ANNALES
POLONICI MATHEMATICI
XL (1983)

On typical-real functions

by ZBIGNIEW JERZY JAKUBOWSKI (Łódź)

Abstract. In [13] investigations into the family T_α, $\alpha \geqslant 0$, of the functions

$$f(z) = z+a_2z^2+\ldots, \quad a_n = \bar{a}_n, \quad n = 2, 3, \ldots,$$

analytical in the disc $|z| < 1$, and such that

$$\operatorname{Re}\{\alpha(1-z^2)f'(z)+(1-\alpha)(1-z^2)f(z)/z\} > 0 \quad \text{for } |z| < 1$$

have been undertaken. T_0 is the class of typically-real functions, T_1 is the class of functions convex in the direction of the imaginary axis. Moreover, $T_\alpha \subset T$ for all $\alpha \geqslant 0$.

In the present paper some further properties of the classes T_α, $\alpha \geqslant 0$, have been obtained. In particular, interrelations between the families T_0 and T_α, $\alpha > 0$, have been pointed out.

1. Let H_R denote the family of all functions

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n, \quad a_n = \bar{a}_n, \quad n = 2, 3, \ldots, \tag{1}$$

analytical in the disc $K = K_1$, where $K_r = \{z\colon |z| < r\}$.

The geometrical properties of certain subclasses of the family H_R have already been investigated by many authors. The most interesting properties seem to be those of T-class of typically-real functions [11], of S_R-class of univalent functions [5], and of C_R-class of the functions convex in the direction of the imaginary axis [10]. Obviously $C_R \subset S_R \subset T \subset H_R$. It is also known that a function of the form (1) belongs to T or C_R if and only if the conditions

$$\operatorname{Re}\{(1-z^2)z^{-1}f(z)\} > 0 \tag{2}$$

or

$$\operatorname{Re}\{(1-z^2)f'(z)\} > 0 \tag{3}$$

respectively, hold in K.

In [13], a method employing (2)–(3), and permitting simultaneous investigation of the classes T and C_R has been worked out (cf. [8]). Let T_α, $\alpha \geqslant 0$ – arbitrary fixed, be the family of all functions of the form (1) satisfying in the disc K the condition

$$\operatorname{Re} J(\alpha, z, f) > 0, \tag{4}$$

where

$$J(\alpha, z, f) = \alpha(1-z^2)f'(z)+(1-\alpha)(1-z^2)z^{-1}f(z). \tag{5}$$

It is immediately seen from (1)–(5) that $T_0 = T$, $T_1 = C_R$. Moreover, $f \in T_\alpha, \alpha \geqslant 0$, if and only if f is a special solution of the differential equation

$$\alpha z f'(z)+(1-\alpha)f(z) = F(z), \quad z \in K, \quad F \in T. \tag{6}$$

From equation (6), and from (4), (5) the following theorems can be obtained [13]:

THEOREM 1. *The function* $f \in T_\alpha, \alpha > 0$, *if and only if*

$$f(z) = \frac{1}{\alpha}\int_0^1 t^{-2+1/\alpha} F(tz)\,dt, \quad z \in K, \tag{7}$$

where $F \in T$.

THEOREM 2. *If* $0 \leqslant \alpha_1 \leqslant \alpha_2$, *then*

$$T_{\alpha_2} \subset T_{\alpha_1}. \tag{8}$$

COROLLARY 1. *Every function* $f \in T_\alpha, \alpha \geqslant 0$, *is typically-real.*

COROLLARY 2. *Any function* $f \in T_\alpha, \alpha \geqslant 1$, *is univalent and convex in the direction of the imaginary axis (with real coefficients).*

In the present paper we undertake further investigations into the family $T_\alpha, \alpha \geqslant 0$. Special emphasis will be put on interrelations between the classes T and $T_\alpha, \alpha > 0$.

