

Multi-valued solutions of a functional equation

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Abstract. The existence of a solution Φ_0 of the inequality

$$\Phi(x) \supset H(x, \Phi[f(x)])$$

implies the existence of a minimal solution $\Phi \subset \Phi_0$ of the equation

$$(1) \quad \Phi(x) = H(x, \Phi[f(x)])$$

in some classes of multi-valued mappings. Moreover, if f and H are single-valued, then (under suitable conditions) the minimal continuum-valued solution of (1) is almost everywhere a single-valued mapping.

Let X be a non-empty set and let Y be a topological Hausdorff space. We introduce the following classes of subsets of Y :

$$n(Y) = \{A \subset Y: A \neq \emptyset\},$$

$$c(Y) = \{A \subset Y: A \neq \emptyset \text{ and } A \text{ is compact}\},$$

$$C(Y) = \{A \subset Y: A \neq \emptyset \text{ and } A \text{ is a continuum}\}.$$

Let Z be a topological space. We say that $\Phi: Z \rightarrow n(Y)$ is a closed mapping if, whenever $z_0 \in Z$, $y_0 \in Y$, $y_0 \notin \Phi(z_0)$, there exist two open neighbourhoods $U(z_0)$ and $V(y_0)$ such that $\Phi(z) \cap V(y_0) = \emptyset$ for every $z \in U(z_0)$.

Let $z_0 \in Z$. We say that Φ is *upper semi-continuous* (u.s.c.) at z_0 if for each open set G containing $\Phi(z_0)$ there is an open neighbourhood $U(z_0)$ such that $\Phi(z) \subset G$ for every $z \in U(z_0)$.

A mapping $\Phi: Z \rightarrow n(Y)$ is u.s.c. if it is u.s.c. at every $z_0 \in Z$.

We consider the following equation:

$$(1) \quad \Phi(x) = H(x, \Phi[f(x)]),$$

where the functions $f: X \rightarrow n(X)$, $H: X \times Y \rightarrow n(Y)$ are given and $H(x, \cdot)$ is closed for every $x \in X$, and $\Phi: X \rightarrow n(Y)$ is the unknown function.

LEMMA 1. Let F be a family of functions $\Phi: X \rightarrow c(Y)$ such that:

(I) if $\Phi \in F$, then $H(\cdot, \Phi[f(\cdot)]) \in F$ and $\Phi[f(x)] \in c(Y)$ for $x \in X$;

(II) if $\Phi_n \in F$ and $\Phi_{n+1} \subset \Phi_n$ for $n = 1, 2, \dots$, then $\bigcap_{n=1}^{\infty} \Phi_n \in F$.

If a function

$$(2) \quad \Phi_0 \in F$$

fulfils the inequality

$$(3) \quad \Phi_0(x) \supset H(x, \Phi_0[f(x)]),$$

then the formulas

$$(4) \quad \Phi_{n+1} = H(\cdot, \Phi_n[f(\cdot)])$$

and

$$(5) \quad \bar{\Phi} = \bigcap_{n=1}^{\infty} \Phi_n$$

define a solution $\bar{\Phi} \subset \Phi_0$ of (1) in the family F . If $\Psi: X \rightarrow n(Y)$ is a solution of (1) such that $\Psi \subset \Phi_0$, then $\Psi \subset \bar{\Phi}$.

Proof. It follows by (2), (I) and (4) that $\Phi_n \in F$ for $n = 0, 1, 2, \dots$. According to (2), (3) and (4) we have $\Phi_{n+1}(x) \subset \Phi_n(x)$ for $n = 0, 1, 2, \dots$ and for every $x \in X$. Therefore by (II) and (5) we get $\bar{\Phi} \in F$.

It is obvious that $H(x, \bar{\Phi}[f(x)]) \subset H(x, \Phi_n[f(x)]) \subset \Phi_n(x)$ for $n = 0, 1, 2, \dots$ and for $x \in X$. This implies the inclusion

$$H(x, \bar{\Phi}[f(x)]) \subset \bar{\Phi}(x)$$

for every $x \in X$.

