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WEAK CARTESIAN PRODUCT OF GRAPHS

BY

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1. Introduction. Sabidussi [1] proves that for connected graphs with certain finiteness conditions, cartesian multiplication satisfies unique prime factorization. These finiteness conditions were essentially imposed to prove a theorem concerning the structure of the automorphism group of a connected graph. In this same paper he introduces the weak cartesian product of an arbitrary family of rooted graphs to show the existence of connected graphs that are idempotent with respect to cartesian multiplication. Shapiro [2] was the first to introduce the cartesian product and Szamkołowicz [3] poses the question (due to Mycielski) of unique prime factorization for the cartesian and cardinal product. The purpose of this paper* is to prove an extension of Sabidussi's theorem, namely, for connected graphs weak cartesian multiplication satisfies unique prime factorization.

By a *graph* X we mean an ordered pair $(V(X), E(X))$, where $V(X)$ is a set and $E(X)$ is a set of unordered pairs of distinct elements of $V(X)$. (We can consider a graph to be a set together with a symmetric, irreflexive relation on the set.) We shall denote an unordered pair by brackets. The elements of $V(X)$ will be called the *vertices* of X and the elements of $E(X)$ the *edges* of X. We denote the cardinal of the set $V(X)$ by $|X|$. The *empty graph*, i.e., the graph with empty vertex set, will be denoted by $\emptyset$. An edge e is said to be *incident* with a vertex x if and only if $e = [x, y]$ for some vertex y. Two edges $e = [x, y]$ and $e' = [x', y']$ are said to be *adjacent* if and only if exactly two of the vertices x, y, x', y' are equal, i.e., two edges are adjacent if and only if they are distinct and incident with a common vertex.

A *subgraph* Y of a graph X is a graph whose vertex and edge sets are respectively subsets of the vertex and edge sets of X. A subgraph Y

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of X is called *saturated* if and only if $x, y \in V(Y)$ with $[x, y] \in E(X)$ implies $[x, y] \in E(Y)$. Let X and Y be graphs. By $X \cup Y$ and $X \cap Y$ we mean the graphs defined by $V(X \cup Y) = V(X) \cup V(Y)$, $E(X \cup Y) = E(X) \cup E(Y)$, and $V(X \cap Y) = V(X) \cap V(Y)$, $E(X \cap Y) = E(X) \cap E(Y)$.

If $x \in V(X)$, we let (x) denote the subgraph of X for which $V\big((x)\big) = \{x\}$ and $E\big((x)\big) = \emptyset$. If $e = [x, y] \in E(X)$, (e) denotes the subgraph of X for which $V\big((e)\big) = \{x, y\}$ and $E\big((e)\big) = \{e\}$. Whenever there is no likelihood of confusion we shall write x for (x) and e for (e). If Y is a subgraph of X we define the *relative complement* $X \setminus Y$ of Y in X to be the smallest subgraph with $E(X \setminus Y) = E(X) - E(Y)$.

Let X and Y be graphs. By a *homomorphism* of X into Y we mean a function $\varphi: V(X) \to V(Y)$ such that $[\varphi x, \varphi y] \in E(Y)$ whenever $[x, y] \in E(X)$. For a homomorphism $\varphi: V(X) \to V(Y)$ we shall write $\varphi: X \to Y$. A *monomorphism* of X into Y is a one-one homomorphism. If $\varphi: X \to Y$ is a homomorphism, then φ induces a function $\varphi^{\#}: E(X) \to E(Y)$ as follows: for $[x, y] \in E(X)$ define

$$\varphi^{\#}[x, y] = [\varphi x, \varphi y].$$

A homomorphism $\varphi: X \to Y$ is called an *epimorphism* if and only if φ and $\varphi^{\#}$ are both onto. By an *isomorphism* of X onto Y we mean a monomorphism $\varphi: X \to Y$ such that φ and $\varphi^{\#}$ are both onto. We shall frequently write φe for $\varphi^{\#} e$.

Given graphs X and Y let φ be a function from $V(X)$ to $V(Y)$. If A is a subgraph of S, we let φA denote the subgraph of Y defined by $V(\varphi A) = \varphi\big(V(A)\big)$, $E(\varphi A) = \{[y, y'] \in E(Y) \mid y = \varphi x, y' = \varphi x'$ for some $[x; x'] \in E(A)\}$. If φ is a homomorphism, then $E(\varphi A) = \varphi^{\#}\big(E(A)\big)$. The only functions we consider that are not homomorphisms are projections.

Let $x, y \in V(X)$. A *path* of X joining x and y is a subgraph P of X such that $V(P)$ is the set of elements of a finite sequence $(x_0, x_1, \ldots, x_n)$ of distinct vertices of X with $x_0 = x$ and $x_n = y$, and

$$E(P) = \{[x_i, x_{i+1}] \mid 0 \leqslant i \leqslant n-1\}.$$

We shall denote the path P by $[x_0, x_1, \ldots, x_n]$. n is called the *length* of P. If $P = [x_0, \ldots, x_n]$ and $x = x_i$, $y = x_j \in V(P)$, then P_{xy} will denote the path $[x_i, x_{i+1}, \ldots, x_j]$. A path P is called *proper* if the length of P is greater than or equal to 1. A graph X is called *connected* if any two vertices of X are joined by a path in X, otherwise it is called *disconnected*. A path P joining x and y is called a *shortest path* if and only if for any path Q joining x and y the length of P does not exceed the length of Q. Let X be connected, $x, y \in V(X)$. By the *distance* $d_X(x, y)$ of x and y in X we mean the length of a shortest path joining x and y in X. When no confusion is likely we shall write $d(x, y)$ for $d_X(x, y)$.

By a *circuit* of a graph X we mean a subgraph C of X such that $V(C)$ is the set of elements of a sequence $(x_1, x_2, \dots, x_n)$ of distinct vertices of X, and $E(C) = \{[x_i, x_{i+1}] \mid i = 1, \dots, n\}$, where subscripts are taken modulo n, $n \geqslant 3$. We shall denote the circuit C by $[x_1, \dots, x_n]$. n is called the *order* of C. A graph X is called *acyclic* if X contains no circuits. A *tree* is a connected acyclic graph.

Let $(X_a)_{a \in A}$ be a family of graphs. We will denote the cartesian product of the set $\{V(X_a) \mid a \in A\}$ by $\prod_{a \in A} V(X_a)$, and the projection into the b^{th} factor by $\mathrm{pr}_b \colon \prod_{a \in A} V(X_a) \to X_b$. We define the *cartesian product* $X = \prod_{a \in A} X_a$ of the graphs $(X_a)_{a \in A}$ by:

$$V(X) = \prod_{a \in A} V(X_a),$$

$$E(X) = \big\{[x, y] \mid x, y \in V(X),\ [\mathrm{pr}_a x, \mathrm{pr}_a y] \in E(X_a) \text{ for exactly one } a \in A,\ \mathrm{pr}_b x = \mathrm{pr}_b y \text{ for all } b \in A - \{a\}\big\}.$$

Since it is clear that the cartesian product of an infinite number of non-trivial connected graphs is disconnected, we consider the following product. For each $a \in A$ let $x_a \in V(X_a)$. The *weak cartesian product* $\prod_{a \in A} (X_a, x_a)$ *of the family of rooted graphs* $(X_a, x_a)_{a \in A}$ is defined to be the graph X such that

$$V(X) = \{x \in \prod_{a \in A} V(X_a) \mid \mathrm{pr}_a x \neq x_a \text{ for at most finitely many } a \in A\},$$

$$E(X) = \big\{[x, y] \mid x, y \in V(X),\ [\mathrm{pr}_a x, \mathrm{pr}_a y] \in E(X_a) \text{ for exactly one } a \in A, \text{ and } \mathrm{pr}_b x = \mathrm{pr}_b y \text{ for all } b \in A - \{a\}\big\}.$$

It is obvious that the weak cartesian product is connected if each of the factors are connected. Moreover, if the index set A is finite, then the weak cartesian product is independent of the roots and is equal to the cartesian product. For a finite family of graphs $(X_i)_{i=1,2,\dots,n}$ we shall frequently denote the cartesian product by $X_1 \times X_2 \times \dots \times X_n$.

A graph X is called *prime* or *indecomposable* with respect to (weak) cartesian multiplication if and only if X is non-trivial and $X \cong Y \times Z$ implies that either Y or Z is trivial.

In order to show that every connected graph has a weak cartesian decomposition into prime factors that is unique to within isomorphisms, we will investigate a particular set of equivalence relations (the acyclic equivalences (Definition 2) which contain $\alpha \cup \beta$ (Definition 3)) in the lattice of all equivalence relations on $E(X)$ and show that this is a principal filter with the following property: each equivalence in this filter gives rise to a weak cartesian decomposition of X such that two edges are equivalent if and only if they project to the same factor and the least element of the filter decomposes the graph X into prime factors. We will, moreover, show that to each decomposition of X as a weak cartesian product there corresponds an equivalence relation in this filter with the property that two edges are equivalent if and only if they project to the same factor. The least element will correspond to a prime decomposition. Unless otherwise stated X, Y, ... will denote arbitrary graphs.