2. First, let us consider the "limit" cases: $\alpha \to +\infty, \alpha \to 0^+$.

It follows from (4)–(5) that the identity function belongs to each family $T_\alpha, \alpha \geqslant 0$. Moreover, from (7), for each function $f \in T_\alpha, \alpha > 0$,

$$f(z)-z = \frac{1}{\alpha}\int_0^1 t^{1/\alpha-2}\big(F(zt)-tz\big)\,dt, \quad z \in K, \tag{9}$$

$F \in T$. Since in the family T the following estimations are valid [11]:

$$|a_n| \leqslant n, \quad n = 2, 3, \ldots, \tag{10}$$

from (9) we obtain: for any $\varepsilon > 0$ and $r \in (0, 1)$ there is α' such that for all $\alpha \geqslant \alpha'$ and for each function $f \in T_\alpha$, the condition

$$|f(z)-z| < \varepsilon \quad \text{for } z \in K_r \tag{11}$$

holds. Thus, when α increases infinitely, the classes T_α are "contracting" (cf. Theorem 2, (8)) to the set $T_{+\infty}$ whose only element is the identity function.

From Theorem 1 it also follows that for each function $f \in T_\alpha, 0 < \alpha < 1$, there exists a function $F \in T$ such that in the disc K

$$f(z)-F(z) = \frac{\alpha}{1-\alpha}F(z)-\frac{1}{1-\alpha}\int_0^1 t^{1/\alpha-1}zF'(tz)\,dt. \tag{12}$$

Making use of (12) and (10) we infer that in every disc $K_r, 0 < r < 1$, the difference $f(z)-F(z)$ is arbitrarily small, provided that α is sufficiently close to zero.

3. We shall now investigate the problem contrary in a sense to that solved in Corollary 1. Let $f \in T$. Put

$$\alpha = \alpha_f = \sup\{\beta\colon\ f \in T_\beta, \beta \geqslant 0\}.$$

The function f will be called *α-typically-real*. We shall then write $f \in T(\alpha)$ (cf. [7]). Obviously for every $f \in T, 0 \leqslant \alpha_f \leqslant +\infty$.

Observe that from Theorem 2 and (4)–(5) it follows that if $f \in T(\alpha)$, then for all $0 \leqslant \beta < \alpha$ and $z \in K$

$$\operatorname{Re} J(\beta, z, f) > 0.$$

Passing to the limit with $\beta \to \alpha^-$ we obtain

$$\operatorname{Re} J(\alpha, z, f) \geqslant 0.$$

Since $\operatorname{Re} J$ is harmonic in K and $J(\alpha, 0, f) = 1$, from the maximum principle for harmonic functions we get (4). Thus $f \in T_\alpha$. Hence

THEOREM 3. *The function f belongs to $T(\alpha), 0 \leqslant \alpha < +\infty$, if and only if $f \in T_\beta$ for every $\beta \in \langle 0, \alpha\rangle$ and $f \notin T_\beta$ for $\beta > \alpha$.*

The function f is in $T(+\infty)$ if and only if $f \in T_\alpha$ for every $\alpha \geqslant 0$.

COROLLARY 3. $T = \bigcup_{\alpha \in \langle 0, +\infty\rangle} T(\alpha)$.

Also, from (11), $T(+\infty) = T_{+\infty}$.

Consider the functions $f_1(z) = z(1+z)^{-2}$, $f_2(z) = z(1+z)^{-1}, z \in K$. Directly from the definition of the class T_α and Theorem 3 it follows that $f_1 \in T$, $f_1 \notin T_\alpha, \alpha > 0$. Thus $f_1 \in T(0)$. Analogously, $f_2 \in T(1)$. Therefore the families $T(0)$, $T(1)$ are not empty. Moreover, from Theorem 3 we get

COROLLARY 4. *For every $\alpha \in \langle 0, +\infty\rangle$, $T(\alpha) \neq \emptyset$.*

Indeed, it is enough to investigate the case $\alpha \in \langle 0, +\infty)$. Let

$$f_\alpha(z) = z+\frac{1}{2(1+\alpha)}z^2, \quad z \in K.$$

It can be seen that the function satisfies (6), where $F(z) = z+\frac{1}{2}z^2$. Since $F \in T$, $f_\alpha \in T_\alpha$ for every $\alpha \geqslant 0$.