Now, we take $z \in \bar{\Phi}(x)$. Then $z \in \Phi_{n+1}(x) = H(x, \Phi_n[f(x)])$ for $n = 0, 1, 2, \dots$. There exists a sequence $\{y_n\}$ such that

$$y_n \in \Phi_n[f(x)] \subset \Phi_0[f(x)], \quad n = 0, 1, 2, \dots$$

and

$$(6) \quad z \in H(x, y_n).$$

If $m \geq n$, then $y_m \in \Phi_n[f(x)]$. Thus

$$(7) \quad A_n := \text{cl}\{y_n, y_{n+1}, \dots\} \subset \Phi_n[f(x)], \quad n = 0, 1, 2, \dots,$$

for the sets $\Phi_n[f(x)]$ are closed (being compact). We have $A_{n+1} \subset A_n$, $n = 0, 1, 2, \dots$, and so there exists

$$(8) \quad y \in \bigcap_{n=0}^{\infty} A_n.$$

We shall prove that

$$(9) \quad z \in H(x, y).$$

Suppose, conversely, that $z \notin H(x, y)$. Then there exist two neighbourhoods U and V of points y and z , respectively, such that

$$H(x, U) \cap V = \emptyset.$$

It follows by (8) that there exists k , $k = 0, 1, 2, \dots$, such that $y_k \in U$. Since by (6)

$$z \in H(x, y_k) \cap V \subset H(x, U) \cap V,$$

we have a contradiction. To prove the first part of the lemma it is sufficient to notice that according to (9), (8), (7) and (5) we have

$$z \in H(x, y) \subset H(x, \bigcap_{n=0}^{\infty} A_n) \subset H(x, \bigcap_{n=0}^{\infty} \Phi_n[f(x)]) = H(x, \bar{\Phi}[f(x)]).$$

The second part of the lemma is obvious.

The set of all functions $\Phi: X \rightarrow n(Y)$ is partially ordered by the relation:

$$\Phi_1 \subset \Phi_2 \quad \text{iff} \quad \Phi_1(x) \subset \Phi_2(x) \quad \text{for every } x \in X.$$

LEMMA 2. Let F be a family of mappings $\Phi: X \rightarrow c(Y)$ such that conditions (I) of Lemma 1 and

(II') if $\Phi_i \in F$ for every $i \in I$ and the family $\{\Phi_i\}_{i \in I}$ is a chain, then $\bigcap_{i \in I} \Phi_i \in F$ are fulfilled. Let a mapping $\Phi_0 \in F$ fulfil inequality (3). Then there exists a minimal solution $\bar{\Phi} \subset \Phi_0$ of equation (1) in the family F .

Proof. Let

$$F_0 := \{\Phi \in F: \Phi \subset \Phi_0, \Phi \text{ is a solution of (1)}\}.$$

It follows by Lemma 1 that the family F_0 is not empty. Let T be a chain in F_0 , and let $\Phi_T = \bigcap_{\Phi \in T} \Phi$. According to (II'), $\Phi_T \in F$. Moreover, if $\Phi \in T$, then we have

$$\Phi(x) = H(x, \Phi[f(x)]) \supset H(x, \Phi_T[f(x)])$$

and

$$\Phi_T(x) \supset H(x, \Phi_T[f(x)]).$$

Lemma 1 guarantees the existence of a solution $\bar{\Phi}_T \subset \Phi_T$ of (1) in the family F . This shows that $\bar{\Phi}_T$ is a minorant of T in F_0 . The family F_0 possesses a minimal element in virtue of Kuratowski-Zorn lemma.

LEMMA 3. Let X be a topological space and let $\Phi: X \rightarrow n(Y)$ be u.s.c. at $x_0 \in X$. Then the mapping $x \rightarrow (x, \Phi(x))$ is u.s.c. at x_0 .

Proof. Let G be an open set in Y such that $\Phi(x_0) \subset G$ and let V be an open set in X such that $x_0 \in V$. Then there exists an open neighbourhood

$U(x_0)$ of x_0 such that if $x \in U(x_0)$, then $\Phi(x) \subset G$, because Φ is u.s.c. at x_0 . For every $x \in U(x_0) \cap V$ we have $(x, \Phi(x)) \subset V \times G$.