2. ϱ-compatible graphs and acyclic equivalence relations. In this section we introduce the concept of an acyclic equivalence relation on the edge set of a graph and prove an equivalent formulation (Proposition 1). The lemmas required to prove Proposition 1 are used elsewhere in the paper. By a *cover* of a graph X we mean a collection $\boldsymbol{B}$ of subgraphs such that

(i) $\bigcap_{B \in \boldsymbol{B}} B = X$, and

(ii) $E(B) \cap E(B') = \emptyset$ for $B, B' \in \boldsymbol{B}$ with $B \neq B'$.

Definition 1. Let ϱ be an equivalence relation on $E(X)$. A subgraph Y of X will be called *ϱ-compatible* if Y has a cover $\boldsymbol{B}$ such that

(i) every $B \in \boldsymbol{B}$ is a proper path, and

(ii) for $B, B' \in \boldsymbol{B}$, $E(B) \times E(B') \subset \varrho$ or $\bar{\varrho}$ according as $B = B'$ or $B \neq B'$. (Here $\bar{\varrho}$ denotes the complement of ϱ in $E(X) \times E(X)$.)

It will be convenient to apply the term ϱ-compatible to the cover $\boldsymbol{B}$ as well.

Convention 1. Let ϱ be an equivalence relation on $E(X)$. When a ϱ-compatible path P is written in the form $P = P_1 \cup \ldots \cup P_n$ it is automatically understood that

(i) P_i is a proper path, $i = 1, \ldots, n$,

(ii) $P_i \cap P_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$, and

(iii) $E(P_i) \times E(P_j) \subset \varrho$ or $\bar{\varrho}$ according as $i = j$ or $i \neq j$.

Similarly, if a ϱ-compatible circuit C is written in the form $C = P_0 \cup \ldots \cup P_n$ it is understood that

(i) P_i is a proper path, $i = 0, i, \ldots, n$,

(ii) $P_i \cap P_j \neq \emptyset$ if and only if either $|i-j| \leqslant 1$ or $|i-j| = n$, and

(iii) $E(P_i) \times E(P_j) \subset \varrho$ or $\bar{\varrho}$ according as $i = j$ or $i \neq j$.

Remark 1. Let C be a circuit such that $E(C) \times E(C) \not\subset \varrho$, where ϱ is an equivalence on $E(X)$. Then clearly C can be uniquely expressed as the union of proper paths, each path being maximal with respect to its edges belonging to one equivalence class modulo ϱ, i.e., $C = P_0 \cup \cup \ldots \cup P_n$, where P_i is a maximal proper path such that $E(P_i) \times E(P_i) \subset \varrho$, $i = 0, \ldots, n$. This decomposition will be called the *ϱ-decomposition* of C or the *decomposition of C determined by* ϱ, and $n+1$ will be called the *ϱ-degree of C*. We will denote the ϱ-degree of C by $\deg_\varrho C$. Whenever the ϱ-decomposition of a circuit C is written in the form

$$C = P_0 \cup \ldots \cup P_n$$

it will automatically be understood that $P_i \cap P_j \neq \emptyset$ if and only if either $|i-j| \leqslant 1$ or $|i-j| = n$. By the maximality of the P_i we have $E(P_i) \times E(P_j) \subset \bar{\varrho}$ if $|i-j| = 1$ or $|i-j| = n$. If $C = P_0 \cup \ldots \cup P_n$ is not ϱ-compatible there exist integers $i < j$ such that $E(P_i) \times E(P_j) \subset \varrho$ and $P_{i+1} \cup P_{i+2} \cup \ldots \cup P_j$ is a ϱ-compatible path.

LEMMA 1. *Let Y be a connected ϱ-compatible subgraph of X. Then given any two distinct vertices $x, y \in Y$ there exists a ϱ-compatible path joining x and y in Y.*

Proof. Since Y is connected there is a path $[x_0, \ldots, x_m] \subset Y$ such that $x_0 = x$, $x_m = y$. Let $\boldsymbol{B}$ be a ϱ-compatible cover of Y and let W_i be that path belonging to $\boldsymbol{B}$ which contains the edge $[x_{i-1}, x_i]$, $i = 1, \ldots, m$. This means $W_1, \ldots, W_m$ are paths in $\boldsymbol{B}$ such that $x \in W_1$, $y \in W_m$, and $W_i \cap W_{i+1} \neq \emptyset$, $i = 1, \ldots, m-1$. Now let n be the smallest integer for which $\boldsymbol{B}$ contains n paths $P^{(1)}, \ldots, P^{(n)}$ such that

$$x \in P^{(1)}, \quad y \in P^{(n)}, \tag{1}$$

and

$$P^{(j)} \cap P^{(j+1)} \neq \emptyset, \quad j = 1, , \ldots, n-1. \tag{2}$$

Then $P^{(k)} \cap P^{(j+1)} = \emptyset$ for all $k < j < n$. For if there exists a $k < j$ with $p^{(k)} \cap P^{(j+1)} \neq \emptyset$, then

$$P^{(1)}, \ldots, P^{(k)}, P^{(j+1)}, \ldots, P^{(n)}$$

is a set of fewer than n paths in $\boldsymbol{B}$ with properties (1) and (2). Now put $y_0 = x$, $y_n = y$, and for $i = 1, \ldots, n-1$ define y_i inductively to be a vertex in $P^{(i)} \cap P^{(i+1)}$ such that no other vertex of $Q_i = P^{(i)}_{y_{i-1}y_i}$ also belongs to $P^{(i+1)}$. Then

$$P = \bigcup_{i=1}^{n} Q_i$$

is a ϱ-compatible path joining x and y in Y.

LEMMA 2. *Let Y be a ϱ-compatible subgraph of X which is not acyclic. Then Y contains a ϱ-compatible circuit.*

Proof. Since Y is not acyclic there exists a finite circuit $C = [x_0, \ldots, x_n] \subset Y$. Let $\boldsymbol{B}$ be a ϱ-compatible cover of Y. Let $W^{(0)}$ be that path belonging to $\boldsymbol{B}$ which contains the edge $[x_0, x_n]$ and $W^{(i)}$ that path belonging to $\boldsymbol{B}$ which contains the edge $[x_{i-1}, x_i]$, $i = 1, \ldots, n$. $W^{(0)} \neq W^{(i)}$ for at least one i, $1 \leqslant i \leqslant n$. Otherwise $C \subset W^{(0)}$, a contradiction to $W^{(0)}$ being a path. Hence there exist $x_h, x_k \in W^{(0)} \cap C$, $0 \leqslant h < k \leqslant n$, such that $x_t \notin V(W^{(0)})$ for $h < t < k$.

$$Z = \bigcup_{i=h+1}^{k} W^{(i)}$$

is a connected ϱ-compatible subgraph containing x_h and x_k and hence there exists a ϱ-compatible path P joining x_h and x_k in Z. Either $W^{(0)}_{x_h x_k} \cup P$

is the desired ϱ-compatible circuit or there exists an $x \in W^{(0)}_{x_h x_k} \cap P$ such that $W^{(0)}_{x_h x} \cup P_{x x_h}$ is the required circuit.

Definition 2. An equivalence relation ϱ on $E(X)$ is called *acyclic* if every ϱ-compatible subgraph of X is acyclic.

PROPOSITION 1. *A necessary and sufficient condition that ϱ be acyclic is that X contain no ϱ-compatible circuit.*

Proof. Necessity. Assume that ϱ is not acyclic. By definition there exists a ϱ-compatible subgraph of X which is not acyclic and hence by Lemma 2, X contains a ϱ-compatible finite circuit.

Sufficiency. Trivial.

3. The binary relations α and β and the construction of ladders. The following two binary relations α and β on $E(X)$ were introduced by Sabidussi ([1], (2.1)) and are of considerable importance in our subsequent considerations.

Definition 3. Let X be a graph, $e, e' \in E(X)$. We write $e\alpha e'$ if

(i) e and e' are adjacent, and

(ii) among the saturated subgraphs of X which contain e and e' there is no 4-circuit.

And we write $e\beta e'$ if

(i) e and e' are not adjacent, and

(ii) among the saturated subgraphs of X which contain e and e' there is a 4-circuit.

In general, neither α nor β is an equivalence relation. By ϱ_0 we shall denote the smallest equivalence on $E(X)$ which contains $\alpha \cup \beta$. Note that if X is connected and every 4-circuit of X has a diagonal, then $\varrho_0 = E(X) \times E(X)$.

By a *ladder* we mean a graph which is isomorphic to the cartesian product of an edge with a proper path.