Observe that functions of the form $F(z) = z+az^2$, $z \in K$, $a = \bar{a}$, are typically-real if and only if $-\frac{1}{2} \leqslant a \leqslant \frac{1}{2}$. Hence and from (6) $f_\alpha \notin T_\beta$ for any $\beta > \alpha$. Thus for every $\alpha \in \langle 0, +\infty)$, $f_\alpha \in T(\alpha)$.

4. We shall now consider some special cases. Let

$$F(z;\, \tau) = z(1-2\tau z+z^2)^{-1}, \quad z \in K,\ -1 \leqslant \tau \leqslant 1. \tag{13}$$

Clearly, for every admissible τ, $F \in T$.

For an arbitrary parameter $\tau \in \langle -1, 1\rangle$ let us define in K the functions

$$f(z;\, \tau, \alpha) = \frac{1}{\alpha}\int_0^1 t^{1/\alpha-2} F(zt;\, \tau)\,dt, \quad \text{when } \alpha > 0, \tag{14}$$

$$f(z;\, \tau, 0) = F(z;\, \tau). \tag{15}$$

By (12), definition (15) is a natural consequence of (14).

From (13) and Theorem 1 we get

COROLLARY 5. *For any* $\alpha \geqslant 0$ *and* $-1 \leqslant \tau \leqslant 1$

$$f(z;\, \tau, \alpha) \in T_\alpha. \tag{16}$$

We shall prove a stronger result. To this aim, denote by $S_R^* \subset S_R$ the subclass of starlike functions, [1] (with real coefficients). Clearly $F(z;\tau) \in S_R^*$.

The following theorem holds true:

THEOREM 4. *For any function* $F \in S_R^*$ *and every* $0 < \alpha \leqslant 1$ *the function* f *defined by* (7) *belongs to* S_R^*.

Proof (cf. [9]). Let $0 < \alpha \leqslant 1$, $F \in S_R^*$, and $G(z) = z\big(F(z)/z\big)^\alpha$, $z \in K$. Then $\operatorname{Re}\{zG'(z)G^{-1}(z)\} > 1-\alpha$, $z \in K$.

Put

$$H(z) = \left[\frac{1}{\alpha}\int_0^z G^{1/\alpha}(\zeta)\frac{d\zeta}{\zeta}\right]^\alpha, \quad z \in K.$$

From the properties of the function G we obtain $H(0) = H'(0)-1 = 0$, $H(z)\cdot H'(z) \neq 0$ for $0 < |z| < 1$, and

$$\operatorname{Re}\left\{(1-\alpha)\frac{zH'(z)}{H(z)}+\alpha\left(1+\frac{zH''(z)}{H'(z)}\right)\right\} > 1-\alpha, \quad z \in K.$$

Thus, [12], $\operatorname{Re}\left\{\dfrac{zH'(z)}{H(z)}\right\} > 1-\alpha$ in K. From (7) and the definitions of the functions G and H we have

$$H(z) = z\left(\frac{f(z)}{z}\right)^\alpha, \quad z \in K,$$

and consequently $\operatorname{Re}\{zf'(z)/f(z)\} > 0, z \in K$. Hence f is a starlike function. Moreover, it follows from (7) that f is of form (1), and thus $f \in S_R^*$.

From Theorem 4 we obtain an extension of Corollary 2.

COROLLARY 6. *For any $\alpha \in (0, 1)$ and every function $F \in S_R^*$ function* (7) *is univalent.*

The following question remains open: Is the respective function (7) univalent for every $\alpha \in (0, 1)$ and any $F \in S_R$?

