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Categories, groupoids, pseudogroups and analytical structures

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INTRODUCTION

The aim of this paper is to discuss some fundamental notions of local geometry and to explain the relationships existing between them. In Chapter II we give the axioms of the groupoid theory without making use of the Eilenberg and MacLane category notion (see [4]). In this chapter we prove that a multiplicative system satisfies those axioms if and only if it is a groupoid in the sense of Ehresmann (see [3]) and thus it is a particular case of the category notion. The Ehresmann groupoid notion appears to be closely related to another groupoid notion introduced earlier by H. Brandt (see [2]). Nijenhuis in his paper [12] explains the structure of Brandt groupoids by formulating the relations existing between the notion of a groupoid and that of a group. The theorem proved in paragraph 3 of Chapter II of the present paper enables us to explain the structure of Ehresmann groupoids with the aid of Brandt's notion of groupoid.

Speaking of a category and a groupoid (see [3]) we assume that the algebraical operations defined in them are defined in sets and not in classes in the sense of N. Bourbaki. Thus the considerations contained in this paper may be asserted in terms of the set theory.

On the other hand, we prove in Chapter II a representation theorem for categories, whence we deduce an analogous theorem for Ehresmann groupoids. This theorem in turn gives the relation between Ehresmann groupoids and an earlier notion of a Gołąb pseudogroup (see [8]). It appears that any Ehresmann groupoid may be reduced by an isomorphism to a Gołąb pseudogroup in a suitably chosen topological space. In Chapter III we prove that the sets of all possible functional elements determined by functions belonging to a given pseudogroup form a groupoid with the composition of elements as the algebraic operation. Chapter IV is entirely devoted to the problem of generating pseudogroups by arbitrary sets of local homeomorphisms in a given topological space. As S. Gołąb has proved, each element of a pseudogroup is a local homeomorphism. On the other hand, the set of all local homeomorphisms of a given space forms a pseudogroup in that space. Thus it is a pseudogroup including in it all the pseudogroups of the given space. Thus, in a natural manner, the problem arises how to define, for a set, given a priori, of local homeomorphisms, the pseudogroup generated by this set. This problem is much more complicated than the corresponding problem in the group theory, since it appears that if is not for every set of local homeomorphisms that there exists a pseudogroup, including a given set of local homeomorphisms, which is the smallest, in the sense of inclusion. In this chapter we prove that of all the pseudogroups including a given set of local homeomorphisms the generated pseudogroup gives the smallest possible set of functional elements. Chapter V contains some generalization of the Gołąb pseudogroup and an attempt at generalizing the notions of the differentiable manifold and of analytical structure. The notion of analytical structure introduced here is independent of the notions of analysis, such as the notions of the derivative and of Cartesian space.

I. TERMS AND NOTATION

The symbols: $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\bigvee$, $\bigwedge$ denote respectively the alternative, the conjunction, the implication, the equivalence of sentences, the existential quantifier and the generality quantifier.

If R is a set whose elements are also sets, then by $\bigcup R$ and $\bigcap R$ we mean the union and the intersection of all sets belonging to R respectively. By $\subset$ we mean the inclusion of the sets. The product of the sets A and B will be denoted by $A \times B$.

By a *function* we always mean a binary relation unambiguous with respect to the second argument and by a *relation* we mean a subset of the product of a certain set and the same set (see [11]). For a given function f by D_f we mean the *domain* of the function f. For a given set $A \subset D_f$ by $f(A)$ we mean the *image* of the set A by the function f. In particular, the set $f(D_f)$ is the set of all values of the function f. For a given set $A \subset D_f$, by $f|A$ we denote a function of the form $f \cap (A \times f(D_f))$. This is what we call a function reduced to the set A. We obviously have $D_{f|A} = A$. The notion of an operation is a particular case of the notion of a function. By an *operation defined in the set A* we mean a function f satisfying the conditions $D_f \subset A \times A$ and $f(D_f) \subset A$. The set D_f will be called the *domain of the operation f*.

By $\{x: P(x)\}$ we denote the set of all x satisfying the condition $P(x)$.

If F is a function whose values are sets, and if $P(t)$ is the condition such that

$$\bigwedge_t (P(t) \Rightarrow t \in D_F);$$

then by $\{F(t): P(t)\}$ we mean

$$\{A: \bigvee_t (P(t) \wedge A = F(t))\}.$$

For a given topological space X we denote by $\underline{X}$ and $\omega(X)$ the set of all the points and the set of all the open sets of that space respectively. In this paper we only assume that

$$\underline{X} \in \omega(X), \qquad \bigwedge_{M \subset \omega(X)} \left(\bigcup M \in \omega(X)\right), \qquad \bigwedge_{A,B \in \omega(X)} \left(A \cap B \in \omega(X)\right).$$

For a given topological space X and a set $A \subset \underline{X}$ we denote by $X|A$ a space induced by the space X in the set A. Hence it follows that $\omega(X|A) = \omega(X)|A$.

By $(X \to Y)$ we mean the set of all functions mapping continuously a topological space X into a topological space Y. Similarly by $(X \leftrightarrow Y)$ we mean the set of all the homeomorphisms of the space X into the space Y.

II. GROUPOIDS AND CATEGORIES

1. The notion of groupoid. An ordered pair $(C,\cdot)$, where C denotes a set and an operation defined in this set, will be called a *multiplicative system* (see [5]). The operation may be defined merely for some ordered pairs $(x, y) \in C\times C$. By R we denote the domain of the operation Let C_0 denote a set defined by the equality

$$(1.0) \qquad C_0 = \{e\colon\ e \in C \wedge (e, e) \in R \wedge e\cdot e = e\}.$$

An ordered pair $(C,\cdot)$ will be called a *groupoid* if it satisfies the following axioms:

$$(1.1) \qquad \bigwedge_{x,y,z}\Big(\big((x, y) \in R \wedge (y, z) \in R\big) \Rightarrow (x, y\cdot z) \in R\Big),$$

$$(1.2) \qquad \bigwedge_{x,y,z}\Big(\big((x, y) \in R \wedge (y, z) \in R\big) \Rightarrow (x\cdot y, z) \in R\Big),$$

$$(1.3) \qquad \bigwedge_{x,y,z}\Big(\big((y, z) \in R \wedge (x, y\cdot z) \in R\big) \Rightarrow (x, y) \in R\Big),$$

$$(1.4) \qquad \bigwedge_{x,y,z}\Big(\big((x, y) \in R \wedge (x\cdot y, z) \in R\big) \Rightarrow (y, z) \in R\Big),$$

$$(1.5) \qquad \bigwedge_{x,y,z}\Big(\big((x, y) \in R \wedge (y, z) \in R \wedge (x\cdot y, z) \in R \wedge (x, y\cdot z) \in R\big) \Rightarrow (x\cdot y)\cdot z = x\cdot(y\cdot z)\Big),$$

$$(1.6) \qquad \bigwedge_{x,y,z}\Big(\big((x, y) \in R \wedge (x, z) \in R \wedge x\cdot y = x\cdot z\big) \Rightarrow y = z\Big),$$

$$(1.7) \qquad \bigwedge_{x,y,z}\Big(\big((y, x) \in R \wedge (z, x) \in R \wedge y\cdot x = z\cdot x\big) \Rightarrow y = z\Big),$$

$$(1.8) \qquad \bigwedge_{x\in C} \bigvee_{y\in C}\big((x, y) \in R \wedge x\cdot y \in C_0\big).$$

We shall prove that

$$(1.9) \qquad \bigwedge_{e\in C_0} \bigwedge_{x}\Big(\big((x, e) \in R \Rightarrow x\cdot e = x\big) \wedge \big((e, x) \in R \Rightarrow e\cdot x = x\big)\Big)$$

Proof. Suppose $e \in C_0$. Let $x \in C$ and $(x, e) \in R$. From the definition of the set C_0 it follows that $(e, e) \in R$. Hence by axioms (1.1) and (1.2) we obtain $(x\cdot e, e) \in R$ and $(x, e\cdot e) \in R$. From (1.5) it follows that

$$(x\cdot e)\cdot e = x\cdot(e\cdot e) = x\cdot e.$$

Hence by (1.7) $x\cdot e = x$. Suppose now that $(e, x) \in R$. From (1.1), (1.2) and (1.5) it follows that $(e\cdot e, x) \in R$ and

$$e\cdot(e\cdot x) = (e\cdot e)\cdot x = e\cdot x.$$

Hence by (1.6) we get $x = e\cdot x$, which completes the proof.

We shall prove the following proposition:

(1.10) $$\bigwedge_{x\in C}\bigwedge_{e_1,e_2\in C_0}\Big(\big((x,e_1)\in R\wedge(x,e_2)\in R\big)\Rightarrow e_1=e_2\Big).$$

Proof. It follows from (1.9) that $x\cdot e_1 = x = x\cdot e_2$. Hence by the axiom (1.6) we get $e_1 = e_2$.

One can prove similarly that

(1.11) $$\bigwedge_{x\in C}\bigwedge_{e_1,e_2\in C_0}\Big(\big((e_1,x)\in R\wedge(e_2,x)\in R\big)\Rightarrow e_1=e_2\Big).$$

We shall prove that

(1.12) $$\bigwedge_{x,y}\big((x,y)\in R\wedge x\cdot y\in C_0\big)\Rightarrow\big((y,x)\in R\wedge y\cdot x\in C_0\big).$$

Proof. Consider two arbitrary elements x and y of the set C such that $(x,y)\in R$ and $x\cdot y\in C_0$. It follows from the definition of the set C_0 that $(x\cdot y, x\cdot y)\in R$. Hence by (1.1)-(1.5) and (1.10) we get $(y, x\cdot y)\in R$, $(y,x)\in R$, $(x, y\cdot x)\in R$, $(y\cdot x, y\cdot x)\in R$, $\big(y, x\cdot(y\cdot x)\big)\in R$ and

$$(y\cdot x)\cdot(y\cdot x) = y\cdot\big(x\cdot(y\cdot x)\big) = y\cdot\big((x\cdot y)\cdot x\big) = y\cdot x,$$

which ends the proof.

(1.13) $$\bigwedge_{x\in C}\bigvee_{e_1,e_2\in C_0}\big((x,e_1)\in R\wedge(e_2,x)\in R\big).$$

Proof. Consider an arbitrary element x of the set C. Axiom (1.8) implies the existence of a $y\in C$ such that $(x,y)\in R$ and $x\cdot y\in C_0$. By (1.12) we deduce that $(y,x)\in R$ and $y\cdot x\in C_0$. To finish the proof it suffices to observe that $(x, y\cdot x)\in R$ and $(x\cdot y, x)\in R$.

Now we shall prove the following equality:

(1.14) $$C_0 = \{e\colon\ e\in C\wedge\bigwedge_{x\in C}\Big(\big((x,e)\in R\Rightarrow x\cdot e = x\big)\wedge\big((e,x)\in R\Rightarrow e\cdot x = x\big)\Big)\},$$

Proof. Suppose that e belongs to the set occurring in the right member of equality (1.14). By (1.13) there exists an $e_1\in C_0$ such that $(e, e_1)\in R$. Hence it follows that $e\cdot e_1 = e_1$. On the other hand, by (1.9) we have $e\cdot e_1 = e$. Hence $e = e_1$. Thus $(e,e)\in R$ and $e\cdot e = e$. In other words $e\in C_0$. It follows immediately from (1.9) that the set C_0 is included in the set occurring in the right member of equality (1.14), which completes the proof.

(1.15) $$\bigwedge_{x,y,z}\Big(\big((x,y)\in R\wedge(x,z)\in R\wedge x\cdot y\in C_0\wedge x\cdot z\in C_0\big)\Rightarrow y = z\Big).$$

(1.16) $$\bigwedge_{x,y,z}\Big(\big((y,x)\in R\wedge(z,x)\in R\wedge y\cdot x\in C_0\wedge z\cdot x\in C_0\big)\Rightarrow y = z\Big).$$

Proof of (1.15). Consider arbitrary elements x, y, z satisfying the conditions contained in the assumption of implication (1.15). From the assertion of (1.12) it follows that $(z, x) \in R$ and $z \cdot x \in C_0$. Hence by axioms (1.1) and (1.2) we deduce that $(z, x \cdot y) \in R$ and $(x \cdot z, x \cdot y) \in R$. Thus, since $(x \cdot z, x \cdot z) \in R$ by the assertion of (1.10), we obtain the equality $x \cdot y = x \cdot z$. Hence by axiom (1.6) we obtain $y = z$.

The proof of assertion (1.16) is similar to that of assertion (1.15).

For an arbitrary $x \in C$ we shall denote by x^{-1} an element of the set C, unique by (1.15), which satisfies the conditions $(x, x^{-1}) \in C$ and $x \cdot x^{-1} \in C_0$. The existence of such an element follows from axiom (1.8). It follows from (1.12) and (1.16) that the element x^{-1} is also characterized by the condition $(x^{-1}, x) \in R$ and $x^{-1} \cdot x \in C_0$. We call the element x^{-1} an *element inverse to* x.

2. Equivalence of the definition of groupoid to the definition of Ehresmann. In their book devoted to the foundations of algebraic topology S. Eilenberg and N. Steenrod formulate the definition of the notion of abstract category as follows:

"DEFINITION 2.1. A set C of elements $\{\gamma\}$ is called a *multiplicative system* if, for some pairs $\gamma_1, \gamma_2 \in C$, a product $\gamma_2\gamma_1 \in C$ is defined. An element $\varepsilon \in C$ is called an *identity* (or a *unit*) if $\varepsilon\gamma_1 = \gamma_1$ and $\gamma_2\varepsilon = \gamma_2$ whenever $\varepsilon\gamma_1$ and $\gamma_2\varepsilon$ are defined. The multiplicative system is called an *abstract category* if the following axioms are satisfied:

(1) *The triple product $\gamma_3(\gamma_2\gamma_1)$ is defined if and only if $(\gamma_3\gamma_2)\gamma_1$ is defined.* When either is defined the associative law

$$\gamma_3(\gamma_2\gamma_1) = (\gamma_3\gamma_2)\gamma_1$$

holds. This triple product will be written as $\gamma_3\gamma_2\gamma_1$.