LEMMA 4. *Let X be a topological space. If $\Phi: X \rightarrow n(Y)$ is a closed mapping, then the image $\Phi(K)$ of a compact subset K of X is a closed set.*

Proof. We shall prove that the set $Y \setminus \Phi(K)$ is open. Let $y \notin \Phi(K)$. Then there exist neighbourhoods $V^x(y)$ and $U(x)$ such that

$$\Phi(U(x)) \cap V^x(y) = \emptyset$$

for every $x \in K$. Since the set K is compact, there exist elements x_1, x_2, \dots, x_n in K such that $K \subset U(x_1) \cup \dots \cup U(x_n)$. Putting $V(y) = V^{x_1}(y) \cap \dots \cap V^{x_n}(y)$ we have

$$\Phi(K) \cap V(y) = \emptyset.$$

Therefore $Y \setminus \Phi(K)$ is open.

DEFINITION. We say that equation (1) has *property (P)* in the family F iff the following statement is true:

If $\Phi_0 \in F$ is a solution of (3), then formulas (4) and (5) define a solution $\bar{\Phi} \subset \Phi_0$ of (1) in F and there exists a minimal solution $\underline{\Phi} \subset \Phi_0$ of (1) in F .

THEOREM 1.1. *If $f: X \rightarrow X$, then equation (1) has property (P) in the family of all functions $\Phi: X \rightarrow c(Y)$.*

2. *If $f: X \rightarrow X$ and H with connected values is u.s.c. with respect to the second variable, then equation (1) has property (P) in the family of all functions $\Phi: X \rightarrow C(Y)$.*

3. *If X is a topological Hausdorff space, $f: X \rightarrow c(X)$ is u.s.c. and H is u.s.c., then equation (1) has property (P) in the family of all u.s.c. functions $\Phi: X \rightarrow c(Y)$.*

4. *If X is a topological Hausdorff space, $f: X \rightarrow C(Y)$ is u.s.c. and H with connected values is u.s.c., then equation (1) has property (P) in the family of all u.s.c. functions $\Phi: X \rightarrow C(Y)$.*

Proof. In order to prove this theorem we must test that suitable families fulfil conditions (I) and (II') of Lemmas 1 and 2.

1. Let $F_1 = \{\Phi: \Phi \subset \Phi_0, \Phi: X \rightarrow c(Y)\}$, where $\Phi_0: X \rightarrow c(Y)$ is a given solution of (3). For every $\Phi \in F_1$ and $x \in X$ the sets $\Phi[f(x)]$ are compact and $H(x, \Phi[f(x)])$ are closed, as the images of compact sets by closed mappings. The inclusions

$$(10) \quad H(x, \Phi[f(x)]) \subset H(x, \Phi_0[f(x)]) \subset \Phi_0(x)$$

imply that $H(x, \Phi[f(x)]) \in c(Y)$. Hence

$$H(\cdot, \Phi[f(\cdot)]) \in F_1.$$

We suppose that $\{\Phi_i\}_{i \in I}$ is a chain in F_1 . Then $\Phi(x) = \bigcap_{i \in I} \Phi_i(x) \in c(Y)$ and $\Phi \subset \Phi_0$. Therefore conditions (I) and (II') of Lemmas 1 and 2 are fulfilled. The first part of the theorem is proved.

2. Let $f: X \rightarrow X$ and let H with connected values be u.s.c. with respect to the second variable. Let $F_2 = \{\Phi: \Phi \subset \Phi_0, \Phi: X \rightarrow C(Y)\}$, where $\Phi_0: X \rightarrow C(Y)$ is a solution of (3). The image of a connected set by an u.s.c. mapping with connected values is connected (cf. [4]). The set $H(x, \Phi[f(x)])$ is closed by Lemma 4, and by the inclusions (10) it is compact. Thus we have $H(\cdot, \Phi[f(\cdot)]) \in F_2$. The family F_2 has property (II'), because an intersection of a chain of continuums is a continuum (cf. [3]).