Let $S_\alpha = T_\alpha \cdot S_R$, $\alpha \geqslant 0$. Obviously $S_0 = S_R$, $S_\alpha = T_\alpha$ for $\alpha \geqslant 1$, and $S_\alpha \subset T_\alpha$ for $0 \leqslant \alpha < 1$. Also from Theorem 4, we get

COROLLARY 7. *For any $\alpha \geqslant 0$, $-1 \leqslant \tau \leqslant 1$,*

$$f(z; \tau, \alpha) \in S_\alpha.$$

The question arises whether $S_\alpha \neq T_\alpha, 0 \leqslant \alpha < 1$. Consider the functions

$$F_0(z) = z\frac{1+z^2}{(1-z^2)^2}, \quad z \in K,$$

(17)
$$f_0(z) = \frac{1}{\alpha}\int_0^1 t^{-2+1/\alpha} F_0(zt)\,dt, \quad z \in K,\ 0 < \alpha < 1.$$

Each family T_α being convex, from (13)–(16) it follows that $F_0 \in T_0$, $f_0 \in T_\alpha$. Moreover, $F_0 \notin S_R$. We are going to show that $f_0 \notin S_\alpha$. Indeed, for $r \in (0, 1)$

$$F_0(ir) = ir\frac{1-r^2}{(1+r^2)^2}, \quad f_0(ir) = \frac{ir}{\alpha}\int_0^1 t^{1/\alpha - 1}\frac{1-r^2 t^2}{(1+r^2 t^2)^2}\,dt.$$

It can easily be seen that for each fixed $\alpha \in (0, 1)$ the equation

(18)
$$\frac{1-r^2}{(1+r^2)^2} = \frac{1-\alpha}{\alpha}\int_0^1 t^{1/\alpha - 1}\frac{1-r^2 t^2}{(1+r^2 t^2)^2}\,dt$$

has got one solution in the interval (0, 1). Denote it by r_α. Thus

$$F_0(ir_\alpha) = (1-\alpha) f_0(ir_\alpha).$$

Therefore from (6)

$$f_0'(ir_\alpha) = 0.$$

This shows that f_0 is not univalent in K, i.e.

$$S_\alpha \neq T_\alpha, \quad 0 \leqslant \alpha < 1.$$

From (18) it follows that $r = r_\alpha$ satisfies the equation

(19)
$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^2}{2n\alpha + 1} r^{2n} = 0.$$

Thus in the limit case we have

$$r_1 = 1, \quad r_0 = \sqrt{2} - 1.$$

Observe that r_0 is the radius of conformity (univalency) of the class T (see for instance [15]), $r_1 = 1$ and $T_1 = S_1$. The above results allow to formulate the following hypothesis: for every $\alpha \in (0, 1)$ the radius of conformity of the family T_α is equal to the solution $r_\alpha \in (0, 1)$ of equation (19).

5. Let M denote the family of functions μ defined on the interval $\langle -1, 1\rangle$, non-decreasing and normed by the condition $\int_{-1}^{1} d\mu(\tau) = 1$. Thus M is the set of probabilistic measures on the segment $\langle -1, 1\rangle$.

From Theorem 1, the known structural formula in the family T, [6], and Fubini's theorem one obtains a structural formula for the class $T_\alpha, \alpha > 0$, [13]. Summing up the basic properties of the family T_α we get

THEOREM 5. *For an arbitrary fixed* $\alpha, \alpha > 0$, *the following conditions are equivalent*:

(a) $f \in T_\alpha$;

(b) *there is only one function* $F \in T$ *such that* f *is a special solution of the differential equation* (6);

(c) *there is only one function* $F \in T$ *such that* f *satisfies* (7);

(d) *there is only one function* $\mu \in M$ *such that*

$$f(z) = \int_{-1}^{1} f(z; \tau, \alpha) d\mu(\tau), \quad z \in K, \tag{20}$$

where $f(z; \tau, \alpha)$ *is defined by* (13)–(14).

Obviously, from (12)–(15) it follows that in the limit case $\alpha = 0$, (20) reduces to the structural formula in the family T.

From (d) and Brickman's theorem, [3], one immediately obtains the set E_{T_α} of extremal points of the family T_α:

$$E_{T_\alpha} = \{f(z; \tau, \alpha): -1 \leqslant \tau \leqslant 1\}, \tag{21}$$

where $f(z; \tau, \alpha)$ are defined in (13)–(15).