Construction 1. Let ϱ be an equivalence on $E(X)$ which contains $\alpha \cup \beta$. Let x, y, x' be distinct vertices of X, $e = [x, x'] \in E(X)$, and let P be a path joining x and y with $E(P) \times \{e\} \subset \bar{\varrho}$. We will now give a method for constructing a ladder in X from e and P provided that one of the following conditions holds:

(i) P is a shortest path joining x and y,

(ii) ϱ is acyclic and $P = P_1 \cup \ldots \cup P_n$ is a ϱ-compatible path joining x and y (here P need not be a shortest path joining x and y).

Denote the consecutive vertices of P by $x_0 = x, x_1, \ldots, x_{s-1}, x_s = y$ and let $e_i = [x_{i-1}, x_i]$, $i = 1, \ldots, s$. $e\bar{\varrho}e_1$ implies $e\bar{\alpha}e_1$. Since e and e_1 are adjacent, X contains a saturated 4-circuit $C_1 = [x_0, x_1, x_1', x_0' = x']$ such that $e, e_1 \in E(C_1)$. Let $e^{(0)} = e$, $e^{(1)} = [x_1, x_1']$ and $e_1' = [x', x_1']$. Then

$e_1\beta e_1'$, and $e\beta e^{(1)}$, so that $e^{(1)}\bar{\varrho} e_2$ (otherwise $e\beta e^{(1)}\varrho e_2$, contrary to $e\bar{\varrho} e_2$). This implies $e^{(1)}\bar{\alpha} e_2$, hence again X contains a saturated 4-circuit $C_2 = [x_1, x_2, x_2', x_1']$ such that $e^{(1)}, e_2 \in E(C_2)$. Thus we obtain two new edges $e^{(2)} = [x_2, x_2']$ and $e_2' = [x_1', x_2']$ such that $e_2\beta e_2'$ and $e^{(1)}\beta e^{(2)}$. It is obvious from the construction that e_1' and e_2' are either equal or adjacent and it can easily be shown that, since $\varrho \supset \alpha \cup \beta$, e_1' and e_2' cannot be equal. Proceeding in this manner we produce two new sequences,

$$e_1', \ldots, e_s' \quad \text{and} \quad e^{(0)}, \ldots, e^{(s)}$$

of edges of X such that

$$e_1' \text{ and } e_{i+1}' \text{ are adjacent}, \quad i = 1, \ldots, s-1, \tag{4}$$

$$e_i\beta e_i', \quad i = 1, \ldots, s, \tag{5}$$

$$e^{(0)}\beta e^{(1)}\beta \ldots \beta e^{(s)}. \tag{6}$$

Let $Q = e_1' \cup e_2' \cup \ldots \cup e_s'$.

Now if we assume that P is a shortest path joining x and y we can show the above construction yields a ladder. We first prove that $P \cap Q = \emptyset$. Assume instead that there exist $x_i \in P$, $0 \leqslant i \leqslant s$, $x_j' \in Q$, $0 \leqslant j \leqslant s$, with $x_i = x_j'$, and without loss of generality we may take $i < j$. $j \neq i+1$ since $C_i = [x_i, x_{i+1}, x_{i+1}', x_i']$ was a saturated 4-circuit and hence $x_i \neq x_{i+1}'$. Therefore $j-i \geqslant 2$ and hence

$$[x = x_0, x_1, \ldots, x_{i-1}, x_i = x_j', x_j, \ldots, x_s = y]$$

is a path joining x and y of length less than s. This is a contradiction to the minimality of the length of P. Hence $P \cap Q = \emptyset$. Next we show Q is a path of length s. Assume instead that there exist $x_i', x_j' \in Q$, $0 \leqslant i \leqslant s$, $0 \leqslant j \leqslant s$ with $x_i' = x_j'$, and take $i < j$. Since e_i' and e_{i+1}' are adjacent, $i+2 < j$, and hence

$$[x = x_0, x_1, \ldots, x_i, x_i' = x_j', \ldots, x_s = y]$$

is a path of length less than s joining x and y, contradicting the minimality of the length of P. Therefore Q is a path of length s and $P \cup Q \cup e^{(0)} \cup \cup \ldots \cup e^{(s)}$ is isomorphic to $P \times \{e\}$.

Now assume instead that ϱ is acyclic and that $P = P_0 \cup \ldots \cup P_n$ is a ϱ-compatible path joining x and y. (Here P need not be a shortest path.) We shall only consider the case $n = 1$. The reader will have no difficulty in extending the argument to $n \geqslant 2$. Suppose there exist vertices $x_i', x_j' \in Q$ with $x_i' = x_j'$ and $i < j$. Since $\alpha \cup \beta \subset \varrho$, (6) implies $e^{(i)}\varrho e^{(j)}$. Hence $e_{i+1} \cup \ldots \cup e_j, e^{(j)} \cup e^{(i)}$ would form a ϱ-compatible circuit contradicting the acyclicity of ϱ. Therefore Q is again a path of length s.

If $P \cap Q \neq \emptyset$, then there exist $x_i \in P$, $x_j' \in Q$, $0 \leqslant i \leqslant s$, $0 \leqslant j \leqslant s$ with $x_i = x_j'$. Again we may assume without loss of generality that $i < j$.

Hence $e_{i+1} \cup \ldots \cup e_j, e^{(j)}$ form a ϱ-compatible circuit contradicting the acyclicity of ϱ. Thus condition (ii) also insures that the construction yields a ladder.

In the above construction we will refer to Q as the path *opposite* P and to $e^{(i)}$ as the *i-th rung* of the ladder.

4. Application of ladders. Proposition 2 below will be proved by a straight forward application of the previous construction. This proposition will be used later to show that the collection of all acyclic equivalences on $E(X)$ which contain $a \cup \beta$ is a filter.

PROPOSITION 2. *Let ϱ be an equivalence on $E(X)$ containing $a \cup \beta$, $P = P_1 \cup \ldots \cup P_n$ a ϱ-compatible path joining x and y. If ϱ is acyclic, or if P is a shortest path joining x and y (here ϱ need not be acyclic), then there exists a ϱ-compatible path $Q = Q_1 \cup \ldots \cup Q_n$ joining x and y such that*

(i) $|P_i| = |Q_{i+1}|$, $i = 1, \ldots, n-1$,

(ii) $|P_n| = |Q_1|$,

(iii) $E(P_i) \times E(Q_{i+1}) \subset \varrho$, $i = 1, \ldots, n-1$,

(iv) $E(P_n) \times E(Q_1) \subset \varrho$.

Proof. We shall only consider the case $n = 2$, the reader will have no difficulty in extending the argument to $n \geqslant 3$. Denote the consecutive vertices of P_2 by $x_0, x_1, \ldots, x_r = y$ and let $e_i = [x_{i-1}, x_i]$, $i = 1, \ldots, r$. If P is a shortest path joining x and y, then $P_1 \cup e_1$ is a shortest path joining x and x_1. Hence if ϱ is acyclic or P is a shortest path joining x and y, Construction 1 implies that a ladder can be formed from e_1 and P_1. Let $e_1' = [x, x_1']$ be the final rung of the ladder and $P_1^{(1)}$ the path opposite P_1. Again it is clear that a ladder can be constructed from e_2 and $P_1^{(1)}$. Let $e_2' = [x_1', x_2']$ be the final rung of the ladder and $P_1^{(2)}$ the path opposite $P_1^{(1)}$. Continuing in this manner we get a path $P_1^{(r)}$, which we shall denote Q_2, such that $E(Q_2) \times E(P_1) \subset \varrho$, and $|Q_2| = |P_1|$, and a sequence $e_1', \ldots, e_r'$ of edges of X such that

(7) $\quad e_i'$ and e_{i+1}' are either equal or adjacent for $i = 1, \ldots, r-1$,

and

(8) $$e_i \varrho e_i', \quad i = 1, \ldots, r.$$

Let $Q_1 = e_1' \cup e_2' \cup \ldots \cup e_r'$. Then (8) implies $E(Q_1) \times E(P_2) \subset \varrho$.

If ϱ is acyclic and $x_i' = x_j'$, $i \neq j$, then $P_1^{(i)} \cup P_1^{(j)}$ and the segment of P_2 determined by x_i and x_j form a ϱ-compatible circuit contradicting the acyclicity of ϱ. Hence Q_1 is a path with $|Q_1| = |P_2|$.

Also it is clear that $Q = Q_1 \cup Q_2$ is a path joining x and y (otherwise we again get a contradiction to the acyclicity of ϱ).

Now we assume that P is a shortest path joining x and y. By (7), $Q = Q_1 \cup Q_2$ is a connected subgraph of X joining x and y; hence Q is a path and $|Q_1| = |P_2|$, otherwise P is not a shortest path joining x and y.

5. The principal filter of all acyclic equivalence relations containing $\alpha \cup \beta$. We shall denote by $\boldsymbol{E}(X)$ *the collection of all acyclic equivalence relations on* $E(X)$ *which contain* $\alpha \cup \beta$. $\boldsymbol{E}(X)$ is non-empty, since $E(X)\times \times E(X) \in \boldsymbol{E}(X)$.