3. We suppose that X is a topological Hausdorff space, $f: X \rightarrow c(X)$ is u.s.c. and H is u.s.c.. Let $F_3 = \{\Phi: \Phi \subset \Phi_0, \Phi: X \rightarrow c(Y), \Phi \text{ is u.s.c.}\}$, where $\Phi_0: X \rightarrow c(Y)$ is an u.s.c. solution of (3). The set $\Phi[f(x)]$ is compact for all x because Φ is an u.s.c. mapping with compact values and $f(x)$ is compact. Since the composition of two u.s.c. mappings is an u.s.c. mapping (cf. [2]), then by Lemmas 3 and 4 and by (10) we get $H(\cdot, \Phi[f(\cdot)]) \in F_3$. The family F_3 has property (II') because the intersection of a chain of compact sets is compact and the intersection of a family of u.s.c. mappings with compact values is u.s.c. (cf. [2]).

4. We suppose that X is a topological Hausdorff space, $f: X \rightarrow C(X)$ is u.s.c. and H with connected values is u.s.c. Let $F_4 = \{\Phi: \Phi \subset \Phi_0, \Phi: X \rightarrow C(Y), \Phi \text{ is u.s.c.}\}$, where $\Phi_0: X \rightarrow C(Y)$ is an u.s.c. solution of (3). The family F_4 is included in F_3 . It follows by the connectedness of $H(x, \Phi[f(x)])$ that $H(\cdot, \Phi[f(\cdot)]) \in F_4$. The family F_4 has property (II'), because the intersection of a chain of continuums is a continuum.

THEOREM 2. *Let $f: X \rightarrow X$ and $H: X \times \mathbf{R} \rightarrow \mathbf{R}$ be continuous and increasing with respect to the second variable. The function $\Phi: X \rightarrow C(\mathbf{R})$ is a solution of (1) if and only if $\varphi(x) = \min \Phi(x)$ and $\psi(x) = \max \Phi(x)$ fulfil (1).*

Proof. Necessity. The following equalities hold according to continuity and monotonicity of H with respect to the second variable and by (1):

$$\begin{aligned} [H(x, \varphi[f(x)]), H(x, \psi[f(x)])] &= H(x, [\varphi[f(x)], \psi[f(x)]]) \\ &= H(x, \Phi[f(x)]) = \Phi(x) = [\varphi(x), \psi(x)]. \end{aligned}$$

Thus $H(x, \varphi[f(x)]) = \varphi(x)$ and $H(x, \psi[f(x)]) = \psi(x)$. Hence φ and ψ are solutions of (1).

The proof of the sufficiency is also easy.

As a corollary we get the following

THEOREM 3. *Let $f: X \rightarrow X$ and $H: X \times \mathbf{R} \rightarrow \mathbf{R}$ be an increasing and continuous function with respect to the second variable. If $\Phi_0: X \rightarrow C(\mathbf{R})$ is a solution of (3), then there exist a minimal solution $\Phi: X \rightarrow C(\mathbf{R})$ of (1), $\Phi \subset \Phi_0$, and this solution is a single-valued mapping.*

We have by a direct verification

LEMMA 5. Let X be a topological space. If a mapping $\Phi: X \rightarrow c(\mathbf{R})$ is u.s.c., then $\varphi(x) = \min \Phi(x)$ is lower semi-continuous and $\psi(x) = \max \Phi(x)$ is upper semi-continuous. A mapping $\Phi: X \rightarrow C(\mathbf{R})$ is u.s.c. if and only if $\varphi(x) = \min \Phi(x)$ is lower semi-continuous and $\psi(x) = \max \Phi(x)$ is upper semi-continuous.

DEFINITION. Let X and Y be topological spaces and let $\Phi: X \rightarrow n(Y)$ be a mapping. The closure of Φ is the mapping $\text{cl } \Phi$ defined by

$$\text{cl } \Phi(x) = \{y \in Y: (x, y) \in \text{cl } \Gamma_\Phi\} \quad \text{for } x \in X,$$

where Γ_Φ is the graph of Φ .

If X and Y are metric spaces, the above definition is equivalent to the condition:

$y \in \text{cl } \Phi(x)$ if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow x$, $y_n \rightarrow y$ and $y_n \in \Phi(x_n)$ for every n .