Let $\overline{\mathrm{co}}\, A$ stand for the closed convex hull of a set A. It is known that $\overline{\mathrm{co}}\, S_R = T$, [3]. Since (cf. [13]) the clas T_α is convex and compact, and $S_\alpha = T_\alpha, \alpha \geqslant 1$, thus $\overline{\mathrm{co}}\, S_\alpha = T_\alpha$ for $\alpha \geqslant 1$. We shall prove

THEOREM 6. *For every* $\alpha \in (0, 1)$

$$\overline{\mathrm{co}}\, S_\alpha = T_\alpha. \tag{22}$$

Proof. From the properties of the classes S_α and T_α we have

$$\overline{\mathrm{co}}\, S_\alpha \subset \overline{\mathrm{co}}\, T_\alpha = T_\alpha.$$

On the other hand, by Corollary 7 and (21) we have

$$E_{T_\alpha} \subset S_\alpha.$$

Thus, by Krein–Milman's theorem (cf. [14], p. 172), we get $T_\alpha = \overline{\text{co}}\, E_{T_\alpha} \subset \overline{\text{co}}\, S_\alpha$, which completes the proof.

Thus (22) holds true for every $\alpha \geqslant 0$.

6. Let $V_n(\alpha)$, $n \geqslant 2$, $\alpha \geqslant 0$, be a region of values of the system $(a_2, a_3, \ldots, a_n)$ of coefficients of functions of the class T_α. The set $V_n(0)$ is known. It is a linear image of a respective region of values of the system of coefficients of the Carathéodory functions with real coefficients ($P(z) = (1-z^2)F(z)/z$, $z \in K$, $F \in T$). Directly from (d) and the Carathéodory theorem, [4], we obtain $V_n(\alpha)$ for every $\alpha > 0$, [13]. From the properties of the set $V_3(\alpha)$ we get the following sharp estimations of Golusin's functional:

COROLLARY 8. *If $f \in T_\alpha$, $\alpha \geqslant 0$, then*

$$a_3 - \beta a_2^2 \leqslant \begin{cases} 3(2\alpha+1)^{-1} & \text{for } \beta > 0, \\ 3(2\alpha+1)^{-1} - 4\beta(1+\alpha)^{-2} & \text{for } \beta \leqslant 0. \end{cases} \tag{23)–(24}$$

COROLLARY 9. *If $f \in T_\alpha, \alpha \geqslant 0$, then*

$$a_3 - \beta a_2^2 \geqslant \begin{cases} 3(2\alpha+1)^{-1} - 4\beta(1+\alpha)^{-2} & \text{for } \beta \geqslant \dfrac{(1+\alpha)^2}{1+2\alpha}, \\ -(1+2\alpha)^{-1} & \text{for } \beta < \dfrac{(1+\alpha)^2}{1+2\alpha}. \end{cases} \tag{25)–(26}$$

In the case of estimations (24)–(26) the function $f(z;\ \pm 1, \alpha)$ or $f(z;\ 0;\ \alpha)$ is an extremal function. From Corollary 7 it follows that the estimations are sharp also in the class $S_\alpha, 0 \leqslant \alpha < 1$.

The equality sign in (23) takes place for function (17), because

$$f_0(z) = z + \frac{3}{1+2\alpha} z^3 + \ldots, \quad z \in K.$$

As known, $f_0 \notin S_\alpha$ for $0 \leqslant \alpha < 1$.

7. From general properties of the classes of functions defined by structural formulas (see e.g. [2]) and from (d) it follows, [13], that the region D of the values of $f(z), f \in T_\alpha$, is the convex hull of the curve

$$w = f(z; \tau, \alpha) \quad -1 \leqslant \tau \leqslant 1 \ (z \in K, \text{const}).$$

Moreover,

COROLLARY 10. *If $\Phi(w)$ is a given real continuous function in $D^* \supset D$, then $\max_{f \in T_\alpha} \Phi(f(z))$ is attained only by the functions*

$$f(z) = \lambda f(z; \tau_1, \alpha) + (1-\lambda) f(z; \tau_2, \alpha),$$

where $\lambda \in \langle 0, 1\rangle, \tau_1, \tau_2 \in \langle -1, 1\rangle$.