(2) *The triple product $\gamma_3\gamma_2\gamma_1$ is defined whenever both products $\gamma_3\gamma_2$ and $\gamma_2\gamma_1$ are defined.*

(3) *For each $\gamma \in C$ there exist identities $\varepsilon_1, \varepsilon_2 \in C$ such that $\gamma\varepsilon_1$ and $\varepsilon_2\gamma$ are defined.*"

From this definition it follows that the conjunction of the axioms considered in it is equivalent to the conjunction of axioms (1.1)-(1.5) and of the propositions (1.10), (1.11) and (1.13). In this connection the set C_0 occurring in the axioms (1.10), (1.11) and (1.13) is to be defined by equality (1.14). Thus every groupoid is an abstract category.

The notion of an abstract category is a modification of the notion of category introduced by S. Eilenberg and S. MacLane in their paper [4]. The notion of abstract category, termed briefly *category*, also appears in paper [3] by Ch. Ehresmann.

If $(C, \ )$ is a category, then by (1.10), (1.11) and (1.13) for every $x \in C$ there exist exactly one element $\alpha(x) \in C_0$ such that $(x, \alpha(x)) \in R$ and exactly one element $\beta(x) \in C_0$ such that $(\beta(x), x) \in R$.

Ch. Ehresmann, in paper [3], uses the name groupoid for a category satisfying the following condition:

$$(2.1) \qquad \bigwedge_{x\in C} \bigvee_{y\in C} \big((x, y)\in R \wedge (y, x)\in R \wedge x\cdot y = \beta(x) \wedge y\cdot x = \alpha(x)\big).$$

We shall prove the following theorem.

THEOREM. *In order that a multiplicative system be a groupoid defined by axioms* (1.1)-(1.8) *it is necessary and sufficient that it is a groupoid in the sense of Ehresmann.*

Proof. Suppose that the multiplicative system $(C, \cdot)$ satisfies axioms (1.1)-(1.8). Then it is a category. Consider an arbitrary element $x\in C$. Then $(x, x^{-1}\cdot x)\in R$, $\big(x, \alpha(x)\big)\in R$, $x^{-1}\cdot x\in C_0$ and $\alpha(x)\in C_0$. Hence by (1.10) we deduce that $x^{-1}\cdot x = \alpha(x)$. Similarly from (1.11) it follows that $x\cdot x^{-1} = \beta(x)$. Thus axiom (2.1) is satisfied.

Suppose now that the multiplicative system $(C, \cdot)$ is an Ehresmann groupoid. Axioms (1.1)-(1.5) are satisfied, for $(C, \cdot)$ is a category. In order to prove that axiom (1.6) is also satisfied consider arbitrary elements x, y, z of the set C such that $(x, y)\in R$, $(x, z)\in R$ and $x\cdot y = x\cdot z$. Axiom (2.1) implies the existence of an element $u\in C$ such that $(u, x)\in R$ and $u\cdot x = \alpha(x)$. Hence $\big(\alpha(x), y\big)\in R$, $\big(\alpha(x), z\big)\in R$ and

$$y = \alpha(x)\cdot y = (u\cdot x)\cdot y = u\cdot(x\cdot y) = u\cdot(x\cdot z) = (u\cdot x)\cdot z = \alpha(x)\cdot z = z.$$

Axiom (1.6) is thus satisfied. The proof that axiom (1.7) is also satisfied is analogous. Further we shall prove that if the set C_0 is defined by equality (1.14), then the equality

$$(2.2) \qquad C_0 = \{e\colon e\in C \wedge (e, e)\in R \wedge e\cdot e = e\}$$

holds.

Indeed, suppose that $e\in C_0$. Axiom (2.1) implies the existence of an $a\in C$ such that $(a, e)\in R$, $(e, a)\in R$, $a\cdot e = \alpha(e)$ and $e\cdot a = \beta(e)$. Hence $(e\cdot a, e)\in R$, $(e, a\cdot e)\in R$ and $\beta(e) = e\cdot a = (e\cdot a)\cdot e = e\cdot(a\cdot e) = e\cdot\alpha(e) = e$. Thus $(e, e) = \big(\beta(e), e\big)\in R$ and $e\cdot e = \beta(e)\cdot e = e$. Suppose $(e, e)\in R$ and $e\cdot e = e$. Let x be an arbitrary element of the set C such that $(x, e)\in R$. Then $x\cdot e = x\cdot(e\cdot e) = (x\cdot e)\cdot e$. Since axiom (1.7) is satisfied, then $x = x\cdot e$. Similarly $(e, x)\in R$ implies $e\cdot x = x$. Thus $e\in C_0$. Equality (2.2), which has just been proved, and axiom (2.1) immediately imply that axiom (1.8) is satisfied.

3. Relationship between the notion of an Ehresmann groupoid and the notion of a Brandt groupoid. In his paper [2] H. Brandt introduced the notion of a groupoid. The notion of an Ehresmann groupoid is more general than that of a Brandt groupoid. A Brandt groupoid may be defined as an Ehresmann groupoid satisfying the axiom

$$(3.1) \qquad \bigwedge_{x,y\in C} \bigvee_{u\in C} \big((x, u)\in R \wedge (u, y)\in R\big).$$

Consider a multiplicative system $(C, \cdot)$ and an arbitrary set $A \subset C$. Let R be the domain of the operation By $\cdot_A$ we denote the operation defined as follows:

$$x \cdot_A y = x \cdot y \quad \text{for} \quad (x, y) \in R \cap (A \times A).$$

The domain of the operation is the set $R \cap (A \times A)$.

THEOREM 1. *In order that a multiplicative system $(C, \cdot)$ be an Ehresmann groupoid it is necessary and sufficient that there exists a division $\mathfrak{A}$ of the set C into disjoint sets such that*

$$R \subset \bigcup \{M \times M \colon M \in \mathfrak{A}\}. \tag{3.2}$$

and that for every $A \in \mathfrak{A}$ the multiplicative system $(A, \cdot_A)$ be a groupoid in the sense of Brandt. If $(C, \cdot)$ is an Ehresmann groupoid, then such a division is unique.

Proof. Suppose that the multiplicative system $(C, \cdot)$ is an Ehresmann groupoid. Consider a relation ϱ defined as follows:

$$\varrho = \{(x, y) \colon (x, y) \in C \times C \colon \bigvee_{u \in C} \big((x, u)\big) \in R \wedge (u, y) \in R\}.$$

We shall prove that the relation ϱ is an equivalence in the set C. Let x be an arbitrary element of the set C. Since $(x, x^{-1}) \in R$ and $(x^{-1}, x) \in R$, then $(x, x) \in \varrho$. Suppose now $(x, y) \in \varrho$. Then there exists a $u \in C$ such that $(x, u) \in R$ and $(u, y) \in R$. Hence it follows that $(u^{-1}, x^{-1}) \in R$ and $(y^{-1}, u^{-1}) \in R$. Thus also $(y^{-1} \cdot u^{-1}, x^{-1}) \in R$. Thus

$$\big(y, (y^{-1} \cdot u^{-1}) \cdot x^{-1}\big) \in R \quad \text{and} \quad \big((y^{-1} \cdot u^{-1}) \cdot x^{-1}, x\big) \in R.$$

Consequently $(y, x) \in \varrho$.

Suppose now that $(x, y) \in \varrho$ and $(y, z) \in \varrho$. Then there exist $u, v \in C$ such that the ordered pairs (x, u), (u, y), (y, v) and (v, z) belong to R. From axioms (1.1)-(1.5) it follows that $\big(x, u \cdot (y \cdot v)\big)$ and $(u \cdot (y \cdot v), z)$, belong to R. Hence follows $(x, z) \in \varrho$. Thus we have also proved that the relation ϱ is an equivalence in C.

Denote by C/ϱ the family of equivalence classes of the relation ϱ.

Consider an arbitrary set $A \in C/\varrho$. We shall prove that $(A, \cdot_A)$ is a groupoid in the sense of Brandt for $A \in C/\varrho$. Let (x, y) and (y, z) be arbitrary ordered pairs of the set $R \cap (A \times A)$. Hence it follows that $(x \cdot y, z) \in R$ and that $(x, y \cdot z) \in R$. From the fact that $(C, \cdot)$ is a category it follows that $\alpha(x \cdot y) = \beta(z)$. Hence $\big(x \cdot y, \beta(z)\big) \in R$ and $\big(\beta(z), z\big) \in R$. Thus $(x \cdot y, z) \in \varrho$. Similarly we can show that $(x, y \cdot z) \in \varrho$. Since $x \in A$ and $z \in A$, then $x \cdot y \in A$ and $y \cdot z \in A$. Thus the ordered pairs $(x \cdot_A y, z)$ and $(x, y \cdot_A z)$ belong to the set $R \cap (A \times A)$. So we have proved that the axioms (1.1) and (1.2) are satisfied. It can similarly be proved that axioms (1.3) and (1.4) are also satisfied. Axioms (1.5), (1.6) and (1.7) are satisfied, since the domain of the operation $\cdot_A$ is a part of the domain of the operation

Let x be an arbitrary element of the set A. Since the pairs (x, x^{-1}) and $(x\cdot x^{-1}, x)$ belong to R, then $(x, x^{-1})\in\varrho$ and $(x\cdot x^{-1}, x)\in\varrho$. So $x^{-1}\in A$ and $x\cdot x^{-1}\in A$. Hence it follows that the ordered pairs (x, x^{-1}) and $(x\cdot x^{-1}, x\cdot x^{-1})$ belong to the set $R\cap(A\times A)$ and that

$$x\cdot x^{-1}\cdot_A x\cdot x^{-1} = (x\cdot x^{-1})\cdot(x\cdot x^{-1}) = x\cdot x^{-1}.$$

So axiom (1.8) is satisfied. Let x and y be arbitrary elements of the set A. It follows from the definition of the relation ϱ that there exists an $u\in C$ such that $(x, u)\in R$ and $(u, y)\in R$. Hence we deduce that $(x, u)\in\varrho$. Then $u\in A$. Hence it follows that the ordered pairs (x, u) and (u, y) belong to the set $R\cap(A\times A)$. Thus the multiplicative system $(A, \cdot_A)$ also satisfies axiom (3.1).

Now we proceed to the verification of condition (3.2). Accordingly, consider an arbitrary ordered pair $(x, y)\in R$. It follows that $(x, y)\in\varrho$. There exists a set $A\in C/\varrho$ such that $x\in A$. Consequently also $y\in A$. Thus (x, y) belongs to the set $R\cap(A\times A)$. Thus the proof of necessity of the condition asserted in the theorem has been completed.

To prove the sufficiency of this condition suppose that there exists a division of the set C such that the multiplicative system $(A, \cdot_A)$ is a groupoid in the sense of Brandt for $A\in\mathfrak{A}$ and that condition (3.2) is satisfied.

Consider the ordered pairs (x, y) and (y, z) belonging to the set R. Thus there exist sets $A, B\in\mathfrak{A}$ such that $(x, y)\in A\times A$ and $(y, z)\in B\times B$. Hence it follows that $y\in A\cap B$. So $A = B$. Since the multiplicative system $(A, \cdot_A)$ satisfies axioms (1.1) and (1.2), the ordered pairs $(x\cdot y, z)$ and $(x, y\cdot z)$ belong to the set $R\cap(A\times A)$. Thus these pairs belong to R. Consequently the multiplicative system $(C, \cdot)$ satisfies axioms (1.1) and (1.2). Now let the pairs (y, z) and $(x, y\cdot z)$ belong to R. As before, there exist sets A and B belonging to $\mathfrak{A}$ such that $(y, z)\in A\times A$ and $(x, y\cdot z)\in B\times B$. Since $(A, \cdot_A)$ is a multiplicative system, we have $y\cdot z = y\cdot_A z\in A$. On the other hand, $y\cdot z\in B$. Hence $A = B$. Hence it follows that (x, y) belongs to $R\cap(A\times A)$ and thus to the domain of the operation $\cdot_A$, which proves that axiom (1.3) is satisfied. The proof that the multiplicative system $(C, \cdot)$ satisfies axiom (1.4) is analogous. From the fact that axioms (1.5), (1.6) and (1.7) are satisfied by the groupoids $(A, \cdot_A)$ for $A\in\mathfrak{A}$ and from the disjontedness of the sets of the family $\mathfrak{A}$ it immediately follows that the multiplicative system $(C, \cdot)$ satisfies axioms (1.5), (1.6) and (1.7). Since $C = \bigcup\mathfrak{A}$ and $(A, \cdot_A)$ satisfies axiom (1.8), the multiplicative system $(C, \cdot)$ also satisfies this axiom.

To prove the uniqueness of the division considered in the theorem take an arbitrary such division $\mathfrak{A}$ and observe that

(3.3) $$\bigwedge_{x,y}\Big(\big(x\in A\in\mathfrak{A}\wedge(x, y)\in R\big)\Rightarrow y\in A\Big).$$

Indeed, suppose $x \in A \in \mathfrak{A}$ and $(x, y) \in R$. Then there exists a set $B \in \mathfrak{A}$ such that $(x, y) \in B \times B$. Thus $x \in A \cap B$. Since all sets belonging to $\mathfrak{A}$ are pairwise disjoint, we have $A = B$. Consequently $y \in A$.

Now let the divisions $\mathfrak{A}_1$ and $\mathfrak{A}_2$ satisfy the condition stated in the theorem. Consider an arbitrary set $A_1 \in \mathfrak{A}_1$ and an arbitrary element x of A_1. There exists a set $A_2 \in \mathfrak{A}_2$ such that $x \in A_2$. Consider an arbitrary element y of the set A_2. Since the multiplicative system $(A_2, \cdot_{A_2})$ satisfies axiom (3.1), there exists a $u \in A_2$ such that the ordered pairs (x, u) and (u, y) belong to $R \cap (A_2 \times A_2)$. By (3.3) $u \in A_1$. From the same remark it follows that $y \in A_1$. Consequently $A_2 \subset A_1$. Similarly, from $A_1 \cap A_2 \neq \emptyset$ it follows that $A_1 \subset A_2$. Thus $A_1 = A_2 \in \mathfrak{A}_2$. We have proved that $\mathfrak{A}_1 \subset \mathfrak{A}_2$. Similarly we can show that $\mathfrak{A}_2 \subset \mathfrak{A}_1$. Hence $\mathfrak{A}_1 = \mathfrak{A}_2$.