LEMMA 6. Let X be a topological space. Then the mapping $\text{cl } \Phi$ is closed for every mapping $\Phi: X \rightarrow n(Y)$.

Proof. Let $y_0 \notin \text{cl } \Phi(x_0)$. Then $(x_0, y_0) \notin \text{cl } \Gamma_\Phi$. There exist neighbourhoods $U(x_0)$ and $V(y_0)$ such that

$$(11) \quad U(x_0) \times V(y_0) \cap \Gamma_\Phi = \emptyset.$$

We shall prove that

$$\text{cl } \Phi(x) \cap V(y_0) = \emptyset$$

for every $x \in U(x_0)$. Let $z \in \text{cl } \Phi(x) \cap V(y_0)$ for some $x \in U(x_0)$. Then $(x, z) \in \text{cl } \Gamma_\Phi$. There exist points $x' \in U(x_0)$ and $z' \in V(y_0)$ such that $(x', z') \in \Gamma_\Phi$. Thus $(x', z') \in U(x_0) \times V(y_0) \cap \Gamma_\Phi$, but this is impossible by (11).

LEMMA 7. Let X be a topological space and let Y be compact. If $H: X \times Y \rightarrow Y$ is continuous, f is an open and continuous function from X into X and if $\Phi: X \rightarrow n(Y)$ is a solution of (1), then $\text{cl } \Phi$ is an u.s.c. solution of (1) with compact values.

Proof. The set $\text{cl } \Phi(x)$ is closed, for the mapping $\text{cl } \Phi$ is closed. Thus $\text{cl } \Phi$ has compact values and is u.s.c. The mapping $\text{cl } \Phi \circ f$ is u.s.c. as the composition of u.s.c. mappings. Further, $H(\cdot, \text{cl } \Phi[f(\cdot)])$ is u.s.c. by Lemma 3 and by the continuity of H . The inclusion

$$\Phi(x) \subset H(x, \text{cl } \Phi[f(x)])$$

implies

$$\text{cl } \Phi(x) \subset \text{cl } H(x, \text{cl } \Phi[f(x)]) = H(x, \text{cl } \Phi[f(x)]),$$

since an u.s.c. mapping with compact values is closed. Now, we shall prove the inclusion

$$(12) \quad H(x, \text{cl } \Phi[f(x)]) \subset \text{cl } \Phi(x).$$

Let

$$(13) \quad y \in H(x, \text{cl } \Phi[f(x)])$$

and let U and V be any neighbourhoods of points x and y , respectively. It follows by (13) that there exists

$$(14) \quad z \in \text{cl } \Phi[f(x)]$$

such that

$$y = H(x, z).$$

There exist neighbourhoods $U_1 \subset U$ and W of points x and z , respectively, such that

$$H(U_1 \times W) \subset V.$$

The function f is open; thus the set $f(U_1)$ is a neighbourhood of $f(x)$. Condition (14) implies

$$\Phi[f(U_1)] \cap W \neq \emptyset.$$

Hence, there exist $u \in U_1$ and $w \in W$ such that $w \in \Phi[f(u)]$. Consequently

$$H(u, w) \in H(u, \Phi[f(u)]) = \Phi(u)$$

and

$$H(u, w) \in H(U_1 \times W) \subset V.$$

Thus

$$\Phi(U) \cap V \neq \emptyset,$$

which completes the proof of (12).

LEMMA 8. Let X be a topological space, $f: X \rightarrow X$ and let $H: X \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and increasing with respect to the second variable. If $\Phi: X \rightarrow c(\mathbb{R})$ is a solution of (1), then $\hat{\Phi}(x) := \text{conv}[\Phi(x)]$ is a solution of (1) with compact and connected values. Moreover, if Φ is u.s.c., then $\hat{\Phi}$ is u.s.c., too.