PROPOSITION 3. $\boldsymbol{E}(X)$ *is closed under intersection of chains, and hence contains a minimal element.*

Proof. Let $\boldsymbol{B}$ be a chain in $\boldsymbol{E}(X)$, $\varrho = \bigcap_{\sigma \in \boldsymbol{B}} \sigma$. Clearly ϱ contains $\alpha \cup \beta$. It remains to show that ϱ is acyclic. Suppose there exists a ϱ-compatible circuit $C = P_0 \cup \ldots \cup P_n$. Let $E_{ij} = E(P_i)\times E(P_j)$. $E_{ii} \subset \varrho$ implies $E_{ii} \subset \sigma$ for every $\sigma \in \boldsymbol{B}$. $E_{ij} \subset \bar{\varrho}$ (for $i \neq j$) implies that there is a $\sigma_{ij} \in \boldsymbol{B}$ with $E_{ij} \subset \bar{\sigma}_{ij}$. Let

$$\sigma_0 = \bigcap_{0 \leqslant i \leqslant j \leqslant n} \sigma_{ij}.$$

Then $\sigma_0 \in \boldsymbol{B}$, and $E_{ij} \subset \bar{\sigma}_0$ whenever $i \neq j$. Also $E_{ii} \subset \sigma_0$, $i = 0, \ldots, n$, so that σ_0 is not acyclic, a contradiction.

PROPOSITION 4. $\boldsymbol{E}(X)$ *is closed under finite intersections.*

Proof. Assume that there exist $\varrho_1, \varrho_2 \in \boldsymbol{E}(X)$ such that $\varrho = \varrho_1 \cap \varrho_2 \notin \boldsymbol{E}(X)$. Since $\alpha \cup \beta \subset \varrho$ this implies that ϱ is not acyclic. If $E(C)\times \times E(C) \subset \varrho_i$, $i = 1, 2$, then $E(C)\times E(C) \subset \varrho$. Hence we may assume without loss of generality that among the ϱ-compatible circuits of minimal degree there exists at least one that has a ϱ_1-decomposition. Among all ϱ-compatible circuits of minimal ϱ-degree, choose one, $C = P_0 \cup \ldots \cup P_n$ say, whose ϱ_1-degree is minimal. Let $C = Q_0 \cup \ldots \cup Q_r$ be the decomposition of C determined by ϱ_1 and let the notation be so chosen that $P_0 \subset Q_0$. Note that $\varrho \subset \varrho_1$ implies

$$(9) \qquad Q_i = \bigcup \{P_j \mid E(P_j) \cap E(Q_j) \neq \emptyset\}, \qquad i = 0, \ldots, r.$$

ϱ_1 is acyclic. Hence Remark 1 implies without loss of generality that there exists an integer s, $0 < s < r$, such that $E(Q_0)\times E(Q_s) \subset \varrho_1$ and $Y = Q_1 \cup Q_2 \cup \ldots \cup Q_s$ is a ϱ_1-compatible path.

Let the end vertices of Y be x and y. By Proposition 2, there exists a ϱ_1-compatible path $Y' = Q_1' \cup \ldots \cup Q_s'$ joining x and y such that

$$E(Q_i)\times E(Q_{i+1}') \subset \varrho_1, \qquad i = 1, \ldots, s-1,$$

and

$$E(Q_s)\times E(Q_1') \subset \varrho_1.$$

Let

$$C' = Q_0 \cup Q_1' \cup Q_2' \cup \ldots \cup Q_s' \cup Q_{s+1} \cup \ldots \cup Q_r.$$

We now show that C' is a ϱ-compatible circuit. Since the notation was chosen so that $P_0 \subset Q_0$, (9) implies that

$$Y = P_k \cup P_{k+1} \cup \ldots \cup P_m, \qquad 0 < k < m < n.$$

$C = P_0 \cup \ldots \cup P_n$ is a ϱ-compatible circuit and hence $Y = P_k \cup \cup P_{k+1} \cup \ldots \cup P_m$ is a ϱ-compatible path. By the construction, Y' is also a ϱ-compatible path with $E(Y')$ belonging to the same set of equivalence classes modulo ϱ as $E(Y)$. Hence C' is a ϱ-compatible subgraph with a ϱ-compatible cover of cardinality $n+1$. Again by (9) and the fact that $C = P_0 \cup \ldots \cup P_n$ is a ϱ-compatible circuit we have

$$E(Q_0) \times E(Q_s) \subset \bar{\varrho}.$$

By the construction of Q_1', $E(Q_1')$ is contained in the same set of equivalence classes modulo ϱ as $E(Q_s)$. Hence

$$E(Q_0) \times E(Q_1') \subset \bar{\varrho}.$$

Therefore $E(Q_0) \cap E(Q_1') = \emptyset$ and C' is not acyclic. If C' is not a circuit, then, by Lemma 2, C' contains a ϱ-compatible circuit which can be covered by less than $n+1$ paths. This is a contradiction to the minimality of n. Hence C' is a ϱ-compatible circuit with a ϱ-compatible cover of cardinality $n+1$. But $E(Q_0) \times E(Q_1') \subset \varrho_1$. Hence the ϱ_1-decomposition of C' has less than $r+1$ paths. This is a contradiction to our choice of r. Therefore $\boldsymbol{E}(X)$ is closed under finite intersections.

PROPOSITION 5. *Let $\varrho \in \boldsymbol{E}(X)$ and let σ be any equivalence containing ϱ. Then $\sigma \in \boldsymbol{E}(X)$.*

Proof. Suppose $\sigma \notin \boldsymbol{E}(X)$. Then $\alpha \cup \beta \subset \varrho \subset \sigma$ implies that σ is not acyclic. Let C be any σ-compatible circuit. $E(C) \times E(C) \not\subset \sigma$ and $\varrho \subset \sigma$ imply $E(C) \times E(C) \not\subset \varrho$. Hence every σ-compatible circuit has a ϱ-decomposition. Among all σ-compatible circuits of minimal order choose one, say C, whose ϱ-degree is minimal. Let $C = P_0 \cup \ldots \cup P_m$ be the σ-decomposition of C and let $C = Q_0 \cup \ldots \cup Q_r$ be the ϱ-decomposition of C. Note that $\varrho \subset \sigma$ implies that

$$P_i = \bigcup \{Q_j \mid E(P_i) \cap E(Q_j) \neq \emptyset\}, \quad i = 1, \ldots, m.$$

Since ϱ is acyclic, Remark 1 implies that there exist Q_i, Q_j, $i < j$, such that $E(Q_i) \times E(Q_j) \subset \varrho$ and $Y = Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_j$ is a ϱ-compatible path.

$\varrho \subset \sigma$ implies $Y \subset P_k$ for some k, $0 \leqslant k \leqslant m$. Let the end vertices of Y be x and y. By Proposition 2, there exists a ϱ-compatible path $Y' = Q_{i+1}' \cup Q_{i+2}' \cup \ldots \cup Q_j'$ such that

$$E(Q_{i+k}) \times E(Q_{i+k+1}') \subset \varrho, \quad k = 1, \ldots, j-i-1,$$

and

$$E(Q_j) \times E(Q_{i+1}') \subset \varrho.$$

Let

$$C' = Q_0 \cup \ldots \cup Q_i \cup Q_{i+1}' \cup \ldots \cup Q_j' \cup Q_{j+1} \cup \ldots \cup Q_r.$$

By the construction of Y' we have $E(Y') \times E(P_k) \subset \sigma$. Hence it is clear that C' is a σ-compatible circuit of minimal order. Since $E(Q_i)\times \times E(Q'_{i+1}) \subset \varrho$, the ϱ-decomposition of C' has less than $r+1$ paths. This is a contradiction to the minimality of r and hence $\sigma \in \boldsymbol{E}(X)$.

THEOREM 1. *$\boldsymbol{E}(X)$ is a principal filter in the lattice of all equivalence relations on $E(X)$.*

Proof. Proposition 4 and Proposition 5 imply that $\boldsymbol{E}(X)$ is a filter. Proposition 3 implies that it is a principal filter.

PROPOSITION 6. *Let ϱ be the least element of $\boldsymbol{E}(X)$ and let E_a, $a \in A$, denote the equivalence classes of $E(X)$* mod ϱ. *Then $\boldsymbol{E}(X)$ is isomorphic to the lattice of all equivalence relations on A.*

Proof. The proposition is readily established even if we replace $\boldsymbol{E}(X)$ by an arbitrary principal filter.

COROLLARY 1. *Let ϱ be the least element of $\boldsymbol{E}(X)$. $\boldsymbol{E}(X)$ is finite if and only if $E(X)$ consists of a finite number of equivalence classes* mod ϱ. *If $E(X)$ has $n \geqslant \aleph_0$ equivalence classes* mod ϱ, *then* $|\boldsymbol{E}(X)| = 2^n$.

We will complete this section by giving a different characterization of $\boldsymbol{E}(X)$.