4. Categories of functions and representation theorems. We shall consider a multiplicative system $(F, \circ)$, where F is a set of functions and $\circ$ denotes the composition of functions. In this case the set

$$S = \{(g, f) \in F \times F \colon f(D_f) \subset D_g\}.$$

is the domain of the operation $\circ$. By $i|A$ we denote the identity function defined in the set A.

THEOREM 1. *Every ordered pair* $(F, \circ)$ *satisfying the following conditions*:

(4.1) $$\bigwedge_{g,f \in F}\big((D_g = D_f \vee D_g \cap D_f = \emptyset) \wedge D_f \neq \emptyset\big),$$

(4.2) $$\bigwedge_{f \in F} \bigvee_{g \in F}\big(f(D_f) \subset D_g\big),$$

(4.3) $$\bigwedge_{f \in F}(i|D_f \in F),$$

(4.4) $$\bigvee_{g,f \in F}\big(f(D_f) \subset D_g \Rightarrow g \circ f \in F\big)$$

is a category.

Every abstract category is isomorphic with some functional category $(F, \circ)$ *satisfying conditions* (4.1)-(4.4).

Proof. Suppose that an ordered pair $(F, \circ)$ satisfies conditions (4.1)-(4.4). From (4.4) it follows that the ordered pair $(F, \circ)$ is a multiplicative system. From the definition of composition of functions it immediately follows that axioms (1.1), (1.2), (1.4) and (1.5) are satisfied. To prove that axiom (1.3) is also satisfied consider arbitrary functions f, g and h such that $f(D_f) \subset D_g$ and $g\big(f(D_f)\big) \subset D_h$. (4.2) implies the existence of a function $k \in F$ such that $g(D_g) \subset D_k$. Hence

$$\emptyset \neq g\big(f(D_f)\big) \subset D_h \cap D_k.$$

From (4.1) it follows that $D_h = D_k$. Thus $g(D_g) \subset D_h$. Consequently (h, g) belongs to S. Now we shall prove that

(4.5) $$\Big\{e \colon e \in F \wedge \bigwedge_{f \in F}\Big(\big((f, e) \in S \Rightarrow f \circ e = f\big) \wedge \big((e, f) \in S \Rightarrow e \circ f = f\big)\Big)\Big\} = \{i|D_f \colon f \in F\}.$$

In fact, let e belong to the set appearing in the left member of equality (4.5). From condition (4.3) it follows that $i|D_e \in F$. The condition $(e, i|D_e) \in S$ implies the equality $i|D_e = e \circ i|D_e = e$. Consider now a function of the form $i|D_f$ where $f \in F$. Let g be an arbitrary function from the set F such that $(g, i|D_f) \in S$. Hence it follows that $D_f \subset D_g$. From condition (4.1) we obtain the equality $D_f = D_g$. Thus $g = g \circ i|D_f$. From the definition of the set S we deduce that if $(i|D_f, g) \in S$, then $i|D_f \circ g = g$. Thus equality (4.5) has been proved. We shall prove that the multiplicative system $(F, \circ)$ satisfies axioms (1.10), (1.11) and (1.13). Let f be an arbitrary function from the set F. Let g and h be arbitrary functions from the set F such that $(f, i|D_g) \in S$ and $(f, i|D_h) \in S$. Since $D_f \supset D_g$, it follows from (4.1) that $D_f = D_g$. Similarly we obtain the equality $D_f = D_h$. Thus axiom (1.10) is satisfied. Suppose that the ordered pairs $(i|D_g, f)$ and $(i|D_h, f)$ belong to the set S. Hence it follows that $D_f \subset D_g \cap D_h$. Hence by (4.1) we deduce $D_g = D_h$, which proves that axiom (1.11) is satisfied. Since condition (4.2) is satisfied and equality (4.5) holds, axiom (1.13) is also satisfied. So the ordered pair $(F, \circ)$ is a category.

We now pass to the proof of the second part of theorem 1. Consider an arbitrary abstract category $(C, \circ)$. For every $u \in C$ denote by $I(u)$ a function defined as follows:

$$I(u) = \{z\colon\ \bigvee_x \big((u, x) \in R \wedge z = (x, u \cdot x)\big)\},$$

where R is the domain of the operation, and assume

$$F = I(C). \tag{4.6}$$

From the definition of the function $I(u)$ it follows that

$$D_{I(u)} = \{x\colon\ (u, x) \in R\}$$

and

$$\big(I(u)\big)(x) = u \cdot x \quad \text{for} \quad x \in D_{I(u)}.$$

Let $(u, v) \in R$ and consider an arbitrary element y of the set $\big(I(v)\big)(D_{I(v)})$. Then there exists an $x \in C$ such that $(u, v) \in R$ and $y = v \cdot x$. Hence we obtain $(u, v \cdot x) \in R$ or, in other words, $y \in D_{I(u)}$. Suppose now that the set $\big(I(v)\big)(D_{I(v)})$ is contained in the set $D_{I(u)}$. Since $a(v) \in D_{I(v)}$ and $\big(I(v)\big)\big(a(v)\big) = v$, we have $v \in D_{I(u)}$. Hence $(u, v) \in R$. Let us observe that

$$\big(I(u) \circ I(v)\big)(t) = u \cdot (v \cdot t) = (u \cdot v) \cdot t = \big(I(u \cdot v)\big)(t) \quad \text{for} \quad t \in D_{I(v)}.$$

Thus we have proved the condition

$$\bigwedge_{u,v} \Big((u, v) \in R \Leftrightarrow \big(I(u), I(v)\big) \in S \Rightarrow I(u) \circ I(v) = I(u \cdot v)\Big), \tag{4.7}$$

where S is the set defined by equality (4.0).

Now let $u, v \in C$ and $I(u) = I(v)$. Then $a(v) \in D_{I(u)}$. Thus $(u, a(v)) \in R$. Hence it follows that

$$(I(u))(a(v)) = u \cdot a(v) = u.$$

On the other hand,

$$(I(v))(a(v)) = v \cdot a(v) = v.$$

Thus $u = v$. So we have proved that function I gives a one to one mapping of the set C onto the set F. From (4.6) it follows that I is an isomorphism mapping the category $(C, \cdot)$ onto the category $(F, \circ)$, where F denotes the set defined by equality (4.6).

Further we shall prove that the set F satisfies conditions (4.1)-(4.4).

In fact, (4.6) immediately implies (4.4). To prove condition (4.1) suppose that $D_{I(u)} \cap D_{I(v)} \neq \emptyset$. Consider an arbitrary $x \in D_{I(u)}$. There exists a $t \in D_{I(u)} \cap D_{I(v)}$. Hence it follows that $a(u) = \beta(t) = a(v)$ and $a(u) = \beta(x)$. Thus $a(v) = \beta(x)$. Consequently $(v, x) \in R$ or, on the other hand, $x \in D_{I(v)}$. Thus we have proved that $D_{I(u)} \subset D_{I(v)}$. Similarly we obtain inclusion $D_{I(v)} \subset D_{I(u)}$. Hence $D_{I(u)} = D_{I(v)}$. Since $(u, a(u)) \in R$, we have $D_{I(u)} = \emptyset$. Let v belong to the set $(I(u))(D_{I(u)})$. Thus there exists an $x \in D_{I(u)}$ such that $v = u \cdot x$. Since $\beta(v) = \beta(u)$ and $(\beta(v), v) \in R$, we have $v \in D_{I(\beta(u))}$. Consequently

$$(I(u))(D_{I(u)}) \subset D_{I(\beta(u))}.$$

So condition (4.2) is satisfied.

Let $u \in C$. Then $(u, a(u)) \in R$. Thus $a(u) \in D_{I(u)}$. On the other hand, since $u = u \cdot a(u)$, it follows from (1.4) that $(a(u), a(u)) \in R$. Consequently $a(u) \in D_{I(a(u))}$. From (4.1) we obtain $D_{I(u)} = D_{I(a(u))}$. Hence

$$i \,|\, D_{I(u)} = i \,|\, D_{I(a(u))} = I(a(u)) \in F.$$

Thus condition (4.5) is satisfied, which completes the proof.

THEOREM 2. *Every ordered pair* $(F, \circ)$ *satisfying condition* (4.1) *and the condition*

$$\bigwedge_{g,f \in F} \big(D_g = D_f \Rightarrow (f^{-1} \in F \wedge g \circ f^{-1} \in F)\big) \tag{4.7}$$

is a groupoid. Every groupoid is isomorphic with some functional groupoid $(F, \circ)$ *satisfying conditions* (4.1) *and* (4.7).

Proof. (4.2) follows from (4.7). Let $g, f \in F$ and $f(D_f) \subset D_g$. Since $f^{-1} \in F$, it follows from (4.1) that $D_{f^{-1}} = D_g$. From (4.7) we conclude that $g \circ f \in F$. Thus condition (4.4) is satisfied. (4.7) and (4.1) imply (4.3). So $(F, \circ)$ is a category. Consider now an arbitrary function $f \in F$. From the equalites $f^{-1} \circ f = i \,|\, D_f$ and $f \circ f^{-1} = i \,|\, D_{f^{-1}}$ and from equality (4.5) it follows that $(F, \circ)$ satisfies axiom (2.1).

Theorem 1 implies the existence of an isomorphism I mapping the groupoid $(C, \cdot)$ onto a functional category $(F, \circ)$ satisfying conditions

(4.1)-(4.4). Consider an arbitrary $e \in C_0$. Let $(I(e), f)$ belong to S, where S denotes the domain of the operation $\circ$ in the groupoid $(F, \circ)$. There exists an $x \in C$ such that $f = I(x)$. Then $(e, x) \in R$. Consequently $I(e) \circ f = I(e \cdot x) = I(x) = f$. Similarly we could prove that $(f, I(e)) \in S$ implies $f \circ I(e) = f$. By equality (4.5) we deduce the existence of a function $h \in F$ such that $I(e) = i | D_h$.

Suppose now that g and f are arbitrary functions from the set F such that $D_g = D_f$. There exist $x, y \in C$ such that $f = I(x)$ and $g = I(y)$. Let $h = I(x^{-1})$. Obviously $(h, f) \in S$, $(f, h) \in S$ and

$$h \circ f = I(x^{-1} \cdot x) = I(a(x)) = i | D_k$$

for some $k \in F$. But this implies that $h \circ f = i | D_f$. Similarly $f \circ h = i | D_h$. Hence it follows that the function f is reversible and $f^{-1} = h \in F$. From (4.4) it follows that $g \circ f^{-1} \in F$, which completes the proof.

5. The algebraic product of sets and the closure of a set in the multiplicative system. Let $(C, \cdot)$ be an arbitrary multiplicative system and R the domain of the operation Consider arbitrary subsets M and N of the set C. By $M \cdot N$ we shall denote the set defined by the equality

$$M \cdot N = \{z \colon \bigvee_{x,y} ((x, y) \in R \cap (M \times N) \wedge z = x \cdot y)\}. \tag{5.1}$$

By M^p we mean a set defined as follows:

$$M^1 = M \quad \text{and} \quad M^p = M \cdot M^{p-1} \quad \text{for} \quad p = 1, 2, \ldots \tag{5.2}$$

THEOREM 1. *If a multiplicative system* $(C, \cdot)$ *satisfies axioms* (1.1), (1.4) *and* (1.5), *then for any subsets* M_1, M_2, M_3 *of the set* C *we have*

$$(M_1 \cdot M_2) \cdot M_3 \subset M_1 \cdot (M_2 \cdot M_3) \tag{5.3}$$

Proof. Consider an arbitrary element x of the set $(M_1 \cdot M_2) \cdot M_3$. There exist elements $x_i \in M_i$ $(i = 1, 2, 3)$ such that $(x_1, x_2) \in R$, $(x_1 \cdot x_2, x_3) \in R$ and $x = (x_1 \cdot x_2) \cdot x_3$. From the axiom (1.4) it follows that $(x_2, x_3) \in R$. Hence by axiom (1.1) we obtain $(x_1, x_2 \cdot x_3) \in R$. From axiom (1.5) it follows that $x = x_1 \cdot (x_2 \cdot x_3)$. Thus $x \in M_1 \cdot (M_2 \cdot M_3)$, which completes the proof.

THEOREM 2. *If a multiplicative system* $(C, \cdot)$ *satisfies conditions* (1.1), (1.4) *and* (1.5), *then any set* $M \subset C$ *satisfies the condition*

$$M^p \cdot M^q \subset M^{p+q} \quad \textit{for} \quad p, q = 1, 2, \ldots \tag{5.4}$$

Proof. For $p = 1$, condition (5.4) is satisfied. Let p be an arbitrary positive integer and suppose that $M^{p-1} \cdot M^q \subset M^{p+q-1}$. From theorem 1 and equality (5.2) it follows that

$$M^p \cdot M^q = (M \quad M^{p-1}) \cdot M^q \subset M \cdot (M^{p-1} \cdot M^q) \subset M \cdot M^{p+q-1} = M^{p+q}.$$

Thus the theorem has been proved.

For an arbitrary $M \subset C$ we denote by $\mathrm{CA}(M, \cdot)$ a set defined by the equality

(5.5) $$\mathrm{CA}(M, \cdot) = \bigcup_{p=1}^{\infty} M^p.$$

This set we shall call the *algebraical closure* of the set M with respect to the operation

THEOREM 3. *If a multiplicative system* $(C, \cdot)$ *satisfies conditions* (1.1), (1.4) *and* (1.5), *then for any set* $M \subset C$ *the set* $\mathrm{CA}(M, \cdot)$ *is the smallest of all sets* N *satisfying the condition*

(5.6) $$M \cup N \cdot N \subset N \subset C.$$

Proof. Because of $M^1 = M$ we have the set $M \subset \mathrm{CA}(M, \cdot)$. From equalites (5.1), (5.5) and inclusion (5.4) it follows that

$$\mathrm{CA}(M, \cdot) \cdot \mathrm{CA}(M, \cdot)$$
$$= \bigcup_{p=1}^{\infty} M^p \cdot \bigcup_{q=1}^{\infty} M^q = \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} M^p \cdot M^q \subset \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} M^{p+q} = \bigcup_{n=2}^{\infty} M^n \subset \mathrm{CA}(M, \cdot).$$

Consider now an arbitrary set $N \subset C$ satisfying condition (5.6). First we prove that $M^p \subset N$ for $p = 1, 2, \ldots$ Accordingly, suppose that $M^{p-1} \subset N$. Multiplying both members of this inclusion by M and making use of (5.6), according to (5.2) we obtain

$$M^p = M \cdot M^{p-1} \subset M \cdot N \subset N \cdot N \subset N.$$

Hence it follows that $\mathrm{CA}(M, \cdot) \subset N$, which completes the proof.