Finally, we shall investigate one more consequence of the structural formula [6]. To this aim, consider the family $\mathfrak{P}$ of the (Carathéodory) functions p analytical in K and such that $p(0) = 1$, $\operatorname{Re} p(z) > 0$, for $z \in K$. As known, [4], the region of the values of $p(z)$, $p \in \mathfrak{P}$, $|z| = r$ is the disc $\overline{K(\varrho_0, r_0)}$, with the middle point $\varrho_0 = (1+r^2)(1-r^2)^{-1}$ and the radius $r_0 = 2r(1-r^2)^{-1}$.

Let

$$g(w) = \alpha w^2 + (1-\alpha)w, \quad w \in \overline{K(\varrho_0, r_0)}, \alpha \geqslant 0.$$

Then for $\alpha > 0$ and $r \in (0, 1)$ we have

$$\operatorname{Re} g(\varrho_0 + r_0 e^{it}) = v(\cos t), \quad t \in \langle 0, 2\pi \rangle,$$

where

$$v(x) = 2\alpha r_0^2 x^2 + r_0(2\alpha\varrho_0 + 1 - \alpha)x + \alpha(\varrho_0^2 - r_0^2) + (1-\alpha)\varrho_0.$$

Thus

$$\min_{-1 \leqslant x \leqslant 1} v(x) = \begin{cases} v(-1), & \text{when} \quad x_0 \leqslant -1, \\ v(x_0), & \text{when} \quad -1 < x_0 < 0, \end{cases}$$

where

$$x_0 = -[1 + \alpha + (3\alpha - 1)r^2]/8\alpha r.$$

Consequently we obtain

$$\min_{0 \leqslant t \leqslant 2\pi} \operatorname{Re} g(\varrho_0 + r_0 e^{it}) = \begin{cases} \dfrac{1-r}{(1+r)^2}\big(1 + (1-2\alpha)r\big), & \text{when } 0 < r \leqslant r_1(\alpha), \\ \dfrac{u(\alpha, r^2)}{8\alpha(1-r^2)^2}, & \text{when } r_1(\alpha) < r < 1, \end{cases} \tag{27}$$

where

$$r_1(\alpha) = \frac{1+\alpha}{4\alpha + \sqrt{13\alpha^2 - 2\alpha + 1}},$$

$$u(\alpha, y) = (7\alpha^2 - 2\alpha - 1)y^2 - 2(11\alpha^2 + 2\alpha - 1)y - (\alpha^2 - 6\alpha + 1). \tag{28}$$

Thus from the form of the function q and from (27) we obtain

$$\min_{w \in \overline{K(\varrho_0, r_0)}} \operatorname{Re} q(w) = \begin{cases} \dfrac{1-r}{(1+r)^2}\big(1 + (1-2\alpha)r\big), & \text{when } 0 < r \leqslant r_1(\alpha), \\ \dfrac{u(\alpha, r^2)}{8\alpha(1-r^2)^2} & \text{when } r_1(\alpha) < r < 1. \end{cases} \tag{29}$$

Put

$$r(\alpha) = \begin{cases} \left(\dfrac{1 + (2\sqrt{2} - 3)\alpha}{1 + (2\sqrt{2} - 1)\alpha}\right)^{1/2} & \text{for } 0 \leqslant \alpha \leqslant \sqrt{2} + 2, \\ (2\alpha - 1)^{-1} & \text{for } \sqrt{2} + 2 \leqslant \alpha. \end{cases} \tag{30}$$

From (29), (28) and (30) we infer that for any $\alpha \geqslant 0$ and any $r \in \langle 0, r(\alpha)\rangle$, $\operatorname{Re} g(w)$ is positive in the set $\overline{K(\varrho_0, r_0)}$, $\varrho_0 = (1-r^2)(1-r^2)^{-1}$, $r_0 = 2r(1-r^2)^{-1}$. If $r = r(\alpha)$, then there is $t_0 \in \langle 0, 2\pi\rangle$ such that $\operatorname{Re} g(\varrho_0 + r_0 e^{it_0}) = 0$.