Proof. We define $\varphi(x) = \min \Phi(x)$, $\psi(x) = \max \Phi(x)$. By (1) and by the monotonicity of H with respect to the second variable we have

$$(15) \quad \begin{aligned} \hat{\Phi}(x) &= \text{conv}[\Phi(x)] = \text{conv}[H(x, \Phi[f(x)])] \\ &= [H(x, \varphi[f(x)]), H(x, \psi[f(x)])]. \end{aligned}$$

Since $\hat{\Phi}(x) = [\varphi(x), \psi(x)]$, we have

$$(16) \quad \begin{aligned} H(x, \hat{\Phi}[f(x)]) &= H(x, [\varphi[f(x)], \psi[f(x)]]) \\ &= [H(x, \varphi[f(x)]), H(x, \psi[f(x)])], \end{aligned}$$

by the continuity and monotonicity of H with respect to the second variable. From (15) and (16) we get

$$\hat{\Phi}(x) = H(x, \hat{\Phi}[f(x)]).$$

Lemma 5 implies that, if Φ is an u.s.c. function, then φ is lower semi-continuous and ψ is an upper semi-continuous numerical function. Thus $\hat{\Phi}(x) = [\varphi(x), \psi(x)]$ is u.s.c. by Lemma 5.

THEOREM 4. *Let X be a metric space and let $f: X \rightarrow X$ be a continuous and open function X into X . Moreover, let a continuous function $H: X \times \mathbf{R} \rightarrow \mathbf{R}$ be increasing with respect to the second variable. If $\Phi_0: X \rightarrow C(\mathbf{R})$ is an u.s.c. solution of inequality (3), and if there exists a compact set $K \subset \mathbf{R}$ such that $\Phi_0(X) \subset K$, then a minimal u.s.c. solution $\Phi: X \rightarrow C(\mathbf{R})$ of (1) such that $\Phi \subset \Phi_0$ is almost everywhere (everywhere except a set of the first category) a single-valued mapping.*

Proof. The existence of a minimal u.s.c. solution $\Phi: X \rightarrow C(\mathbf{R})$ with $\Phi \subset \Phi_0$ is guaranteed by Theorem 1. If $\Phi(x) = [\varphi(x), \psi(x)]$, then (by Theorem 2) the functions φ and ψ fulfil equation (1). According to Lemma 7 the functions $\text{cl } \varphi$ and $\text{cl } \psi$ are u.s.c. solutions of (1). Lemma 8 implies that $\Phi_1(x) = \text{conv}[\text{cl } \varphi(x)]$ and $\Phi_2(x) = \text{conv}[\text{cl } \psi(x)]$ are u.s.c. solution of (1) and $\Phi_1: X \rightarrow C(\mathbf{R})$, $\Phi_2: X \rightarrow C(\mathbf{R})$. By the minimality of Φ we have $\Phi_1 = \Phi_2 = \Phi$. The equality $\text{conv}[\text{cl } \varphi(x)] = [\varphi(x), \psi(x)]$ implies that $\psi(x) \in \text{cl } \varphi(x)$. This means that there exist sequences $\{y_n\}$, $\{x_n\}$ such that $y_n \rightarrow \psi(x)$, $x_n \rightarrow x$, and $y_n = \varphi(x_n)$. Thus there exists a sequence $\{x_n\}$ such that $x_n \rightarrow x$ and $\varphi(x_n) \rightarrow \psi(x)$. If φ is continuous at x , then $\varphi(x) = \psi(x)$. In virtue of the Baire theorem the set of all x at which φ is not continuous is of the first category.

Let Φ_0 be an u.s.c. solution of (3) with values in $C(\mathbf{R})$. K. Baron in [1] proved (under suitable assumptions) the existence of a single-valued u.s.c. solution Φ of (1) such that $\Phi \subset \Phi_0$. The following example indicates that the assumptions of Theorem 4 do not ensure the existence of an u.s.c. single-valued solution Φ of (1) such that $\Phi \subset \Phi_0$.

EXAMPLE. Let $X = Y = \mathbf{R}$, $f(x) = 2x$, $H(x, y) = y$ and let Φ_0 be given by the formula

$$\Phi_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ [0, 1] & \text{for } x = 0, \\ 1 - |2^{n+2}x - 3| & \text{for } x \in [2^{-(n+1)}, 2^{-n}], n = 0, \pm 1, \pm 2, \dots \end{cases}$$

This mapping is an u.s.c. solution of (1) with compact and connected values. Thus Φ_0 fulfils (3), but equation (1) does not possess any u.s.c. single-valued solution.

References

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