III. THE RELATIONSHIP BETWEEN A GOŁĄB PSEUDOGROUP AND AN EHRESMANN GROUPOID

In his paper [8] S. Gołąb introduced the notion of a pseudogroup. This chapter is devoted to investigating the properties of pseudogroups. In particular we prove that the notion of an Ehresmann groupoid may, by means of an isomorphism, be reduced to the notion of a Gołąb pseudogroup.

6. The notions of a Gołąb pseudogroup and of a functional element. Let a topological space X be given. Suppose that a set G of transformations satisfies the following conditions:

(6.0) $$\bigwedge_{f \in G} \big(\emptyset \neq D_f \in \omega(X)\big),$$

(6.1) $$\bigwedge_{f \in G} \bigwedge_{A \in \omega(X)} (\emptyset \neq A \subset D_f \Rightarrow f|A \in G),$$

(6.2) $$\bigwedge_{g,f \in G} \Big(\big(f \in G \wedge D_g = f(D_f)\big) \Rightarrow g \circ f \in G\Big),$$

(6.3) $$\bigwedge_{f \in G} \bigwedge_{x \in D_f} \bigvee_{A \in \omega(X)} \bigvee_{g \in G} \big(x \in A \subset D_f \wedge D_g = f(A) \wedge g \circ f|A = i|A\big),$$

where by X and $\omega(X)$ we denote the set of points and the family of open sets of a given space X respectively (see Chapter I).

S. Gołąb has named a set G with the above properties a *pseudogroup*. Let us observe that if the set G satisfies condition (6.3), then the supposition $\bigwedge_{f\in G}\big(D_f\in\omega(X)\big)$ in condition (6.0) is superfluous. Indeed, let $f\in G$ and $x\in D_f$. Condition (6.3) implies the existence of a set $A\in\omega(X)$. Condition (6.3) may be written in an equivalent form as follows:

$$\bigwedge_{f\in G}\bigvee_{x\in D_f}\bigvee_{A\in\omega(X)}\big(x\in A\subset D_f\wedge(f|A)^{-1}\in G\big). \tag{6.4}$$

In fact, suppose that G satisfies condition (6.3) and let $f\in G$ and $x\in D_f$. Then there exists a set $A\in\omega(X)$ and a function $g\in G$ such that

$$x\in A\subset D_f,\quad D_g=f(A)\quad\text{and}\quad g\circ f|A=i|A.$$

Hence it follows that the function g is reversible and that $f|A=g^{-1}$. It is easily seen that (6.3) follows from (6.4).

In paper [8] S. Gołąb has proved that every function belonging to a set M which satisfies conditions (6.0), (6.1) and (6.3) is continuous. Thus from (6.3) it follows that every function belonging to such a set M is a local homeomorphism. By $L(X)$ we denote the set of all local homeomorphisms of a space X. The set of all pseudogroups of a space X we denote by $\operatorname{psg}X$.

Now we shall define the notion of a functional element. Let functions

$$f\in(X|D_f\to X)\quad\text{and}\quad g\in(X|D_g\to X),$$

where $D_f, D_g\in\omega(X)$ be given. Let $x\in D_f$ and $y\in D_g$. The ordered pairs (f,x) and (g,y) will be considered equivalent if and only if the following conditions are satisfied: $x=y$ and there exists a set $A\in\omega(X)$ such that $x\in A\subset D_f\cap D_g$ and $f|A=g|A$.

The reflexivity of the relation thus defined follows from $D_f\in\omega(X)$. The symmetry of this relation is obvious. Its transitivity follows from the fact that the intersection of two open sets is open. The equivalence classes of the relation thus defined will be called *functional elements*. The equivalence class to which the ordered pair (f,x) belongs will be denoted by $[f,x]$. We observe that $f(x)$ does not depend on the choice of the representative (f,x) of the element $[f,x]$. For a given functional element $p=[f,x]$ we denote by $c(p)$ and $v(p)$ the points x and $f(x)$, respectively, which are uniquely determined by the given element.

Consider two functional elements p and q such that $v(p)=c(q)$. Assume $p=[f,x]$ and $q=[g,y]$. Thus we have $f(x)=y\in D_g$. From the continuity of the function f it follows that $f^{-1}(D_g)\in\omega(X)$. The ordered pair $\big(g\circ f|f^{-1}(D_g),x\big)$ determines a functional element which we denote by qp. We shall prove that the element qp does not depend on the choice of the representatives (f,x) and (g,y) of the elements p and q.

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Indeed, let $p = [h, x]$ and $q = [k, y]$. There exists a set $B \in \omega(X)$ such that $y \in B \subset D_g \cap D_k$ and $g|B = k|B$. From the continuity of the functions f and h it follows that the set $f^{-1}(B) \in \omega(X)$ and $h^{-1}(B) \in \omega(X)$. Thus there exists a set $A \in \omega(X)$ such that

$$x \in A \subset f^{-1}(B) \cap h^{-1}(B) \quad \text{and} \quad f|A = h|A.$$

Hence it follows that

$$\big(f|f^{-1}(D_g)\big)|A = \big(h|h^{-1}(D_k)\big)|A.$$

Thus $\big(k \circ h|h^{-1}(D_k), x\big) \in qp$. The element qp will be called the *composition* of the elements p and q.

Now let p be an arbitrary functional element such that there exists an ordered pair $(f, x) \in p$ satisfying the condition $f|A \in L(X)$ for a certain $A \in \omega(X)$ such that $x \in A$. As before, we can prove that the element $[(f|A)^{-1}, f(x)]$ depends neither on the set A nor on the representative (f, x) of the element p. We denote this element by p^{-1} and call it the element *inverse* to p, while the element p we call *inversible.*

7. The isomorphism of an arbitrary Ehresmann groupoid and a Gołąb pseudogroup of a certain type. Groupoids of functional elements. Let Q denote an arbitrary non-empty family of non-empty disjoint sets and let Q_0 be defined by the equality:

$$Q_0 = \{B\colon \bigvee_{S \subset Q} (B = \bigcup S)\}. \tag{7.1}$$

It follows from the definition of the family Q_0 that the union of any set of sets of the family Q_0 and the intersection of a finite set of sets of the family Q_0 belong to Q_0. The sets $\emptyset$ and $\bigcup Q$ belong to Q_0. Thus the family Q_0 topologizes the set $\bigcup Q$. Moreover, this is the weakest topology of the set $\bigcup Q$ for which all sets of the family Q are open. The topological space thus formed will be denoted by $Q^{\wedge}$ So we have

$$\omega(Q^{\wedge}) = Q_0 \quad \text{and} \quad \underline{Q}^{\wedge} = \bigcup Q. \tag{7.2}$$

THEOREM 1. *For every Ehresmann groupoid $(C, \cdot)$ there exist a topological space X and a pseudogroup G in this space such that the multiplicative system $(G, \circ)$ is a groupoid isomorphic with the groupoid $(C, \cdot)$.*

Proof. Theorem 2 of paragraph 4 implies the existence of a set G of functions which satisfies conditions (4.1) and (4.6) such that the groupoids $(C, \cdot)$ and $(G, \circ)$ are isomorphic. Let Q denote $\{D_f\colon f \in G\}$. Obviously, we may assume that $C \neq \emptyset$. Thus Q is a non-empty family of non-empty disjoint sets. Let $X = Q^{\wedge}$ We shall prove that $G \in \operatorname{psg} X$. Indeed, let $f \in G$. Then $\emptyset \neq D_f \in Q \subset \omega(X)$. Thus G satisfies condition (6.0). Consider an arbitrary set $B \in \omega(X)$ such that $\emptyset \neq B \subset D_f$. From (7.2) and (7.1) it immediately follows that there exists a family $H \subset G$ such that

$B = \bigcup \{D_h: h \in H\}$. Thus there exists a function $h \in H$ such that $\emptyset \neq D_h \subset D_f$. From (4.1) it follows that $D_h = D_f$. Consequently $B = D_f$. Thus $f|B = f \in G$. Thus G satisfies condition (6.1). Consider now arbitrary functions $g, f \in G$ for which $D_g = f(D_f)$. From (4.6) it follows that $g \circ f \in F$. Finally, to prove that G satisfies condition (6.3) it suffices to observe that $f \in G$; then $D_f \in \omega(X)$ and $f^{-1} \in G$, which completes the proof.

The theorem just proved reduces by an isomorphism the notion of a groupoid to that of a Gołąb pseudogroup. We shall prove a theorem which makes possible the construction of groupoids with the aid of pseudogroups.

For an arbitrary set $M \subset L(X)$ we shall denote by M^* a set defined by the equality

$$M^* = \{[f, x]: f \in M \wedge x \in D_f\}. \tag{7.3}$$

THEOREM 2. *If $G \in \operatorname{psg} X$, then the multiplicative system $(G^*, \cdot)$, where $\cdot$ denotes the composition of functional elements in the set G^*, is an Ehresmann groupoid.*

Proof. In view of the theorem proved in paragraph 2 it suffices to verify axioms (1.1)-(1.8) for the system $(G^*, \cdot)$. Let

$$R = \{(q, p): (q, p) \in G^* \times G^* \wedge c(q) = v(p)\}.$$

Thus R is the domain of the operation Let us observe that from the definition of the composition of functional elements it follows that

$$(q, p) \in R \Rightarrow \big(c(q \cdot p) = c(p) \wedge v(q \cdot p) = v(q)\big). \tag{7.4}$$

Let (r, q), $(q, p) \in R$. From (7.4) it follows that $c(r) = v(q \cdot p)$ and $c(r \cdot q) = v(p)$. Thus $(r \cdot q, p)$, $(r, q \cdot p) \in R$. So axioms (1.1) and (1.2) are satisfied. One can prove similarly that axioms (1.3) and (1.4) are also satisfied. From (6.1) and from the continuity of the functions belonging to G it follows that if (r, q), $(q, p) \in R$, then there exist functions $f, g, h \in G$ such that for some $x \in D_f$ we have

$$p = [f, x], \quad q = [g, f(x)], \quad r = \big[h, g\big(f(x)\big)\big],$$
$$f(D_f) \subset D_g \quad \text{and} \quad g(D_g) \subset D_h.$$

Then $[h \circ (g \circ f), x] = r \cdot (q \cdot p)$ and $[(h \circ g) \circ f, x] = (r \cdot q) \cdot p$. Hence it follows that $r \cdot (q \cdot p) = (r \cdot q) \cdot p$. Thus axiom (1.5) is satisfied. Suppose now that (r, p), $(r, q) \in R$ and $r \cdot p = r \cdot q$. In view of the continuity of the functions belonging to G conditions (6.4) and (6.1) imply the existence of functions $f, g, h \in G$ such that $f^{-1}, g^{-1}, h^{-1} \in G$, $f(D_f) = g(D_g) = D_h$ and $p = [f, x]$, $q = [g, x]$, $r = [h, f(x)]$ for some $x \in D_f$. Moreover, $h \circ f = h \circ g$. Hence it follows that $f = g$. Thus $p = q$. The proof that axiom (1.7) is also satisfied is similar. Axiom (1.8) is satisfied because for an arbitrary $p \in G$ there exists a function $f \in G$ such that $f^{-1} \in G$ and $p = [f, x]$ for a certain $x \in D_f$. The element $p^{-1} = [f^{-1}, f(x)]$ satisfies the conditions

$$p^{-1} \cdot p = [i | D_f, x] \quad \text{and} \quad p \cdot p^{-1} = [i | f(D_f), f(x)],$$

which completes the proof.

IV. GENERATING IN GOŁĄB PSEUDOGROUPS AND SOME PROPERTIES OF A SET OF FUNCTIONS

In the theory of groups one proves that for an arbitrary set Z of elements of a certain group there exists a subgroup which is the smallest subgroup of the given group including the set Z. This subgroup is uniquely determined by the set Z and it is called the *subgroup generated by the set* Z. It is clear that for a given topological space X the set $L(X)$ of all the local homeomorphisms constitutes a pseudogroup. This pseudogroup includes all the other pseudogroups of the given space. In this connection the question arises whether or not for every set M of local homeomorphisms of a given space there exists pseudogroup, including the set M, which is the smallest (in the sense of inclusion).

To give an answer to this question consider an example of a family of functions f_a defined for $-\infty < a < +\infty$ by the formula

$$f_a(x) = x + a \quad \text{for} \quad x \in (-\infty, +\infty).$$

Denote by X the space of real numbers. Consider a family G_a defined for every real a by the equality

$$G_a = \{g \colon \bigvee_{A \in \omega(X)} (\emptyset \neq A \wedge g = f_a | A)\},$$

and for $\varepsilon > 0$ the family

$$F_{a,\varepsilon} = \{g \colon g \in F_a \wedge \delta(D_g) < \varepsilon\},$$

where by $\delta(A)$ we denote the diameter of the set A. For $\varepsilon > 0$ we put

$$M = \bigcup \{G_a \colon a \leqslant 0\} \quad \text{and} \quad H_\varepsilon = M \cap \bigcup \{F_{a,\varepsilon} \colon a > 0\}.$$

It can easily be proved that for every $\varepsilon > 0$ the family $H_\varepsilon \in \mathrm{psg} X$. In view of

$$\bigcap \big\{ \bigcup \{F_{a,\varepsilon} \colon a > 0\} \colon \varepsilon > 0 \big\} = \emptyset$$

we obtain the equality $M = \bigcap \{H_\varepsilon \colon \varepsilon > 0\}$. From the definition of the set M it follows that it is not a pseudogroup. Then the smallest pseudogroup including the set M does not exist. So the answer to the question raised is negative. There is, however, a method of generating pseudogroups by an arbitrary family of local homeomorphisms. This chapter is devoted to the discusion of this method.

