose for all $\iota<\varkappa$ that*

(1) $$\binom{\varkappa}{\varkappa^+}\to\left(\binom{\varkappa}{\varkappa}_p\binom{\varkappa}{\iota}_q\right)^{1,1}.$$

Then for all $\lambda<\varkappa$,

(2) $$\binom{\varkappa}{\varkappa^+}\to\left(\binom{\varkappa}{\varkappa}_{p+1}\binom{\varkappa}{\lambda}_{q-1}\right)^{1,1}.$$

Proof. Let $\lambda < \varkappa$ be given. Choose cardinals $\varkappa_r < \varkappa$ for $r < \aleph_0$ such that $\varkappa = \sum\{\varkappa_r;\ r < \aleph_0\}$ and $\lambda < \varkappa_0 < \varkappa_1 \ldots$ Take any partition

$$\varkappa \times \varkappa^+ = \bigcup\{\Delta_l;\ l < p+1\} \cup \bigcup\{\Gamma_k;\ k < q-1\}, \tag{3}$$

and suppose that for all $l < p$ there are no sets $H_0 \epsilon [\varkappa]^\varkappa$, $H_1 \epsilon [\varkappa^+]^\varkappa$ with $H_0 \times H_1 \subseteq \Delta_l$, and also that for all $k < q-1$ there are no sets $H_0 \epsilon [\varkappa]^\varkappa$, $H_1 \epsilon [\varkappa^+]^\lambda$ with $H_0 \times H_1 \subseteq \Gamma_k$. To establish (2) we must find sets $H_0 \epsilon [\varkappa]^\varkappa$, $H_1 \epsilon [\varkappa^+]^\varkappa$ such that $H_0 \times H_1 \subseteq \Delta_p$. From (1) it follows that

$$\binom{\varkappa}{\varkappa^+} \to \left(\binom{\varkappa}{\varkappa}_p \binom{\varkappa}{\lambda}_{q-1}\right)^{1,1},$$

and so by Lemma 5.6, we obtain, for any $\iota < \varkappa$,

$$\binom{\varkappa}{\varkappa^+} \to \left(\binom{\varkappa}{\varkappa}_p \begin{matrix}\iota\\ \varkappa^+\end{matrix} \binom{\varkappa}{\lambda}_{q-1}\right)^{1,1}. \tag{4}$$

Take any $r < \aleph_0$ and any sets $A \epsilon [\varkappa]^\varkappa$ and $B \epsilon [\varkappa^+]^{\varkappa^+}$. By (4) and the choice of the partition (3), there are A', B' such that

$$A' \epsilon [A]^{\varkappa_r}, \quad B' \epsilon [B]^{\varkappa^+}, \quad A' \times B' \subseteq \Delta_p. \tag{5}$$

Since $\varkappa_r < \varkappa$, by (1) and the choice of the partition (3), there are A'', B'' such that

$$A'' \epsilon [A]^{\varkappa}, \quad B'' \epsilon [B]^{\varkappa_r}, \quad A'' \times B'' \subseteq \Delta_p. \tag{6}$$

Use induction over $r < \aleph_0$ to define the sets $A_r^* \epsilon [\varkappa]^\varkappa$, $B_r^* \epsilon [\varkappa^+]^{\varkappa^+}$, A_r and B_r as follows. Put $A_0^* = \varkappa$ and $B_0^* = \varkappa^+$. Suppose $A_r^* \epsilon [\varkappa]^\varkappa$ and $B_r^* \epsilon [\varkappa^+]^{\varkappa^+}$ have already been defined for some $r < \aleph_0$. By (5), choose A_r and B_{r+1}^* so that

$$A_r \epsilon [A_r^*]^{\varkappa_r}, \quad B_{r+1}^* \epsilon [B_r^*]^{\varkappa^+}, \quad A_r \times B_{r+1}^* \subseteq \Delta_p.$$

By (6), choose A_{r+1}^* and B_r so that

$$A_{r+1}^* \epsilon [A_r^*]^{\varkappa}, \quad B_r \epsilon [B_{r+1}^*]^{\varkappa_r}, \quad A_{r+1}^* \times B_r \subseteq \Delta_p.$$

Then for all $r < \aleph_0$, the following hold:

$$A_{r+1}^* \subseteq A_r^*, \quad B_{r+1}^* \subseteq B_r^*, \quad |A_r^*| = \varkappa, \quad |B_r^*| = \varkappa^+,$$
$$|A_r| = \varkappa_r, \quad |B_r| = \varkappa_r.$$

Put $A = \bigcup\{A_r;\ r < \aleph_0\}$ and $B = \bigcup\{B_r;\ r < \aleph_0\}$. Then $A \epsilon [\varkappa]^\varkappa$ and $B \epsilon [\varkappa^+]^\varkappa$. Thus to prove the lemma, it suffices to show

$$A \times B \subseteq \Delta_p. \tag{7}$$

Take $r, s < \aleph_0$. We must show that $A_r \times B_s \subseteq \Delta_p$. However, if $r \leqslant s$, then $B_s \subseteq B_{s+1}^* \subseteq B_{r+1}^*$, and so $A_r \times B_s \subseteq A_r \times B_{r+1}^* \subseteq \Delta_p$. If $r > s$, then A_r

$\subseteq A_r^* \subseteq A_{s+1}^*$, and so $A_r \times B_s \subseteq A_{s+1}^* \times B_s \subseteq \Delta_p$. This establishes (7), and completes the proof.

5.13. THEOREM (*). *Suppose* $\mathrm{Cf}(\varkappa) = \aleph_0$. *Then for any* $p < \aleph_0$

(1) $$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \to \begin{pmatrix}\varkappa \\ \varkappa\end{pmatrix}_p^{1,1}.$$

Proof. The case $\varkappa = \aleph_0$ is a consequence of Corollary 5.9, so suppose $\varkappa > \aleph_0$. Then by Theorem 5.11, for all $\lambda < \varkappa$,

$$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \to \begin{pmatrix}\varkappa \\ \lambda\end{pmatrix}_p^{1,1}.$$

Repeated applications of Lemma 5.12 now yield the result.

The result of Theorem 5.13 is clearly the best possible, for if $\mathrm{Cf}(\varkappa) = \aleph_0$, then it is easy to see that

$$\begin{pmatrix}\varkappa \\ \varkappa^+\end{pmatrix} \nrightarrow \begin{pmatrix}\varkappa \\ 1\end{pmatrix}_{\aleph_0}^{1,1}.$$

There is the following corollary to Theorem 5.13. (Compare with Corollary 3.4.)

5.14. COROLLARY (*). *Suppose* $\mathrm{Cf}(\varkappa) = \aleph_0$. *Then* $\varkappa^+ \to {}^2(\varkappa)_p^1$ *for any* $p < \aleph_0$.

Thus for partitions of $\varkappa^+ \times \varkappa^+$, the most simple open questions are seen to be:

5.15. PROBLEM (*). *Is* $\varkappa^+ \to {}^2(\varkappa)_3^1$ *true if* $\mathrm{Cf}(\varkappa) > \aleph_0$? *Is* $\varkappa^+ \to {}^2(\varkappa)_{\aleph_0}^1$ *true* if $\mathrm{Cf}(\varkappa) = \aleph_0$?

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