Definition 4. Let ϱ be any equivalence on $E(X)$ and let C be a circuit with ϱ-decomposition $C = P_0 \cup \ldots \cup P_n$. C is called *weakly ϱ-compatible* if there exists an i, $0 \leqslant i \leqslant n$, such that $E(P_i)\times E(P_j) \subset \bar{\varrho}$ for all j, $i \neq j$, $0 \leqslant j \leqslant n$. ϱ is called *strongly acyclic* if X contains no weakly ϱ-compatible circuits.

Sabidussi [1], (2.10), proves that the equivalence relation that he constructs in order to decompose a graph into indecomposible factors is a strongly acyclic equivalence. It is clear from the definitions that if ϱ is a strongly acyclic equivalence, then ϱ is acyclic. From the following proposition we infer that if ϱ is an equivalence containing $\alpha \cup \beta$, then ϱ is acyclic if and only if ϱ is strongly acyclic.

PROPOSITION 7. *Let ϱ be an acyclic equivalence on $E(X)$ containing $\alpha \cup \beta$. Then X does not contain any weakly ϱ-compatible circuits.*

Proof. Assume the contrary. Among all weakly ϱ-compatible circuits choose one, $C = P_0 \cup \ldots \cup P_n$ say, whose ϱ-degree is minimal, and let the notation be so chosen that $E(P_0)\times E(P_j) \subset \bar{\varrho}$, $1 \leqslant j \leqslant n$. Since ϱ is acyclic, Remark 1 implies that there exist P_i, P_j $(i<j)$ such that

$$E(P_i)\times E(P_j) \subset \varrho \tag{10}$$

and $Y = P_{i+1} \cup P_{i+2} \cup \ldots \cup P_j$ is a ϱ-compatible path. Let the end vertices of Y be x and y. Proposition 2 implies that there exists a ϱ-compatible path $Y' = P'_{i+1} \cup \ldots \cup P'_j$ joining x and y such that $E(P_{i+k})\times \times E(P'_{i+k+1}) \subset \varrho$ $(k = 1, \ldots, j-i-1)$ and

$$E(P_j)\times E(P'_{i+1}) \subset \varrho. \tag{11}$$

Let

$$C' = P_0 \cup \ldots \cup P_i \cup P'_{i+1} \cup \ldots \cup P'_j \cup P_{j+1} \cup \ldots \cup P_n.$$

If C' is a circuit, then it is weakly ϱ-compatible (since $E(P_0)\times E(P_k) \subset \varrho\,(k = 1, \ldots, i$ and $j+1, \ldots, n)$ and $E(P_0)\times E(P'_k) \subset \bar{\varrho}\,(k = i+1, \ldots, j))$ and $\deg C' = n-1$ (since $E(P_i)\times E(P'_{i+1}) \subset \varrho$ by (10) and (11)), a contradiction to the minimality of n. Suppose C' is not a circuit. Set $Y'' = C'\setminus Y'$; then $C' = Y'' \cup Y'$. Choose $z,\ w \in V(Y'')$ such that

(i) $E(Y''_{zw} \cap P_0) \neq \emptyset$,

and

(ii) $V(Y''_{zw}) \cap V(Y') = \{z, w\}$.

Then $C'' = Y''_{zw} \cup Y'_{zw}$ is a weakly ϱ-compatible circuit with $\deg_\varrho C'' \leqslant n-1$. This is again a contradiction to the minimality of n.

6. Weak cartesian products and acyclic equivalences containing $\alpha \cup \beta$. We now turn our investigations to the relationships between weak cartesian products on the one hand and acyclic equivalences containing $\alpha \cup \beta$ on the other.

Let $X = \prod_{a\in A}(X_a, x_a)$ be the weak cartesian product of the rooted graphs (X_a, x_a). Then for each $e = [x, y]\in E(X)$ there exists exactly one $a\in A$ such that $[\mathrm{pr}_a x, \mathrm{pr}_a y]\in E(X_a)$. We will denote this unique member of A by $a(e)$, and we will denote the edge $[\mathrm{pr}_a x, \mathrm{pr}_a y]$ by $\mathrm{pr}_a e$.

Definition 6. Let $X = \prod_{a\in A}(X_a, x_a)$. For each $b\in A$ and $x\in V(X)$ we define the *injection mapping* $i_b^x\colon V(X_b) \to V(X)$ as follows: for each $x_b \in V(X_b)$,

$$\mathrm{pr}_a i_b^x x_b = \begin{cases} x_b & \text{if } b = a, \\ \mathrm{pr}_a x & \text{if } b \neq a. \end{cases} \qquad (a\in A)$$

It is obvious that $i_b^x\colon X_b \to X$ is a monomorphism and hence $(i_b^x)^{-1}\colon i_b^x X_b \to X_b$ is an isomorphism.

The following two propositions established in [1] for the cartesian product of finitely many graphs immediately extend to the weak cartesian product of our arbitrary family of graphs.

PROPOSITION 8 ([1], (2.5)). *Let $X = \prod_{a\in A}(X_a, x_a)$. Then $e\varrho_0 e'$ implies $a(e) = a(e')$, where ϱ_0 is the smallest equivalence on $E(X)$ containing $\alpha \cup \beta$.*

PROPOSITION 9 ([1], (2.6)). *Let $x\in X$, $e = [x_0, y_0]\in E(X)$, $a(e) = a$. If x and x_0 belong to the same component of X, then $(i_a^x \mathrm{pr}_a e)\varrho_0 e$.*

Definition 6. Let $X = \prod_{a\in A}(X_a, x_a)$ and let σ be an equivalence relation on $E(X_a)$ for some $a\in A$. Then σ can be extended to an equivalence relation $\tilde{\sigma}$ on $E(X)$ as follows: for $e, e'\in E(X)$ define $e\tilde{\sigma}e'$ if either

(i) $a(e) = a = a(e')$ and $\mathrm{pr}_a e \sigma \mathrm{pr}_a e'$

or

(ii) $a(e) \neq a \neq a(e')$.

PROPOSITION 10. *Let* $X = \prod_{a \in A} (X_a, x_a)$ *and let* $\sigma \in \boldsymbol{E}(X_a)$ *for some* $a \in A$. *Then* $\tilde{\sigma} \in \boldsymbol{E}(X)$.

Proof. To show that $\tilde{\sigma}$ is acyclic assume instead that there exists a $\tilde{\sigma}$-compatible circuit $C = P_0 \cup \ldots \cup P_n$ in X. Without loss of generality we may assume that $e \in E(P_0)$ implies $a(e) \neq a$ (otherwise $\mathrm{pr}_a C = \mathrm{pr}_a P_0 \cup \cup \ldots \cup \mathrm{pr}_a P_n$ is a σ-compatible circuit). Let $\{x, y\} = V(P_0) \cap V(P_1 \cup \cup \ldots \cup P_n)$. $x, y \in V(P_0)$ implies $\mathrm{pr}_a x = \mathrm{pr}_a y$. But $x, y \in V(P_1 \cup \ldots \cup P_n)$ implies $\mathrm{pr}_a x \neq \mathrm{pr}_a y$, a contradiction. Hence $\tilde{\sigma}$ is acyclic.

To show that $\alpha \cup \beta \subset \tilde{\sigma}$ we first show that $\alpha \subset \tilde{\sigma}$. Let $e, e' \in E(X)$ and $e\alpha e'$. By Proposition 8 we have $a(e) = a(e')$. If $a(e) \neq a$, then $e \tilde{\sigma} e'$. If $a(e) = a(e') = a$, then $\mathrm{pr}_a e \alpha \mathrm{pr}_a e'$ on X_a and therefore $\mathrm{pr}_a e \sigma \mathrm{pr}_a e'$. Hence in either case $e\alpha e'$ implies $e \tilde{\sigma} e'$. A similar argument shows that $\beta \subset \tilde{\sigma}$.

Remark 2. Let $X = \prod_{a \in A} (X_a, x_a)$ and for each $a \in A$ let $\varrho_a = E(X_a) \times \times E(X_a)$. Then $\varrho = \bigcap_{a \in A} \tilde{\varrho}_a \in \boldsymbol{E}(X)$ and $e \varrho e'$ if and only if $a(e) = a(e')$. Hence $E(X)$ consists of exactly $|A|$ equivalence classes modulo ϱ. Hence *if* ϱ *is least and consists of exactly one equivalence class, then* X *is indecomposable.*

PROPOSITION 11. *Let* $X = \prod_{a \in A} (X_a, x_a)$ *and let* ϱ *be the least element in* $\boldsymbol{E}(X)$. *Then* $e \varrho e'$ *implies* $a(e) = a(e')$.

Proof. Let $\varrho_a = E(X_a) \times E(X_a)$. ϱ being the least element in $\boldsymbol{E}(X)$ and $\tilde{\varrho}_a \in \boldsymbol{E}(X)$ for each $a \in A$ imply $\varrho \subset \tilde{\varrho}_a$ for each $a \in A$. In particular, $\varrho \subset \tilde{\varrho}_{a(e)}$, whence $a(e) = a(e')$. This completes the proof.