8. Some operations with sets of functions. Consider a multiplicative system $(F, \cdot)$, where F denotes a set of functions and the composition of functions defined for ordered pairs (g, f) of functions satisfying the condition

$$D_g = f(D_f). \tag{8.1}$$

The domain of the operation thus defined is the set of those and only those ordered pairs $(g, f) \in F \times F$ which satisfy condition (8.1).

For a given set $M \subset F$ by $M^{\times}$ we denote the set $\mathrm{CA}(M, \cdot)$ (see (5.5)).

THEOREM 1. *For an arbitrary set $M \subset F$ the set $M^{\times}$ satisfies the condition*

$$M \cup M^{\times} \cdot M^{\times} = M^{\times}. \tag{8.2}$$

The set $M^{\times}$ is the smallest of all the sets N satisfying condition

$$M \cup N \cdot N \subset N$$

Proof. The second part of the theorem immediately follows from theorem 3 of paragraph 5. Thus it suffices to prove condition (8.2). First we observe that for any sets M_1, M_2, M_3 included in F the equality

$$(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$$

holds. Indeed, consider an arbitrary function $f \in M_1 \cdot (M_2 \cdot M_3)$. There exist functions $f_i \in M_i$ for $i = 1, 2, 3$ such that $D_{f_2} = f_3(D_{f_3})$ and $D_{f_1} = (f_2 f_3)(D_{f_3})$. Hence it follows that $D_{f_1} = f_2(D_{f_2})$. Thus $f_1 \circ f_2 \in M_1 \cdot M_2$. Consequently $f \in (M_1 \cdot M_2) \cdot M_3$ and

$$M_1 \cdot (M_2 \cdot M_3) \subset (M_1 \cdot M_2) \cdot M_3.$$

The converse inclusion follows from theorem 1 of paragraph 5. Hence we deduce that

$$M^p \cdot M^q = M^{p+q} \quad \text{for} \quad p, q = 1, 2, \ldots$$

and consequently

$$M \cup M^{\times} \cdot M^{\times} = M \cup \bigcup_{p=1}^{\infty} M^p \cdot \bigcup_{q=1}^{\infty} M^q = M \cup \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} M^{p+q} = \bigcup_{p=1}^{\infty} M^p = M^{\times},$$

which completes the proof.

THEOREM 2. *If M satisfies condition* (6.0), *then $M^{\times}$ satisfies conditions* (6.0) *and* (6.2).

Proof. Suppose that M satisfies condition (6.0). From the fact that the domain of a composed function is identical with the domain of the internal function it follows that $M^\times$ also satisfies condition (6.0). From $M^\times \cdot M^\times \subset M^\times$ it immediately follows that condition (6.2) is satisfied.

THEOREM 3. *If M satisfies condition* (6.1) *and* $M \subset L(X)$, *then* $M^\times$ *satisfies condition* (6.1).

Proof. Take an arbitrary positive integer p and suppose that

$$(8.3) \qquad \bigwedge_{f \in M^p} \bigwedge_{A \in \omega(X)} (\emptyset \neq A \subset D_f \Rightarrow f|A \in M^p).$$

Let $h \in M^{p+1}$, $\emptyset \neq A \in \omega(X)$ and $A \subset D_h$. There exist functions $g \in M^p$ and $f \in M$ such that $D_g = f(D_f)$ and $h = g \circ f$. In view of the fact that f is a local homeomorphism it follows that $f(A) \in \omega(X)$. From $D_h = D_f$ we have by (6.1) $f|A \in M$. From (8.3) we deduce that $g|f(A) \in M^p$. The equality $h|A = (g|f(A)) \circ f|A$ yields $h|A \in M^{p+1}$. Thus every positive integer p satisfies condition (6.1).

THEOREM 4. *If the sets M and N of functions satisfy conditions* (6.0), (6.1) *and* (6.3), *then the set* $N \cdot M \cup M \cdot N$ *satisfies condition* (6.3).

Proof. Assuming that the sets M and N satisfy conditions (6.0), (6.1) and (6.3), consider an arbitrary function $h \in N \cdot M$ and an arbitrary point $x \in D_h$. There exist functions $f \in M$ and $g \in N$ such that $D_g = f(D_f)$ and $h = g \circ f$. Since M satisfies condition (6.3), there exists a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $(f|A)^{-1} \in M$. It follows from the assumption of the theorem that f is a local homeomorphism, consequently $f(A) \in \omega(X)$. Since N satisfies conditions (6.1) and (6.3), we have $g|f(A) \in N$ and there exists a set $B \in \omega(X)$ such that $f(a) \in B \subset f(A)$ and $(g|B)^{-1} \in N$. Let us assume $C = (f|A)^{-1}(B)$. In view of the fact that M satisfies condition (6.1) we have $f|C \in M$. Consequently

$$(h|C)^{-1} = (f|C)^{-1} \circ (g|B)^{-1} \in M \cdot N,$$

which completes the proof.

THEOREM 5. *If the set M of functions satisfies conditions* (6.0), (6.1) *and* (6.3), *then the set* $M^\times$ *satisfies condition* (6.3).

Proof. From the fact that the set M satisfies conditions (6.0), (6.1) and (6.3) it follows by theorem 4 that the set M^p satisfies condition (6.3) for $p = 1, 2, \ldots$ Since the union of any set of sets satisfying condition (6.3) also satisfies this condition, it is also satisfied by the set $M^\times$.

From theorems 2, 3 and 5 we immediately deduce the following:

THEOREM 6. *If M satisfies conditions* (6.0), (6.1) *and* (6.3), *then* $M^\times \in \mathrm{psg}\, X$.

This theorem makes it possible to construct pseudogroups including sets, given a priori, of local homeomorphisms.

For any family M of functions having their regions and their sets of values in the set $\underline{X}$ we define the sets $M^{\smile}$, M^{-1}, $M^{\frown}$ by the following equalities:

(8.4) $$M^{\smile} = \{f\colon \bigvee_{g\in M}\bigvee_{A\in\omega(X)}(\emptyset \neq A \subset D_g \wedge f = g|A)\},$$

(8.5) $$M^{-1} = \{f\colon f^{-1}\in M \wedge D_f \cup f(D_f) \subset \underline{X}\},$$

(8.6) $$M^{\frown} = \{f\colon D_f \neq \emptyset \wedge \bigwedge_{x\in D_f}\bigvee_{A\in\omega(X)}(x\in A \subset D_f \wedge f|A\in M)\}.$$

An immediate consequence of definition (8.4) is the following:

THEOREM 7. *For any set M of functions f satisfying condition $D_f \cup f(D_f) \subset \underline{X}$ the set $M^{\smile}$ satisfies conditions* (6.0) *and* (6.1).

We prove the following:

THEOREM 8. *If a set M of functions satisfies conditions* (6.0) *and* (6.1), *then the set $M^{\frown}$ also satisfies conditions* (6.0) *and* (6.1).

Proof. Suppose that the set M satisfies conditions (6.0) and (6.1). From equality (8.6) it immediately follows that the set M satisfies condition (6.0). To prove that $M^{\frown}$ satisfies condition (6.1) consider an arbitrary function $f\in M$ and an arbitrary set $B\in\omega(X)$ such that $\emptyset \subset B \subset D_f$. Let $x\in B$. There exists a set $A\in\omega(X)$ such that $x\in A \subset D_f$ and $f|A\in M$. From (6.1) it follows that

$$f|(A\cap B)\in M \quad\text{and}\quad x\in A\cap B \subset D_f.$$

Thus $f\in M^{\frown}$, which completes the proof.

THEOREM 9. *If we have a set $M \subset L(X)$ and if it satisfies condition* (6.1) *then the set M^{-1} satisfies conditions* (6.0) *and* (6.1) *and the set $M \cup M^{-1}$ satisfies condition* (6.3).

Proof. Let $f\in M^{-1}$. Hence $f^{-1}\in M$. We have

$$D_f = f^{-1}(D_{f^{-1}})\in\omega(X), \quad\text{as}\quad f^{-1}\in L(X).$$

Consequently the set M satisfies condition (6.0). Now let a set A satisfy the conditions $\emptyset \neq A \subset D_f$ and $A\in\omega(X)$. $(f|A)^{-1} = f^{-1}|f(A)$ belongs to M, for M satisfies condition (6.1). Hence $f|A\in M^{-1}$.

To prove that the set $M \cup M^{-1}$ satisfies condition (6.3) consider an arbitrary function $f\in M$ and an arbitrary point $x\in D_f$. Then there exists a set $A\in\omega(X)$ such that $x\in A \subset D_f$ and $(f|A)^{-1}\in L(X)$. Since M satisfies condition (6.1), we have $f|A\in M$. Consequently $(f|A)^{-1}$ belongs to M^{-1}. Now let $f\in M^{-1}$ and $x\in D_f$. From the definition of the set M^{-1} it follows that $f^{-1}\in M$. Then the set $M \cup M^{-1}$ satisfies condition (6.3). Thus our theorem has been proved.

THEOREM 10. *If $M \subset L(X)$, then $\big(M^{\smile} \cup (M^{\smile})^{-1}\big)^{\times}\in \operatorname{psg} X$.*

Proof. From the assumption it follows that $M \subset L(X)$. From theorem 7 we deduce that $M^{\smile}$ satisfies condition (6.1). Hence by theorem 9 we conclude that $(M^{\smile})^{-1}$ satisfies conditions (6.0) and (6.1) and that the set $M^{\smile} \cup (M^{\smile})^{-1}$ satisfies condition (6.3). Since the union of sets satisfying conditions (6.0) and (6.1) also satisfies those conditions, the set $M^{\smile} \cup (M^{\smile})^{-1}$ satisfies conditions (6.0) and (6.1). By making use of theorem 6 we complete the proof.

THEOREM 11. *If* $G \in \mathrm{psg}\, X$, *then* $G^{\frown} \in \mathrm{psg}\, X$.

Proof. In view of theorem 8 it suffices to prove that the set G satisfies conditions (6.2) and (6.3). Consider arbitrary functions $f, g \in G$ such that $D_g = f(D_f)$. Let $h = g \circ f$ and let $x \in D_h$. Then, according to definition (8.6), there exist sets $A, B \in \omega(X)$ such that

$$x \in A \subset D_f, \quad f(x) \in B \subset f(A), \quad f|A \in G \quad \text{and} \quad g|B \in G.$$

Denote by C the set $(f|A)^{-1}(B)$. Hence we have $x \in C \in \omega(X)$, $C \subset A$, $f(C) = B$ and $f|C \in G$. Since G satisfies condition (6.1), we have $f|C \in G$. Hence it follows that

$$h|C = g|f(C) \circ f|C \in G.$$

So $h \in G$. Consequently $G^{\frown}$ satisfies condition (6.2).

Consider an arbitrary function $f \in G$ and an arbitrary point $x \in D_f$. There exists a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $f|A \in G$. From (6.3) it follows that there exists a set $B \in \omega(X)$ such that $x \in B \subset A$ and $(f|B)^{-1} \in G$. In view of the fact that $G \subset G^{\frown}$ we have $(f|B)^{-1} \in G$. This proves that $G^{\frown}$ satisfies condition (6.3). Then $G^{\frown} \in \mathrm{psg}\, X$.

9. A quasi-order of the family of all subsets of the set $L(X)$. We now define a binary relation $\leqslant$ between the sets included in $L(X)$ as follows:

$$(9.1) \qquad M \leqslant N \Leftrightarrow (M \cup N \subset L(X) \wedge$$
$$\wedge \bigwedge_{f \in M} \bigwedge_{x \in D_f} \bigvee_{g \in N} \bigvee_{A \in \omega(X)} (x \in A \subset D_f \cap D_g \wedge f|A = g|A)).$$

From definition (9.1) it immediately follows that the relation $\leqslant$ is reflexive and transitive.

THEOREM 1. *If the sets* M_2 *and* N_2 *satisfy condition* (6.1) *and* $M_1 \leqslant M_2$ *and* $N_1 \leqslant N_2$, *then* $N_1 \cdot M_1 \leqslant N_2 \cdot M_2$.

Proof. Consider an arbitrary function $h \in N_1 \cdot M_1$ and an arbitrary point $x \in D_h$. There exist functions $f_1 \in M_1$ and $g_1 \in N_1$ such that $D_{g_1} = f_1(D_{f_1})$. Since $f_1(x) \in D_g$, there exist a function $g_2 \in N_2$ and a set $B \in \omega(X)$ such that

$$f_1(x) \in B \subset D_{g_1} \cap D_{g_2} \quad \text{and} \quad g_1|B = g_2|B.$$

Similarly, there exist a function $f_2 \in M_2$ and a set $A \in \omega(X)$ such that

$$x \in A \subset D_{f_1} \cap D_{f_2} \quad \text{and} \quad f_1|A = f_2|A.$$

Since f_1 is a local homeomorphism, the set $A' = f_1^{-1}(B) \cap A$ belongs to $\omega(X)$ and $f_1(A') \in \omega(X)$. Moreover, $x \in A'$, $f_1|A' = f_2|A'$ and $f_1(A') \subset B$. Since the sets M_2 and N_2 satisfy condition (6.1), we have

$$f_1|A' \in M_2 \quad \text{and} \quad g_1|f_1(A') = g_2|f_1(A') \in N_2.$$

Hence it follows that

$$h|A' = g_1|f_1(A') \circ f_1|A' \in N_2 \cdot M_2,$$

which completes the proof.

THEOREM 2. *If $M_t \leqslant N_t$ for any $t \in T$, then*

$$\bigcup \{M_t\colon t \in T\} \leqslant \bigcup \{N_t\colon t \in T\}.$$

Proof. Consider an arbitrary function $f \in \bigcup \{M_t\colon t \in T\}$ and an arbitrary point $x \in D_f$. There exists a $t \in T$ such that $f \in M_t$. From the definition of the relation $\leqslant$ it follows that there exist a function $g \in N_t$ and a set $A \in \omega(X)$ such that $x \in A \subset D_f \cap D_g$ and $f|A = g|A$. Then $\bigcup \{M\colon t \in T\} \leqslant \bigcup \{N_t\colon t \in T\}$.

THEOREM 3. *If $M \leqslant N$ and the set N satisfies condition* (6.1), *then $M^\times \leqslant N^\times$.*

Proof. In the proof of theorem 3 of paragraph 7 we have shown that if a set N satisfies condition (6.1), then this condition is satisfied by the set N^p for $p = 1, 2, \ldots$ From theorem 1 it thus follows that $M^p \leqslant N^p$ for $p = 1, 2, \ldots$ Hence by theorem 2 we obtain $M^\times \leqslant N^\times$, which completes the proof.

THEOREM 4. *If $M \leqslant N$ and N satisfies condition* (6.1), *then $M^\frown \leqslant N^\frown$.*

Proof. Consider an arbitrary function $f \in M$ and an arbitrary point $x \in D_f$. From (8.6) it follows that there exists a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $f|A \in M$. Since $M \leqslant N$, there exist a function $g \in N$ and a set $B \in \omega(X)$ such that

$$x \in B \subset A \cap D_g \quad \text{and} \quad f|B = (f|A)|B = g|B.$$

From (6.1) it follows that $g|B \in N$. Then $f \in N^\frown$, which completes the proof.

Equalities (8.1) and (8.6) and the definition of the relation $\leqslant$ yields the following

THEOREM 5. *If $M \subset L(X)$, then $M^\smile \leqslant M$ and $M^\frown \leqslant M$.*

We shall prove

THEOREM 6. *If M $G \in \mathrm{psg}\,X$, then $M^{-1} \leqslant G$.*

Proof. Consider an arbitrary function $f \in M^{-1}$ and an arbitrary point $x \in D_f$. From equality (8.5) it follows that $f^{-1} \in M$. Thus there exists a set $A \in \omega(X)$ such that $f(x) \in A \subset f(D_f)$ and $f^{-1}|A \in G$. From condition (6.4) it follows that there exists a set $B \in \omega(X)$ such that $f(x) \in B \subset A$ and $f|f^{-1}(B) = (f^{-1}|B)^{-1}$ belongs to G. Hence it follows that $M^{-1} \leqslant G$.

For an arbitrary set $M \subset L(X)$ we denote by $[M]$ the set defined by the equality

$$[M] = \big(M^{\smile} \cup (M^{\smile})^{-1}\big)^{\times\frown}.$$

THEOREM 7. *For any set $M \subset L(X)$ the set $[M]$ is a pseudogroup satisfying the following three conditions*:

$$(9.1) \qquad M \subset [M],$$

$$(9.2) \qquad (M \leqslant G \in \operatorname{psg} X) \Rightarrow ([M] \leqslant G),$$

$$(9.3) \qquad (F \leqslant [M]) \Rightarrow (F \subset [M]).$$

These conditions characterise the pseudogroup $[M]$.

Proof. Let $M \subset L(X)$. Condition (9.1) is obvious. From theorems 10 and 11 of paragraph 7 it follows that $[M] \in \operatorname{psg} X$. Suppose that $M \leqslant G \in \operatorname{psg} X$. Then $M^{\smile} \leqslant G^{\smile} \leqslant G$. From theorem 6 it follows that $(M^{\smile})^{-1} \leqslant G$. From theorems 2 and 3 it follows that

$$\big(M^{\smile}(M^{\smile})^{-1}\big)^{\times} \leqslant (G)^{\times} = G.$$

Theorem 5 yields $[M] \leqslant G^{\frown} \leqslant G$. Suppose now that $F \leqslant M$. By theorem 4, $F \subset F^{\frown} \subset [M]$.

Consider now an arbitrary pseudogroup H satisfying the conditions

$$(9.4) \qquad M \subset H,$$

$$(9.5) \qquad (M \leqslant G \in \operatorname{psg} X) \Rightarrow (H \leqslant G),$$

$$(9.6) \qquad (F \leqslant H) \Rightarrow F \subset H.$$

From (9.4) it follows that $M \leqslant H$. Hence by (9.2) we deduce that $[M] \leqslant H$. From condition (9.6) we get $[M] \subset H$. Similarly from (9.1) it follows that $M \leqslant [M]$. Hence by (9.5) we have $H \leqslant [M]$. From (9.3) we obtain $H \subset [M]$ which should be proved.

10. Determining a pseudogroup with the aid of sets of functional elements. This paragraph is devoted to the discussion of some qualities of sets of local homeomorphisms. In particular we explain here the connection between a pseudogroup generated by an arbitrary set of local homeomorphisms and sets of functional elements.

THEOREM 1. *If $M \cup N \subset L(X)$, then $M \leqslant N$ if and only if $M^* \subset N^*$* (see (7.3)).

Proof. Suppose that $M \leqslant N$. Consider an arbitrary element $p \in M^*$. From equality (7.3) we deduce that there exist a function $f \in M$ and a point $x \in D_f$ such that $p = [f, x]$. From (9.1) it follows that there exist a function $g \in N$ and a set $A \in \omega(X)$ such that $x \in A \subset D_f \cap D_g$ and $f|A = g|A$. Hence it follows that $p = [g, x] \in N^*$. Consequently $M^* \subset N^*$.

Suppose now that $M \cup N \subset L(X)$ and $M^* \subset N^*$. Consider an arbitrary function $f \in M$ and an arbitrary point $x \in D_f$. From (7.1) it follows that $[f, x] \in M^*$. So $[f, x] \in N^*$. Thus there exists a function $g \in N$ such that $x \in D_g$ and $[f, x] = [g, x]$. The definition of a functional element implies the existence of a set $A \in \omega(X)$ such that $x \in A \subset D_f \cap D_g$ and $f|A = g|A$. Thus $M \leqslant N$, which completes the proof.

For an arbitrary set P of functional elements by P^{-1} we denote the set of all elements inverse to the inversible elements of the set P. Similarly, for any sets P and Q of functional elements we shall denote by QP the set

$$\{r : r \in (L(X))^* \wedge \bigvee_{p,q}(p \in P \wedge q \in Q \wedge c(q) = v(p) \wedge r = q \cdot p)\}.$$

THEOREM 2. *If $G \in \mathrm{psg}X$, then $(G^*)^{-1} \cup G^*G^* \subset G^*$.*

Proof. Suppose that $G \in \mathrm{psg}X$. Let $p \in (G^*)^{-1}$. Then there exist a function $f \in G$ and a point $x \in D_f$ such that $p^{-1} = [f, x]$. Condition (6.4) implies the existence of a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $(f|A)^{-1} \in G$. Consequently $p = [(f|A)^{-1}, f(x)] \in G$. Now let $q \in G^*$ be a functional element such that $c(q) = v(p)$. Then there exists a function $g \in G$ such that $q = [f, x]$ and $D_g = f(D_f)$. From (6.2) it follows that $g \circ f \in G$. Thus $q \cdot p = [g \circ f, x] \in G^*$, which completes the proof.

For any $P \subset (L(X))^*$ we denote by P' the set defined by the equality

$$P' = \{f : f \in L(X) \wedge \bigwedge_{x \in D_f}([f, x] \in P)\}. \tag{10.1}$$

Making use of definitions (7.3) and (10.1) we shall prove the following theorem.

THEOREM 3. *If $M \subset L(X)$ then $M \subset M^{*\prime}$ and $M^* = M^{*\prime *}$.*

Proof. Consider an arbitrary function $f \in M$ and an arbitrary point $x \in D_f$. Then $[f, x] \in M^*$. Hence it follows that $f \in M^{*\prime}$ Thus $M \subset M^{*\prime}$, and also $M^* \subset M^{*\prime *}$. Consider now an arbitrary functional element $p \in M^{*\prime *}$. There exist a function $f \in M^{*\prime}$ and a point $u \in D_f$ such that $p = [f, u]$. For any $x \in D_f$ we have $[f, x] \in M^*$. In particular, $p \in M^*$. Thus $M^{*\prime *} \subset M^*$, which completes the proof.

THEOREM 4. *If $P^{-1} \cup PP \subset P \subset (L(X))^*$ then $P' \in \mathrm{psg}X$.*

Proof. Consider an arbitrary function $f \in P'$ and an arbitrary set $A \in \omega(X)$ such that $\emptyset \neq A \subset D_f$. Let x denote any point of the set A. Clearly $[f|A, x] = [f, x] \in P$. Thus $f|A \in P'$ The set P' satisfies condition (6.1). Consider functions $g, f \in P'$ such that $D_g = f(D_f)$ and an arbitrary point $x \in D_f$. Thus we have $f(x) \in D_g$, $[f, x] \in P$ and $[g, f(x)] \in P$. Hence

$$[g \circ f, x] = [g, f(x)] \cdot [f, x] \in P.$$

Thus $g \circ f \in P'$. Consequently condition (6.2) is satisfied. To prove that condition (6.3) is also satisfied consider an arbitrary function $f \in P'$ and an arbitrary point $x \in D_f$. Since the function f is a local homeomorphism, there exists a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $(f|A)^{-1} \in L(X)$. Consider an arbitrary point $w \in f(A)$ and assume $v = (f|A)^{-1}(w)$. Thus we have $v \in A$ and

$$[(f|A)^{-1}, w] = [f, v]^{-1} \in P.$$

Consequently $(f|A)^{-1} \in P'$, which completes the proof.

THEOREM 5. *If* $P \subset (L(X))^*$ *and* $F^* \subset P'^*$, *then* $F \subset P'$

Proof. Consider an arbitrary function $f \in F$. Let x be an arbitrary point of the set D_f. Then $[f, x] \in F^*$. Thus $[f, x] \in P'^*$ Hence it follows that there exists a function $g \in P'$ such that $[f, x] = [g, x]$. From the definition of the set P' it follows that $[g, x] \in P$. Thus $[f, x] \in P$. Consequently $f \in P'$, which completes the proof.

THEOREM 6. *If* $M \subset L(X)$, $M^* \subset H^*$ *and* $H \in \operatorname{psg} X$, *then there exists a set* $G \in \operatorname{psg} X$ *such that* $M \subset G$ *and* $H^* = G^*$.

Proof. Assume $G = H^{*\prime}$ From theorem 3 it follows that $G^* = H^*$. Consider an arbitrary function $f \in M$ and an arbitrary point $x \in D_f$. Then $[f, x] \in M^*$. Hence it follows that $[f, x] \in H^*$. Hence $f \in G$. By theorem 2 we have $(G^*)^{-1} \cup G^*G^* \subset G^*$. According to theorem 4 $G \in \operatorname{psg} X$, which completes the proof.

Let us assume for any set $M \subset L(X)$

$$M_0 = (\bigcap \{H^*:\ M \subset H \in \operatorname{psg} X\})', \tag{10.2}$$

(see (7.3) and (10.1)).

THEOREM 7. *The set* M_0 *defined by equality* (10.2) *satisfies the following conditions*:

$$M \subset M_0 \in \operatorname{psg} X, \tag{10.3}$$

$$(F \subset L(X) \wedge F^* \subset M_0^*) \Rightarrow F \subset M_0, \tag{10.4}$$

$$(M^* \subset H^* \wedge H \in \operatorname{psg} X) \Rightarrow M_0^* \subset H^*. \tag{10.5}$$

The pseudogroup $[M]$ *generated by the set* M *is identical with* M_0.

Proof. Denote by P the set $\bigcap\{H^*:\ M \subset H \in \operatorname{psg} X\}$. Consider an arbitrary functional element $p \in P^{-1}$. Then $p^{-1} \in P$. Hence by theorem 2 we deduce that $p \in P$. Consider now an arbitrary functional element $q \in P$ satisfying the condition $c(q) = v(p)$. Then $q \in H^*$ for any pseudogroup $H \supset M$. Hence $q \cdot p \in H^*$. Thus $q \cdot p \in P$. So we have proved that $P^{-1} \cup PP \subset P$. From theorem 4 we deduce that $M_0 = P' \in \operatorname{psg} X$. From the condition $M \subset H \in \operatorname{psg} X$ it follows that $M^* \subset H^*$. Hence $M^* \subset P$. By theorem 3 we obtain

$$M \subset M^{*\prime} \subset P' = M_0.$$

Thus condition (10.3) is satisfied. Consider now an arbitrary set $F \subset L(X)$ such that $F^* \subset M_0^*$. Let G be any pseudogroup including the set M. Then $P \subset G^*$. Hence $M_0 \subset G^{*\prime}$ According to theorem 3 we get

$$F^* \subset G^{*\prime *} = G^*$$

Thus $F^* \subset P$. Consequently $F \subset F^{*\prime} \subset M_0$. To prove implication (10.5) consider an arbitrary pseudogroup H such that $M^* \subset H^*$. From theorem 6 it follows that there exists a pseudogroup G such that $M \subset G$ and $H^* = G^*$. Therefore $P \subset G^*$. Hence $M_0 \subset G^{*\prime}$ Thus

$$M_0^* = G^{*\prime *} = G^* = H^*.$$

According to theorem 1 it follows from conditions (10.3)-(10.5) that the set M_0 satisfies conditions (9.1)-(9.3). Since, however, according to theorem 7 of paragraph 9, there exists exactly one set satisfying those conditions, we have $M_0 = [M]$, which completes the proof.

In the theory of geometrical objects we consider, at a given point of a manifold, a set of local systems of coordinates called the *set of admissible sets of coordinates with respect to a pseudogroup* G (cf. [7]). Local systems of coordinates are regarded as equivalent if one of them may be obtained from another by supplementing it with a functional element determined by some function belonging to a pseudogroup G and a point belonging to the domain of this function. The sets of coordinates systems admissible with regard to the pseudogroup G at a given point are the equivalence classes, of the equivalence relation thus defined. Thus the pseudogroup G induces at a given point a certain local geometry.

If we have a set of local homeomorphisms, then, according to theorem 6, the set of functional elements determined by all possible functions belonging to a pseudogroup including this set is the smallest for the pseudogroup which it generates. The notion of a generated pseudogroup enables us to define the local geometry by means of an arbitrarily chosen set of local homeomorphisms of a certain space (usually a Cartesian space).

11. The problem of the existence of the smallest pseudogroup including a given set of local homeomorphisms. Our previous considerations referred to the construction of a certain pseudogroup called a pseudogroup generated by a family of local homeomorphisms. In this paragraph we give the conditions under which the smallest pseudogroup including a family, given a priori, of local homeomorphisms does not exist. From theorems 7 and 1 of paragraph 8 it immediately follows that for any set M of functions whose domains are included in the set X and whose values belong to $\underline{X}$, the set $(M^{\smile})^{\times}$ satisfies conditions (6.0), (6.1) and (6.2). Moreover, the set $(M^{\smile})^{\times}$ is the smallest one among the sets including the set M and satisfying conditions (6.0), (6.1) and (6.2). The set $(M^{\smile})^{\times}$, however, need not satisfy condition (6.3). The existence of the smallest pseudogroup including the set $(M^{\smile})^{\times}$ is the necessary and sufficient condition for the existence of the smallest pseudogroup including the set M. In view of what has been said, the problem of finding the smallest pseudogroup including a set given a priori can be reduced to finding the smallest pseudogroup including a set given a priori which satisfies conditions (6.0), (6.1) and (6.2).