We will abbreviate the restriction of ϱ to $i_a^x X_a$ by $\varrho_{a,x}$.

Remark 3. If $\varphi: X \to Y$ is an isomorphism and $\varrho \in \boldsymbol{E}(X)$, then $\varrho_\varphi \in \boldsymbol{E}(Y)$, where $e \varrho_\varphi e'$ $(e, e' \in E(Y))$ if and only if $(\varphi^{-1} e) \varrho (\varphi^{-1} e')$. This follows from the fact that both acyclicity and the relations α and β are defined in invariant terms.

PROPOSITION 12. *Let* $X = \prod_{a \in A} (X_a, x_a)$. *If* $\varrho \in \boldsymbol{E}(X)$, *then for each* $x \in X$, $\varrho_{a,x} \in \boldsymbol{E}(i_a^x X_a)$, *and if, moreover,* ϱ *is least, then* $\varrho_{a,x}$ *is least.*

Proof. If $\varrho \in \boldsymbol{E}(X)$, then $\varrho_{a,x}$ is acyclic (in fact, ϱ restricted to any subgraph is acyclic if ϱ is acyclic). Hence we need to show that $\varrho_{a,x}$ contains $\alpha \cup \beta$ on $i_a^x X_a$ to establish that $\varrho_{a,x} \in \boldsymbol{E}(i_a^x X_a)$. Let $e, e' \in E(i_a^x X_a)$ with $e\alpha e'$ on $i_a^x X_a$, i.e., e and e' are adjacent and among the saturated subgraphs of $i_a^x X_a$ there does not exist a 4-circuit. If $\overline{e \varrho_{a,x} e'}$, then $e \overline{\varrho} e'$ and hence $e \overline{\alpha} e'$ on X. But e, e' adjacent implies that e, e' are contained in a saturated 4-circuit, say C. Since $e, e' \in E(i_a^x X_a)$, it is easily verified that $C \subset i_a^x X_a$, a contradiction. Hence $e\varrho_{a,x}e'$, i.e., $\varrho_{a,x}$ contains α on $i_a^x X_a$. If $e\beta e'$ on $i_a^x X_a$, then $e\beta e'$ on X and hence $e\beta e'$. But $e, e' \in E(i_a^x X_a)$ implies $e\varrho_{a,x}e'$. Hence $\varrho_{a,x}$ contains $\alpha \cup \beta$ on $i_a^x X_a$.

Finally, we assume that ϱ is the least element of $E(X)$ and suppose there exists $\sigma \in E(i_a^x X_a)$ with $\sigma \subsetneqq \varrho_{a,x}$. Since the inverse of i_a^x is an isomorphism from $i_a^x X_a$ onto X_a, we see by the remark preceeding the proposition that σ is mapped to an equivalence relation in $E(X_a)$. For notational convenience we shall denote this induced equivalence by σ as well. By Proposition 10, $\tilde{\sigma} \in E(X)$ and hence $\tilde{\sigma} \cap \varrho \in E(X)$. It is easily verified that $\tilde{\sigma} \cap \varrho \subsetneqq \varrho$. This contradicts the minimality of ϱ and hence $\varrho_{a,x}$ is the least element of $E(i_a^x X_a)$.

COROLLARY 2. *Let* $X = \prod_{a \in A} (X_a, x_a)$ *be connected, and for each* $a \in A$ *let* σ_a *be the least element of* $E(X_a)$. *Then* $\bigcap_{a \in A} \tilde{\sigma}_a$ *is the least element of* $E(X)$.

Proof. Since $\tilde{\sigma}_a \in E(X)$ and $E(X)$ is a principal filter, $\bigcap_{a \in A} \tilde{\sigma}_a \in E(X)$. Let $\sigma = \bigcap_{a \in A} \tilde{\sigma}_a$ and let ϱ be the least element of $E(X)$. Let $e, e' \in E(X)$ with $e\sigma e'$. Now $e\sigma e'$ implies $a(e) = a(e') = a$ and $\mathrm{pr}_a e \sigma_a \mathrm{pr}_a e'$. By Proposition 11, $\varrho_{a,x}$ is the least element of $E(i_a^x X_a)$. This together with σ_a the least element of X_a and $i_a^x \colon X_a \to i_a^x X_a$ an isomorphism imply $(i_a^x \mathrm{pr}_a e)$ $\varrho (i_a^x \mathrm{pr}_a e')$. By Proposition 9, $e\varrho_0 (i_a^x \mathrm{pr}_a e)$ and $e' \varrho_0 (i_a^x \mathrm{pr}_a e')$ and hence $e\varrho (i_a^x \mathrm{pr}_a e)$ and $e' \varrho (i_a^x \mathrm{pr}_a e')$. Therefore $e\varrho e'$, that is, $\sigma \subset \varrho$, but ϱ least implies $\varrho = \sigma$.

Example 1. The connectedness of X is actually needed in the previous corollary as is seen in the following example:

Take $X_1 = C(2)$, $X_2 = C(2)$ together with an isolated vertex, where $C(2)$ denotes the complete graph on two vertices. Then $X_1 \times X_2$ is as in fig. 1.

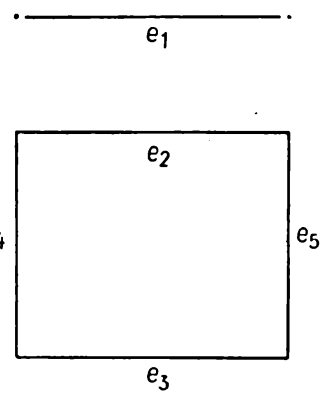

Fig. 1

Now ϱ_i the least element of $E(X_i)$, $i = 1, 2$, implies $\tilde{\varrho}_1 \cap \tilde{\varrho}_2$ partitions $E(X_1 \times X_2)$ into the two classes $\{e_1, e_2, e_3\}$ and $\{e_4, e_5\}$. However, the least element of $E(X_1 \times X_2)$ partitions $E(X_1 \times X_2)$ into the three classes $\{e_1\}$, $\{e_2, e_3\}$ and $\{e_4, e_5\}$.

7. Unique Prime Factorization Theorem. In this section we will show that if X is a connected graph and if ϱ is an equivalence in $\boldsymbol{E}(X)$, then ϱ determines a weak cartesian decomposition of X such that for two edges $e, e' \in E(X)$, $e\varrho e'$ if and only if $a(e) = a(e')$. Moreover, we will show that if ϱ is the least element of $\boldsymbol{E}(X)$, then the factorization determined by ϱ is a prime factorization that is unique up to within isomorphisms. Finally, we will establish a conjecture of Sabidussi's [1], p. 449, namely that if X is a connected idempotent graph (i.e. a graph X for which $X \times X \cong X$), then X does not have a cartesian decomposition into indecomposable factors. We proceed by first proving a proposition based on the following definition:

Definition 7. Let ϱ be any equivalence on $E(X)$. A subgraph Y of X will be called *ϱ-saturated* if (i) Y is connected, and (ii) $e\varrho e'$ for $e \in E(X)$, $e' \in E(Y)$ implies $Y \cup (e)$ is disconnected or $e \in E(Y)$.

This is equivalent to saying that there exists a set $\boldsymbol{A}$ of equivalence classes modulo ϱ such that Y is a maximal connected subgraph of X with $E(Y) \subset \bigcup \boldsymbol{A}$.

Proposition 13. *Let ϱ be any equivalence on $E(X)$ containing $\alpha \cup \beta$, Y a ϱ-saturated subgraph of X. Then any two distinct vertices of Y can be joined by a shortest path in Y which is ϱ-compatible.*

Proof. We fix $x \in Y$ and use induction on $d(x, y)$, the distance of x and y in Y. For $k = 1, 2, \ldots$ put

$$A_k = \{y \in Y \mid d(x, y) = k\}.$$

If $y \in A_1$, then $e = [x, y] \in Y$, hence $P = (e)$ trivially is *a* ϱ-compatible path joining x and y. Assume the proposition true for all $z \in A_k$, and let $y \in A_{k+1}$. Then there is a $z \in A_k$ with $e = [y, z] \in Y$. By the induction hypothesis there is a ϱ-compatible path $P = P_1 \cup \ldots \cup P_n$ such that P is a shortest path joining z and x in Y. If $\{e\} \times E(P_i) \subset \bar{\varrho}$ for $i = 1, \ldots, n$, the proof is complete because $P_1, \ldots, P_n$ and $P_{n+1} = e$ would form the required ϱ-compatible path form x to y. We may therefore assume that $\{e\} \times E(P_m) \subset \varrho$ for some m, $1 \leqslant m \leqslant n$. We may then assume that $2 \leqslant m \leqslant n$, for if $m = 1$, then $e \cup P_1, P_2, \ldots, P_n$ form a ϱ-compatible shortest path from x to y. By Construction 1 there exist paths Q_1, Q_2, $\ldots, Q_{m-1}$ and an edge e' such that

$$Q_1, \ldots, Q_{m-1}, e' \cup P_m, P_{m+1}, \ldots, P_n$$

form a ϱ-compatible shortest path from x to y. (By the maximality of Y, $Q_1, \ldots, Q_{m-1}$ and e' belong to Y.)