In this paragraph the operation has the same meaning as in paragraph 8, that is to say, it is the composition of functions satisfying condition (8.1). Denoting the algebraical product of sets we shall omit the sign

THEOREM 1. *If M and N are sets of function mapping the subsets of a set E into the subsets of the set E satisfying the conditions $MM \subset M$ and $NN \subset N$, then*

$$(M \cup N)^{\times} = M \cup L \cup ML \cup MLM \quad \textit{where} \quad L = N \cap \bigcup_{p=1}^{\infty} (NM)^p N.$$

Proof. Assume $A_{2i-1} = M$ and $A_{2i} = N$ for $i = 1, 2, \ldots$. First we prove that

$$(M \cup N)^{\times} = \bigcup_{i=1}^{\infty} (A_1 \ldots A_i \cup A_2 \ldots A_{i+1}). \tag{11.1}$$

Suppose that

$$(M \cup N)^p \subset \bigcup_{i=1}^{\infty} (A_1 \ldots A_i \cup A_2 \ldots A_{i+1}). \tag{11.2}$$

Multiplying both members of this inclusion by $M \cup N$ we obtain

$$\begin{aligned}(M \cup N)^{p+1} &\subset \bigcup_{i=1}^{\infty} (A_1 \ldots A_i M \cup A_2 \ldots A_{i+1} M \cup A_1 \ldots A_i N \cup A_2 \ldots A_{i+1} N) \\ &\subset \bigcup_{i=1}^{\infty} (A_1 \ldots A_i \cup A_2 \ldots A_{i+1}).\end{aligned}$$

Hence it follows that inclusion (11.2) holds for each $p = 1, 2,$ On the other hand, $A_i \subset M \cup N$. Hence

$$A_1 \ldots A_i \cup A_2 \ldots A_{i+1} \subset (M \cup N)^i \subset (M \cup N)^{\times} \quad \text{for} \quad i = 1, 2, \ldots$$

Thus equality (11.1) has been proved. Let us observe that

$$A_2 \ldots A_{2j+1} = (NM)^j \quad \text{for} \quad j = 1, 2, \ldots$$

Hence it follows that

$$(11.3)\quad \bigcup_{j=1}^{\infty} A_1 \dots A_{2j-1} = A_1 \cup A_1 \bigcup_{j=2}^{\infty} A_2 \dots A_{2j-1} = M \cup M \bigcup_{j=1}^{\infty} (NM)^j$$
$$= M \cup MLM,$$

$$(11.4)\quad \bigcup_{j=1}^{\infty} A_2 \dots A_{2j} = N \cup \bigcup_{j=1}^{\infty} A_2 \dots A_{2j-1}\, N = N \cup \bigcup_{j=1}^{\infty} (NM)^j N = L.$$

From (11.4) we obtain

$$(11.5)\quad \bigcup_{j=1}^{\infty} A_1 \dots A_{2j} = ML \quad \text{and} \quad \bigcup_{j=1}^{\infty} A_2 \dots A_{2j+1} = LM.$$

From equalities (11.1), (11.3), (11.4) and (11.5) we immediately obtain equalities occurring in theorem 1.

THEOREM 2. *If a set* $M \subset L(X)$ *satisfies conditions* (6.0), (6.1) *and* (6.2) *but does not satisfy condition* (6.3) *and if there exists a function* μ *whith real positive values defined on the family* $\omega(X)$ *satisfying the conditions*

$$(11.6)\quad \bigwedge_{A,B\in\omega(X)} \big(A \subset B \Rightarrow \mu(A) \leqslant \mu(B)\big),$$

$$(11.7)\quad \bigwedge_{x'\in X} \bigvee_{\varepsilon>0} \bigvee_{A\in\omega(X)} \big(x \in A \wedge \mu(A) < \varepsilon\big),$$

$$(11.8)\quad \min\Big\{\sup\big\{\mu\big(f(D_f)\big)/\mu(D_f)\colon f\in M\big\}, \sup\big\{\mu(D_f)/\mu\big(f(D_f)\big)\colon f\in M\big\}\Big\} < +\infty,$$

then for any pseudogroup $G \supset M$ *there exists a pseudogroup* H *satisfying the condition*

$$M \subset H \subset G \neq H.$$

Proof. Suppose that G is a pseudogroup including the set M and put for any $\varepsilon > 0$

$$N_\varepsilon = \big\{f\colon f\in G \wedge f^{-1}\in M \wedge \mu(D_f) < \varepsilon \wedge \mu\big(f(D_f)\big) < \varepsilon\big\}.$$

Consider an arbitrary function $f\in N_\varepsilon$ and an arbitrary set $B\in\omega(X)$ satisfying condition $\emptyset \neq B \subset D_f$. Then $f|B\in G$,

$$(f|B)^{-1} = f^{-1}|f(B)\in M, \quad \mu\big((f(B)\big) \leqslant \mu\big(f(D_f)\big) < \varepsilon \quad \text{and} \quad \mu(B) \leqslant \mu(D_f) < \varepsilon.$$

Thus $f|B\in N_\varepsilon$. Now let $h\in N_\varepsilon N_\varepsilon$. There exist functions $g, f\in N_\varepsilon$ such that $D_g = f(D_f)$ and $h = g\circ f$. Hence it follows that $h^{-1} = f^{-1}\circ g^{-1}\in M$,

$$\mu(D_h) = \mu(D_f) < \varepsilon \quad \text{and} \quad \mu\big(h(D_h)\big) = \mu\big(g(D_g)\big) < \varepsilon.$$

Then $h \in N_\varepsilon$. Thus we have proved that N_ε satisfies conditions (6.0) and (6.1) and $N_\varepsilon N_\varepsilon \subset N_\varepsilon$. Consider an arbitrary function $f \in M$ and an arbitrary point $x \in D_f$. Since $M \subset G$, there exists a set $A \in \omega(X)$ such that $x \in A \subset D_f$ and $(f|A)^{-1} \in G$. From condition (11.7) it follows that there exist sets $B, C \in \omega(X)$ such that $x \in B$, $f(x) \in C$, $\mu(B) < \varepsilon$ and $\mu(C) < \varepsilon$. Denote by A_0 the set $A \cap B \cap f^{-1}(C)$. Since the function f is continuous, we have $A_0 \in \omega(X)$. Hence it follows that $(f|A_0)^{-1} \in G$, $f|A_0 \in M$, $\mu(A_0) \leqslant \mu(B) < \varepsilon$ and $\mu(f(A_0)) \leqslant \mu(C) < \varepsilon$. Thus $(f|A_0)^{-1} \in N_\varepsilon$. On the other hand, it follows from the definition of the set N_ε that if $f \in N_\varepsilon$ then $f^{-1} \in M$. Consequently the set $M \cup N_\varepsilon$ satisfies condition (6.3). From theorem 6 of paragraph 8 it follows that $(M \cup N_\varepsilon)^\times \in \operatorname{psg} X$. Clearly $M \subset (M \cup N_\varepsilon)^\times \subset G$ for any $\varepsilon > 0$. We shall prove that there exists an $\varepsilon > 0$ such that $(M \cap N_\varepsilon)^\times \neq G$. For, suppose that $(M \cup N_\varepsilon)^\times = G$ for any $\varepsilon > 0$. From theorem 1 it follows that for any $\varepsilon > 0$

$$G = M \cup L_\varepsilon \cup ML_\varepsilon \cup L_\varepsilon M \cup ML_\varepsilon M, \tag{11.9}$$

where

$$L_\varepsilon = N_\varepsilon \cup \bigcup_{\nu=1}^{\infty} (N_\varepsilon M)^\nu N_\varepsilon. \tag{11.10}$$

Since $M \notin \operatorname{psg} X$, we have $M \neq G$. Thus there exists a function $f_0 \in G - M$. Making use of (11.8), suppose that

$$\sup\{\mu(f(D_f))/\mu(D_f)\colon f \in M\} < +\infty.$$

Hence it follows that there exists a constant $c > 1$ such that

$$\mu(f(D_f)) \leqslant c\mu(D_f) \quad \text{for} \quad f \in M.$$

Let

$$\eta = \min\{\mu(D_f),\ (f_0(D_{f_0}))/c\}.$$

If $f_0 \in L_\eta \cup ML_\eta \cup L_\eta M$, then in view of (11.10) and the definition of the set N_η we would have $\mu(D_{f_0}) < \eta$ or $\mu(f_0(D_{f_0})) < \eta$. This, however, would contradict the definition of the number η. Hence according to equality (11.9) it follows that $f_0 \in ML_\eta M$. Thus there exist functions, $f_1, f_2 \in M$ and $g \in L_\eta$ satisfying the condition

$$f_1(D_{f_1}) = D_g, \quad g(D_g) = D_{f_2} \quad \text{and} \quad f_0 = f_2 \circ g \circ f_1.$$

Hence it follows that

$$\mu(f_0(D_{f_0})) = \mu(f_2(D_{f_2})) \leqslant c\mu(g(D_g)) < c\eta,$$

which also contradicts the definition of the number η. By an analogous argument we conclude from $\sup\{\mu(D_f)/\mu(f(D_f))\colon f \in M\} < +\infty$ that (11.9) and (11.10) cannot hold for any $\varepsilon > 0$ either, which completes the proof.

V. SEMI-PSEUDOGROUPS AND A GENERALIZATION OF THE NOTION OF AN ANALYTICAL STRUCTURE

12. Semi-pseudogroups. In a topological space X consider a set C of functions with their values in $\underline{X}$ which satisfies conditions (6.0), (6.1) and (4.4). Any such set C will be called a *semi-pseudogroup* in the space X. The notion of a semi-pseudogroup is a generalization of the notion of a Gołąb pseudogroup, that is to say, we have the following

THEOREM 1. *Any pseudogroup in a space X is a semi-pseudogroup in that space.*

Proof. Let $G \in \mathrm{psg}X$. Consider arbitrary functions $g, f \in G$ such that $D_g \supset f(D_f)$. Since G is a pseudogroup, f is a local homeomorphism. From (6.0) it follows that $D_f \in \omega(X)$. Thus $f(D_f) \in \omega(X)$. According to (6.1) the function $g|f(D_f)$ belongs to G. From (6.2) we have $g \circ f = g|f(D_f) \circ f \in G$, which completes the proof.

Consider an arbitrary set M of functions. Thus the operation $\circ_M$ means the composition defined for ordered pairs $(g, f) \in M \times M$ which satisfy condition $D_g \supset f(D_f)$. Instead of $\circ_M$ we shall simply write $\circ$, while by $\mathrm{CA}(M, \circ)$ we shall denote the algebraical closure of the set M with respect to the operation $\circ_M$ (see (5.5)). If $f(D_f) \subset \underline{X}$ for any function $f \in M$, then the set $M^{\smile}$ is defined by equality (8.4).

THEOREM 2. *If a set C of functions with their values in the set $\underline{X}$, satisfies condition* (6.0), *then the set* $\mathrm{CA}(C^{\smile}, \circ)$ *is the smallest semi-pseudogroup including the set C.*

Proof. First we show that $\mathrm{CA}(C^{\smile}, \circ)$ is a semi-pseudogroup including the set C. Indeed, since C satisfies condition (6.0), we have $C \subset C^{\smile}$. By theorem 7 of paragraph 8 the set $C^{\smile}$ satisfies conditions (6.0) and (6.1). Since the domain of a composed function is identical with the domain of an internal function, we deduce by (5.2) and (5.5) that the set $\mathrm{CA}(C^{\smile}, \circ)$ satisfies conditions (6.0) and (6.1). From theorem 3 of paragraph 5 it follows that $C^{\smile} \subset \mathrm{CA}(C^{\smile}, \circ)$ and the set $\mathrm{CA}(C^{\smile}, \circ)$ satisfies condition (4.4). It remains to prove that $\mathrm{CA}(C^{\smile}, \circ)$ is the smallest one among the semi-pseudogroups including the set C. Accordingly consider an arbitrary semi-pseudogroup G including the set C. Since the set G satisfies condition (6.1), we have $G^{\smile} \subset G$. In view of the fact that G satisfies condition (4.4), the inclusion $\mathrm{CA}(G, \circ) \subset G$ holds. Thus we have $C \subset C^{\smile} \subset G^{\smile} \subset G \subset \mathrm{CA}(G, \circ) \subset G$, which completes the proof.

THEOREM 3. *If a set C of functions whose values belong to the set $\underline{X}$ satisfies condition* (6.0), *then among the semi-pseudogroups G including the set C and satisfying the condition*

$$(12.1) \qquad \bigwedge_f \Big(\big(D_f \neq \emptyset \wedge \bigwedge_{x \in D_f} \bigvee_{A \in \omega(X)} (x \in A \subset D_f \wedge f|A \in G)\big) \Rightarrow f \in G \Big),$$

there exists one which is the smallest.

Proof. Consider an arbitrary set C of functions with their values in the set $\underline{X}$ which satisfies condition (6.0). By $S(C)$ we denote the set of all semi-pseudogroups including the set C and satisfying condition (12.1). First we prove that the set G_0 of all functions f satisfying the conditions $\emptyset = D_f \in \omega(X)$ and $f(D_f) \subset \underline{X}$ belongs to $S(C)$. Indeed, we immediately see that conditions (6.0), (6.1) and (4.4) are satisfied. To prove that G_0 satisfies condition (12.1) consider an arbitrary function f and suppose that $D_f \neq \emptyset$ and that for any point $x \in D_f$ there exists a set $A_x \in \omega(X)$ such that $x \in A_x \subset D_f$ and $f|A_x \in G_0$. From $D_f = \bigcup \{A_x\colon x \in D_f\}$ it follows that $D_f \in \omega(X)$. Since for every point $x \in D_f$ the condition $f(x) = (f|B_x)(x) \in \underline{X}$ holds, we have $f \in G_0$. Observe now that the set $\bigcap S(C)$ satisfies conditions (6.0), (6.1) and (4.4). Consider an arbitrary function f which for $H = \bigcap S(C)$ satisfies the condition:

$$D_f \neq \emptyset \wedge \bigwedge_{x \in D_f} \bigvee_{B \in \omega(X)} (x \in B \subset D_f \wedge f|B \in H). \tag{12.2}$$

Let G be any semi-pseudogroup belonging to the set $S(C)$. From condition (12.2) it follows that for any point $x \in D_f$ there exists a set $B \in \omega(X)$ such that $x \in B \subset D_f$ and $f|B \in G$. Since G satisfies condition (12.1), we have $f \in G$. Thus $f \in \bigcup S(C)$. This proves that $\bigcap S(C) \in S(C)$, which completes the proof.

The theorem just proved enables us to construct semi-pseudogroups satisfying condition (12.1).

In paragraph 8 for a given set M of functions whose values belong to the set $\underline{X}$ we have defined the set $M^\frown$ by means of equality (8.6) In particular the set $C^\frown$ has been defined for every semi-pseudogroup C.

THEOREM 4. *For any semi-pseudogroup C of functions mapping continuously the space $X|D_f$ into the space X, the set $C^\frown$ is the smallest semi-pseudogroup including C and satisfying condition* (12.1).

Proof. The inclusion $C \subset C^\frown$ immediately follows from (8.6). From theorem 8 of paragraph 8 it follows that the set $C^\frown$ satisfies conditions (6.0) and (6.1). Consider now arbitrary functions $g, f \in C^\frown$ such that $f(D_f) \subset D_g$. Let $x \in D_f$. There exist sets $A, B \in \omega(X)$ satisfying the conditions

$$x \in A \subset D_f, \quad f(x) \in B \subset D_g, \quad f|A \in C \quad \text{and} \quad g|B \in C.$$

From the continuity of the function $f|A$ it follows that the set $(f|A)^{-1}(B)$ is open. From (6.1) it follows that

$$f|\big(A \cap (f|A)^{-1}(B)\big) \in C.$$

Since C satisfies condition (4.4), we have

$$(g \circ f)|\big(A \cap (f|A)^{-1}(B)\big) = g|B \circ f|\big(A \cap (f|A)^{-1}(B)\big) \in C.$$

Hence it follows $g \circ f \in C^{\frown}$. Consequently the set $C^{\frown}$ is a semi-pseudogroup. To prove that the set $C^{\frown}$ satisfies condition (12.1) consider an arbitrary function f satisfying condition (12.2) for $G = C^{\frown}$. Let $x \in D_f$. There exists a set $B \in \omega(X)$ such that $x \in B \subset D_f$ and $f|B \in C^{\frown}$. According to the definition of the set $C^{\frown}$ there exists a set $A \in \omega(X)$ such that $x \in A \subset B$ and $f|A \in C$. Thus $f \in C^{\frown}$.

Suppose that H is any semi-pseudogroup including the set C and satisfies condition (12.1). Consider an arbitrary function $f \in C^{\frown}$. Let $x \in D_f$. There exists a set $B \in \omega(X)$ such that $x \in B \subset D_f$ and $f|B \in C$. Hence it follows that $f|B \in H$. From condition (12.1) we deduce that $f \in H$. Accordingly $C^{\frown} \subset H$, which completes the proof.

For a given set C of functions whose values belong to the set X and which satisfies condition (6.0) we denote by $C^{\sim}$ the smallest of the semi-pseudogroups H including C and satisfying condition (12.1)

THEOREM 5. *For every set C of functions which satisfies condition* (6.0) *and the following condition*

$$\bigwedge_{f \in C} \big(f \in (X|D_f \to X)\big), \tag{12.3}$$

the equality $C^{\sim} = \big(\mathrm{CA}(C^{\smile}, \circ)\big)^{\frown}$ *holds.*

Proof. First we prove that if C satisfies conditions (6.0) and (12.3), then the set $\mathrm{CA}(C^{\smile}, \circ)$ also satisfies those conditions. Consider an arbitrary function $g \in C^{\smile}$. The definition of the set $C^{\smile}$ implies that there exist a function $f \in C$ and a set $A \in \omega(X)$ such that $\emptyset \neq A \subset D_f$ and $g = f|A$. From condition (12.3) it follows that $g \in (X|A \to X)$. Accordingly, the set $C^{\smile}$ satisfies condition (12.3). The continuity of a composition of continuous functions implies that the set $\mathrm{CA}(C^{\smile}, \circ)$ satisfies condition (12.3). From theorem 2 we deduce that $\mathrm{CA}(C^{\smile}, \circ)$ is a semi-pseudogroup including the set C. It follows from theorem 4 that the set $\big(\mathrm{CA}(C^{\smile}, \circ)\big)^{\frown}$ is a semi-pseudogroup satisfying condition (12.1) and including the set C. Thus $C^{\sim} \subset \big(\mathrm{CA}(C^{\smile}, \circ)\big)^{\frown}$. Consider now an arbitrary semi-pseudogroup G including the set C and satisfying condition (12.1). From theorem 2 it follows that $\mathrm{CA}(C^{\smile}, \circ) \subset G$. Theorem 4 yields the inclusion $\big(\mathrm{CA}(C^{\smile}, \circ)\big)^{\frown} \subset G$. Thus $\big(\mathrm{CA}(C^{\smile}, \circ)\big)^{\frown} \subset C^{\sim}$ which completes the proof.

13. The notion of an analytical structure. Let X be a topological space. Speaking of the analytical structure of a given manifold, we usually consider, in a parametrical space X, a set of transformations satisfying, besides conditions (6.0)-(6.2), where X denotes a Cartesian n-dimensional space, some conditions concerning differentiability. One arrives at the notion of an analytical structure by identifing some sets of homeomorphisms which map some open sets of the manifold in question onto open sets of an n-dimensional Cartesian space.

This paragraph contains a generalization of the well-known notion of an analytical structure. We only assume about the space X that it is a topological space, and the assumptions concerning the analytical character connected with the transition from one parametrization to another are replaced by conditions formulated in purely topological terms.

Consider arbitrary topological spaces T and X. Let C be an arbitrary set of functions whose values belong to the set $\underline{X}$ and which satisfies condition (6.0). By $\mathscr{F}(T, X, C)$ we denote the set of all sets F whose elements are functions satisfying the following conditions:

$$\text{(13.1)} \qquad \bigwedge_{f \in F} \big(\emptyset \neq D_f \in \omega(T) \wedge f \in (T|D_f \leftrightarrow X|f(D))\big),$$

$$\text{(13.2)} \qquad \bigcup \{D_f\colon f \in F\} = \underline{T},$$

$$\text{(13.3)} \qquad \bigwedge_{g,f \in F} \big((D_g \cap D_f \neq \emptyset) \Rightarrow g \circ f^{-1}|f(D_f \cap D_g) \in C^{\sim}\big).$$

From conditions (13.1) and (13.3) it follows that if $f \in F$, where $F \in \mathscr{F}(T, X, C)$, then $f \circ f^{-1} \in C^{\sim}$. Since the set $C^{\sim}$ satisfies condition (6.0), we have $f(D_f) \in \omega(X)$.

For the elements of the set $\mathscr{F}(T, X, C)$ we define a relation $\equiv_C$ in the following way:

$$\text{(13.4)} \qquad F_1 \equiv_C F_2 \Leftrightarrow \big(F_1 \in \mathscr{F}(T, X, C) \wedge F_2 \in \mathscr{F}(T, X, C) \wedge F_1 \cup F_2 \in \mathscr{F}(T, X, C)\big).$$

THEOREM 1. *The relation* $\equiv_C$ *is an equivalence in the set* $\mathscr{F}(T, X, C)$.

Proof. The reflexivity and the symmetry of the relation $\equiv_C$ are obvious. To prove the transitivity of this relation consider arbitrary sets F_1, F_2, F_3 belonging to $\mathscr{F}(T, X, C)$ such that $F_1 \cup F_2$ and $F_2 \cup F_3$ belong to $\mathscr{F}(T, X, C)$. Clearly the set $F_1 \cup F_3$ satisfies conditions (13.1) and (13.2). To prove that the set $F_1 \cup F_3$ satisfies condition (13.3) consider arbitrary functions $g, f \in F_1 \cup F_3$ such that $D_f \cap D_g \neq \emptyset$. If $g, f \in F_1$ or $g, f \in F_3$, then by (13.3) the functions f and g satisfy the condition

$$\text{(13.5)} \qquad g \circ f^{-1}|f(D_f \cap D_g) \in C^{\sim}$$

We may thus assume that $g \in F_1$ and $f \in F_3$. Consider an arbitrary point $x \in D_f \cap D_g$. Condition (13.2) implies the existence of a function $h \in F_2$ such that $x \in D_h$. From condition (13.1) in view of $f(D_f) \in \omega(X)$ we deduce that $f(D_f \cap D_g \cap D_h) \in \omega(X)$. From

$$h \circ f^{-1}|f(D_f \cap D_h) \in C^{\sim}$$

it follows that

$$h \circ f^{-1}(D_f \cap D_g \cap D_h) \in C^{\sim}$$

and the set $C^{\sim}$ satisfies condition (6.0). Consequently the function

$$g \circ f^{-1}|f(D_f \cap D_g \cap D_h)$$
$$= \big(g \circ h^{-1}|f(D_h \cap D_g)\big) \circ \big(h \circ f^{-1}|f(D_f \cap D_g \cap D_h)\big) \in C^{\sim}$$

The fact that $C^{\sim}$ satisfies condition (12.1) implies that the functions g and f satisfy condition (13.5). Thus the relation $\equiv_C$ is transitive.

The equivalence classes of the relation $\equiv_C$ will be called *analytical structures* of a class C of a topological space T, with respect to a topological space X.

In particular, if the space X is a Cartesian n-dimensional space with the ordinary topology and C is the class of functions defined on open sets of the space X and possessing derivatives to the kth order inclusively, then the condition $\mathscr{F}(T, X, C) \neq \emptyset$ means that the space T is a manifold of the class C^k.

In general we shall call a topological space T a *manifold of the class C with respect to the space X* if the condition

$$\mathscr{F}(T, X, C) \neq \emptyset$$

is satisfied.

For a given family F of homeomorphisms f mapping the topological space $T|D_f$ onto the topological space $X|f(D_f)$, by F° we denote a set defined by the equality

$$F^{\circ} = \{g \circ f^{-1}|f(D_f \cap D_g) \colon \; g, f \in F \wedge D_f \cap D_g \neq \emptyset\}.$$

THEOREM 2. *If in a topological space X a set C of functions with their values in the set X satisfies condition* (6.0), *then for every set $F \in \mathscr{F}(T, X, C)$ the set $(F^{\circ\smile})^{\times}$* (see (8.6) and (8.1)) *is a pseudogroup in the space X and $F \in \mathscr{F}\big(T, X, (F^{\circ\smile})^{\times}\big)$.*

Proof. Suppose that $F \in \mathscr{F}(T, X, C)$. Theorem 7 of paragraph 8 implies that the set $F^{\circ\smile}$ satisfies conditions (6.0) and (6.1). Consider an arbitrary function $h \in F^{\circ\smile}$. According to definition (8.6) there exist functions $g, f \in F$ and a set $B \in \omega(X)$ such that $\emptyset \neq B \subset f(D_f \cap D_g)$ and $h = g \circ f^{-1}|B$. From condition (13.1) it follows that $f^{-1}(B) \in \omega(T)$. Similarly, in view of $g(D_g) \in \omega(X)$, we have $g\big(f^{-1}(B)\big) \in \omega(X)$. On the other hand, according to the definition of the set F°, the function $f \circ g^{-1}|f(D_g \cap D_f) \in F^{\circ}$ Thus the function

$$h^{-1} = \big(f \circ g^{-1}|f(D_g \cap D_f)\big)\,|\,g\big(f^{-1}(B)\big) \in F^{\circ\smile}$$

Consequently the set $F^{\circ\smile}$ satisfies condition (6.3). By theorem 6 of paragraph 8 we deduce that $F^{\circ\smile\times} \in \mathrm{psg}\, X$.

In view of the fact that the set F satisfies conditions (13.1) and (13.2) it suffices to prove that the set $C = (F^{\circ\smile})^{\times}$ satisfies condition (13.3). Accordingly, consider functions $g, f \in F$ such that $D_g \cap D_f \neq \emptyset$. From the definition of the set F° it follows that

$$g \circ f^{-1} | f(D_f \cap D_g) \in F^{\circ} \subset \left((F^{\circ\smile})^{\times}\right)^{\sim},$$

which completes the proof.

BIBLIOTEKA
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REFERENCES

[1] N. Bourbaki, *Topologie générale. Les structures fondamentales de l'analyse*.

[2] H. Brandt, *Über eine Verallgemeinerung des Gruppenbegriffs*, Math. Annalen 96 (1926), pp. 360-366.

[3] Ch. Ehresmann, *Gattungen von lokalen Strukturen*, Jahresbericht der Deutschen Math. Vereinigung 60 (1957).

[4] S. Eilenberg and S. MacLane, *General theory of natural equivalences*, Trans. of Amer. Math. Soc. Vol. 58, 1945.

[5] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton, New Jersey, 1952.

[6] A. Fraenkel, *Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre*, Math. Annalen 86 (1922). pp. 230-237.

[7] S. Gołąb, *Rachunek tensorowy*, Warszawa, 1956.

[8] — *Über den Begriff der "Pseudogruppe von Transformationen"*, Math. Annalen 116 (1939), pp. 768-780.

[9] A. Grothendieck, *Sur quelques points d'algébre homologique*, Tôhoku Math. J. 9 (1957), pp. 119-221.

[10] R. Isbell, *Subobject, adequacy, completeness and categories of algebras*, Rozprawy Matematyczne XXXVI, Warszawa 1964.

[11] K. Kuratowski i A. Mostowski, *Teoria mnogości*, Monografie Matematyczne, Warszawa - Wrocław 1952.

[12] A. Nijenhuis, *Theory of the geometric object*, Amsterdam, 1952.

[13] Z. Semadeni, *Projectivity, injectivity and duality*, Rozprawy Matematyczne XXXV, Warszawa 1963.

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