COROLLARY 3. *If $\varrho \in E(X)$, then any ϱ-saturated subgraph of X is saturated.*

Proof. Let Y be a ϱ-saturated subgraph of X. Suppose there exist two distinct vertices $x, y \in Y$ such that $e = [x, y] \in E(X) - E(Y)$. By the maximality of Y, e is not equivalent to any edge of Y. By the previous proposition there is a ϱ-compatible path P joining x and y in Y. Hence $P \cup e$ is a ϱ-compatible circuit, contrary to the acyclicity of ϱ.

PROPOSITION 14 ([1], (2.8)). *Let X be a connected graph and let ϱ be an equivalence relation on $E(X)$ containing $\alpha \cup \beta$. Then given any equivalence class E modulo ϱ, there is an $e \in E$ which is incident with x.*

Let X be a connected graph and let $\varrho \in E(X)$. Denote the collection of equivalence classes of $E(X)$ modulo ϱ by E_ν, $0 \leqslant \nu < \nu_0$, ν_0 an ordinal. For any vertex $z \in X$ and any ordinal ν, $0 \leqslant \nu < \nu_0$, *let Y_ν^z be the largest connected subgraph of X such that $z \in Y_\nu^z$ and $E(Y_\nu^z) \subset E_\nu$.* For any vertex $z \in X$ and any ordinal ν, $0 < \nu \leqslant \nu_0$, let *X_ν^z be the largest connected subgraph of X such that $z \in X_\nu^z$ and $E(X_\nu^z) \subset \bigcup_{\mu<\nu} E_\mu$.*

Note that $Y_\mu^z \subset X_\nu^z$ and $X_\mu^z \subset X_\nu^z$ for $0 \leqslant \mu < \nu \leqslant \nu_0$. By Proposition 14, $E(X_\nu^z) \cup E_\mu \neq \emptyset$, for $0 \leqslant \mu < \nu \leqslant \nu_0$. Moreover, we note that Y_ν^z and X_ν^z are ϱ-saturated subgraphs of X.

In our succeeding considerations we will let r be an arbitrary but fixed vertex of X, and for convenience we will denote Y_ν^r by Y_ν and X_ν^r by X_ν. The following proposition will be used in the proof of Theorem 2:

PROPOSITION 15. *Let x and y be distinct vertices of $X_{\nu+1}$, $0 < \nu < \nu_0$. Then X_ν^y and Y_ν^x have exactly one vertex in common.*

Proof. We first show $X_\nu^y \cap Y_\nu^x \neq \emptyset$. Assume the contrary, and let x_0 and y_0 be chosen such that

(i) $x_0 \in X_\nu^y$, $y_0 \in Y_\nu^x$, and

(ii) among all pairs of vertices having property (i), $d(x_0, y_0)$ is minimal, where $d(x_0, y_0)$ is the distance of x_0 and y_0 in $X_{\nu+1}$.

Let $P = P_0 \cup \ldots \cup P_n$ be a shortest path joining x_0 and y_0 in $X_{\nu+1}$ which is ϱ-compatible (by Proposition 13 such a path exists since $X_{\nu+1}$ is a ϱ-saturated subgraph of X). $E(P_0) \subset E_\nu$ and $n \geqslant 1$; otherwise we get a contradiction to (ii). Denote the consecutive vertices of P_0 by $x_0, x_1, \ldots, x_m$ and let $e = [x_{m-1}, x_m]$.

By Construction 1 we can construct a ladder in $X_{\nu+1}$ from e and $P' = P_1 \cup \ldots \cup P_n$. Let $Q = Q_1 \cup \ldots \cup Q_n$ denote the path of the ladder opposite P' and let $e' = [y_0, y_1]$ denote the path of the ladder opposite e. Clearly, $y_1 \in Y_\nu^x$, and $d(x_0, y_1) < d(x_0, y_0)$. This is a contradiction against (ii) and hence $X_\nu^y \cup Y_\nu^x \neq \emptyset$.

Now suppose there exists at least two distinct vertices z_1 and z_2 in $X_\nu^y \cap Y_\nu^x$. By Proposition 15 there exist a ϱ-compatible path P' joining

z_1 and z_2 in X_ν^y and a ϱ-compatible path Q' joining z_1 and z_2 in Y_ν^x. Clearly $P' \cup Q'$ is a connected, ϱ-compatible subgraph of X which is not acyclic, and hence, by lemma 2, $P' \cup Q'$ contains a finite ϱ-compatible circuit. This contradicts the acyclicity of ϱ.

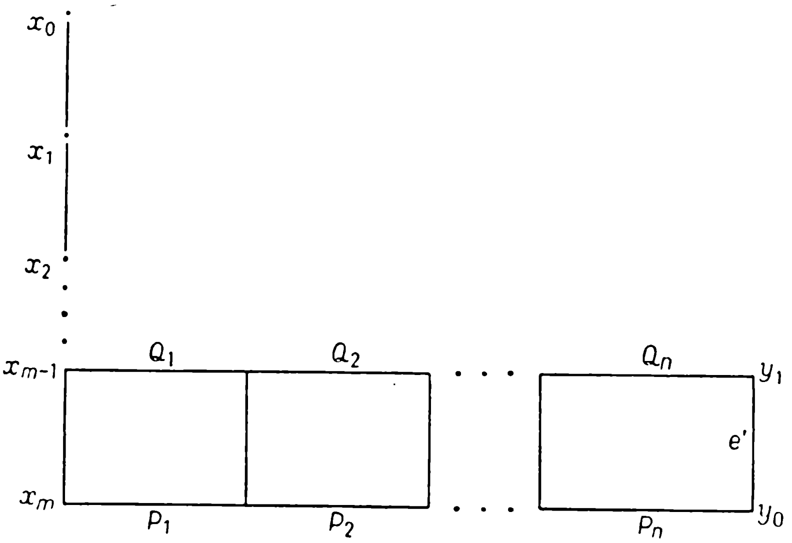

Fig. 2

THEOREM 2. *Let X be a connected graph, $r \in X$, $\varrho \in \boldsymbol{E}(X)$. Then $X = \prod_{\nu<\nu_0} (Y_\nu, r_\nu)$, where $r_\nu = r$ for $0 \leqslant \nu < \nu_0$. Moreover, if ϱ is the least element of $\boldsymbol{E}(X)$, then each Y_ν, $0 \leqslant \nu < \nu_0$, is indecomposable.*

Proof. The proof is by transfinite induction. Let $Z = \prod_{\nu<\nu_0} (Y_\nu, r_\nu)$ and for each ordinal ν, $0 < \nu \leqslant \nu_0$, let $Z_\nu = \prod_{\mu<\nu_0} (Y'_\mu, r'_\mu)$, where $(Y'_\mu, r'_\mu) = (Y_\mu, r_\mu)$ for $\mu < \nu$ and $(Y'_\mu, r'_\mu) = \big((r), r\big)$ for $\nu \leqslant \mu < \nu_0$.

Let ψ_1: $X_1 \to Z$ be defined by

$$\mathrm{pr}_\lambda \psi_1(x) = \begin{cases} x, & \lambda = 0, \\ r, & \lambda \neq 0. \end{cases}$$

Then ψ_1 is clearly a monomorphism such that $\psi_1(X_1) = Z_1$.

Suppose there exists a monomorphism ψ_ν: $X_\nu \to Z$ with $\psi_\nu(X_\nu) = Z_\nu$. Then we can construct a monomorphism $\psi_{\nu+1}$: $X_{\nu+1} \to Z$ with $\psi_{\nu+1}(X_{\nu+1}) = Z_{\nu+1}$ and such that $\psi_{\nu+1} | X_\nu = \psi_\nu$. First define φ_ν:$X_\nu \times Y_\nu \to X_{\nu+1}$ by $\varphi_\nu(x, y) = z_{xy}$, where z_{xy} is the unique vertex in $X_\nu^y \cap Y_\nu^x$ given by the previous proposition. We will show that φ_ν is an isomorphism.

Let z be an arbitrary vertex in $X_{\nu+1}$. By Proposition 15, Y_ν^z and X_ν have exactly one vertex in common, say x, and X_ν^z and Y_ν have exactly one vertex in common, say y; clearly, $Y_\nu^z = Y_\nu^x$ and $X_\nu^z = X_\nu^y$. Hence $z = z_{xy}$. Thus every vertex of $X_{\nu+1}$ has a unique pre image with respect to φ_ν and hence φ_ν is one-one and onto.

To prove that φ_ν is a homomorphism take $[(x, y), (x', y')] \epsilon X_\nu \times Y_\nu$. Then either $[x, x'] \epsilon X_\nu$ and $y = y'$, or $[y, y'] \epsilon Y_\nu$ and $x = x'$. Suppose $[x, x'] \epsilon X_\nu$ and $y = y'$. Let P be a shortest path joining x and z_{xy} in Y_ν^x. Construct a ladder from $[x, x']$ and P, and let e be the final edge opposite $[x, x']$. Then e is incident with z_{xy}, $e \epsilon X_\nu^y$ and hence clearly $e = [z_{xy}, z_{x'y'}] \epsilon X_{\nu+1}$. Similarly, if $[y, y'] \epsilon Y_\nu$ and $x = x'$, then $[z_{xy}, z_{x'y'}] \epsilon X_{\nu+1}$. Hence $[(x, y), (x', y')] \epsilon X_\nu \times Y_\nu$ implies $[z_{xy}, z_{x'y'}] \epsilon X_{\nu+1}$.

To prove that φ_ν is an epimorphism, let $e = [z_{xy}, z_{x'y'}] \epsilon X_{\nu+1}$. Then $e \epsilon \bigcup_{\mu \leqslant \nu} E_\mu$ and hence either $e \epsilon E_\nu$ or $e \epsilon \bigcup_{\mu<\nu} E_\mu$. If $e \epsilon E_\nu$, then $e \epsilon Y_\nu^x$ and $e \epsilon Y_\nu^{x'}$ by the maximality of Y_ν^x, $Y_\nu^{x'}$. Hence $x = x'$. Moreover, it is easy to verify that $[y, y'] \epsilon Y_\nu$. If $e \epsilon \bigcup_{\mu<\nu} E_\mu$, then $e \epsilon X_\nu^y$ and $e \epsilon X_\nu^{y'}$, and hence $y = y'$. Again it is easy to show that $[x, x'] \epsilon X_\nu$. Hence φ_ν is an isomorphism.

Define $\eta_\nu\colon X_\nu \times Y_\nu \to Z$ by

$$\mathrm{pr}_\lambda \eta_\nu(x, y) = \begin{cases} \mathrm{pr}_\lambda \psi_\nu(x), & \lambda < \nu, \\ y, & \lambda = \nu, \\ r, & \nu < \lambda < \nu_0. \end{cases}$$

Set $\psi_{\nu+1} = \eta_\nu \varphi_\nu^{-1}$. Clearly, $\psi_{\nu+1}\colon X_{\nu+1} \to Z$ is a monomorphism with

$$\psi_{\nu+1}(X_{\nu+1}) = Z_{\nu+1} \quad \text{and} \quad \psi_{\nu+1} | X_\nu = \psi_\nu.$$

Next let ν be a limit ordinal and assume that for each ordinal $\mu < \nu$ there exists a monomorphism $\psi_\mu\colon X_\mu \to Z$ with $\psi_\mu(X_\mu) = Z_\mu$ and such that $\psi_\mu | X_\lambda = \psi_\lambda$ for $\lambda < \mu$. $X_\nu = \bigcup_{\mu<\nu} X_\mu$ and hence $x \epsilon X_\nu$ implies $x \epsilon X_\mu$ for some $\mu < \nu$. Set $\psi_\nu(x) = \psi_\mu(x)$. Then clearly $\psi_\nu\colon X_\nu \to Z$ is a monomorphism and $\psi_\nu(X_\nu) = Z_\nu$. Hence $X_\nu \cong \prod_{\mu<\nu} (Y_\mu, r_\mu)$. Since $X = X_{\nu_0}$ and $Z = Z_{\nu_0}$, we have $X \cong \prod_{\mu<\nu_0} (Y_\mu, r_\mu)$.

Finally, we assume that ϱ is the least element of $\boldsymbol{E}(X)$. Set $\psi = \psi_{\nu_0}^{-1}$. Then $\psi\colon Z \to X$ an isomorphism implies that ϱ_ψ is the least element of $\boldsymbol{E}(X)$. Denote ϱ_ψ by σ. Then $\sigma_{\nu,r}$ (recall that $\sigma_{\nu,r}$ is the restriction of σ to $i_\nu^r Y_\nu$) is the least element of $\boldsymbol{E}(i_\nu^r Y_\nu)$. Since $i_\nu^r Y_\nu$ consists of exactly one equivalence class modulo $\sigma_{\nu,r}$ we have $|\boldsymbol{E}(i_\nu^r Y_\nu)| = 1$. Therefore $|\boldsymbol{E}(Y_\nu)| = |\boldsymbol{E}(i_\nu^r Y_\nu)| = 1$ and hence, by Remark 2, Y_ν is indecomposable.

Proposition 16. *Let $X = \prod_{a \epsilon A} (X_a, x_a)$ be connected, and for each $a \epsilon A$ let X_a be indecomposable. Let ϱ be the least element of $\boldsymbol{E}(X)$. Then for $e, e' \epsilon \boldsymbol{E}(X)$, $e \varrho e'$ if only if $a(e) = a(e')$.*

Proof. Since X_a is indecomposable for each $a \epsilon A$, we infer that $\varrho_a = E(X_a) \times E(X_a)$ is the least element of $\boldsymbol{E}(X_a)$. By Corollary 2 we have $\varrho = \bigcap_{a \epsilon A} \tilde{\varrho}_a$ and hence by Remark 2 we see that $e \; \varrho \; e'$ if and only if $a(e) = (e')$.

We are now in a position to prove the following theorem which is our main result:

THEOREM 3. *If X is a connected graph, then X has a weak cartesian decomposition into indecomposable factors which is unique to within isomorphisms.*

Proof. From Theorem 2 we obtain that the least element ϱ of $\boldsymbol{E}(X)$ determines a weak cartesian decomposition of X into indecomposable factors, where the factors are taken to be ϱ-saturated subgraphs with respect to the individual equivalence classes of $E(X)$ modulo ϱ. If we take any other decomposition of X into prime factors, we infer by Proposition 16 that the number of factors in each decomposition is the same. Since the injection mappings are monomorphisms, and the injections of these latter prime factors are ϱ-saturated subgraphs with respect to the individual equivalence classes of $E(X)$ modulo ϱ, the decomposition is unique to within isomorphisms.

Theorem 3 is an extension of Sabidussi's Theorem which we will now quote. We require a further definition.

Definition 8. Let γ be a function from the class of all graphs to the class of all cardinals such that

(i) if $X \cong Y$, then $\gamma(X) = \gamma(Y)$;

(ii) if $X \subset Y$, then $\gamma(X) \leqslant \gamma(Y)$;

(iii) if $\gamma(X) < \aleph_0$, $\gamma(Y) < \aleph_0$, and both X and Y are non-trivial, then $\max(\gamma(X), \gamma(Y)) < \gamma(X \times Y) < \aleph_0$.

A connected graph X is said to be of *finite type* if there exists a γ satisfying (i), (ii) and (iii) for which $\gamma(X) < \aleph_0$.

THEOREM (Sabidussi). *Let X be a connected graph of finite type, or a connected graph containing a vertex of finite degree. Then X has a cartesian decomposition into indecomposable factors that is unique to within isomorphisms.*

We complete this section by showing that if X is a connected indempotent graph, then X does not have a cartesian decomposition into indecomposable factors.

PROPOSITION 17. *Let X be connected. $\boldsymbol{E}(X)$ is finite if and only if X has a cartesian decomposition into indecomposable factors.*

Proof. Let ϱ be the least element of $\boldsymbol{E}(X)$ and let n be equal to the number of equivalence classes of $E(X)$ modulo ϱ. By Corollary 1 we obtain that $\boldsymbol{E}(X)$ is finite if and only if n is finite. First suppose that X has a cartesian decomposition into indecomposable factors. Proposition 16 then implies that n is finite. The sufficiency follows from Theorem 2.

COROLLARY 4. *If X is a connected indempotent graph, then X does not have a cartesian decomposition into indecomposable factors.*

Proof. Let

$$f_n: X \to \prod_{i=1}^{n} X_i$$

be an isomorphism, where $X_i = X$ $(i = 1, \ldots, n)$. For $e = [x, y]$, $e' = [x', y'] \epsilon E(X)$ define $e\varrho^n e'$ if and only if

$$[\mathrm{pr}_i f_n(x), \mathrm{pr}_i f_n(y)] = [\mathrm{pr}_i f_n(x'), \mathrm{pr}_i f_n(y')] \epsilon E(X_i)$$

for some i. $\varrho^n \epsilon \boldsymbol{E}(X)$ and, by Remark 2, ϱ^n has exactly n equivalences classes. Hence $n \neq m$ implies $\varrho^n \neq \varrho^m$. Therefore $\boldsymbol{E}(X)$ is infinite and henoe, by Proposition 17, X does not have a cartesian decomposition.

Added in proof. Recently our Theorem 3 has independently been proved by W. Imrich, *Über das schwache kartesische Produkt von Graphen* (to appear in Journal of Combinatorial Theory).

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