We have just shown:

LEMMA. *For every function $p \in \mathfrak{P}$ and any fixed $\alpha \geqslant 0$ we have*

(31) $$\operatorname{Re}\{\alpha p^2(z) + (1-\alpha)p(z)\} > 0 \quad \textit{for } z \in K_{r(\alpha)},$$

where $r(\alpha)$ is defined by (30). *Property* (31) *in not valid in any disc K_r with a radius $r > r(\alpha)$.*

Let f be an arbitrary function in the family T. Then from (5) and (20) we get

$$J(\alpha, z, f) = \int_{-1}^{1} \big(\alpha p^2(z;\tau) + (1-\alpha)p(z;\tau)\big)\, d\mu(\tau), \quad z \in K,$$

where

$$p(z;\, \tau) = \frac{1-z^2}{1-2\tau z + z^2}, \quad z \in K,\ \tau \in \langle -1, 1\rangle.$$

As $p(z;\,\tau) \in \mathfrak{P}$, we obtain, by Lemma,

(32) $$\operatorname{Re} J(\alpha, z, f) > 0, \quad z \in K_{r(\alpha)},$$

for any function $f \in T$.

The function $f_3(z) = z(1-z)^{-2}$ is in T, and

$$J(\alpha, z, f_3) = \alpha\left(\frac{1+z}{1-z}\right)^2 + (1-\alpha)\left(\frac{1+z}{1-z}\right).$$

Thus property (32) is not preserved in any disc K_r with a radius $r > r(\alpha)$.

Let $R_\alpha(T)$ stand for the radius of the greatest disc K_r in which each function $f \in T$ satisfies condition (4)–(5) in the definition of the family T_α. Then we have

THEOREM 7. *For any $\alpha \geqslant 0$*

$$R_\alpha(T) = r(\alpha),$$

where $r(\alpha)$ is defined in (30).

Obviously $R_0(T) = 1$. Since $R_1(T) = \sqrt{2}-1 = r_0$ (cf. (19)), each function $f \in T$ is not only univalent in the disc K_{r_0} but also convex in the direction of the imaginary axis. It can also easily be seen that $R_\alpha(T) \to 0$ when $\alpha \to +\infty$.

References

[1] J. W. Aleksander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. 17 (1915–1916), p. 12–22.

[2] I. Ya. Asnevic, G. V. Ulina, *On the regions of values which have Stieltjes integral representations*, Vestnik Leningradskogo Univ. 11 (1955), p. 31–42 (in Russian).

[3] L. Brickman, T. H. MacGregor, D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. 156 (1971), p. 91–107.

[4] K. C. Carathéodory, *Über den Variabilitätsbereich der Fourierchen Konstanten von positiv harmonischen Funktion*, Rendiconti di Palermo 32 (1911), p. 193–217.

[5] J. Dieudonné, *Sur les fonctions univalentes*, Compt. Rend. Hébd. Acad. Sci. 192 (1931), p. 1148–1150.

[6] G. M. Golouzin, *On typical real functions*, Mat. Sbornik 27 (69), 2 (1950), p. 201–207 (in Russian).

[7] S. S. Miller, P. T. Mocanu, M. O. Reade, *Bazilevič functions and generalized convexity*, Rev. Roum. Math. Pures et Appl. 19, 2 (1974), p. 213–224.

[8] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) 11 (1969), p. 127–133.

[9] N. N. Pascu, *Janowski alpha-starlike-convex functions*, Studia Univ. Babes–Bolyai, Mathematica (1976), p. 23–27.

[10] M. S. Robertson, *Analytic functions starlike in one direction*, Amer. Journ. Math. 58 (1936), p. 465–472.

[11] W. W. Rogosiński, *Über positive harmonische Entwicklungen und typischreelle Potenzreihen*, Math. Z. 35 (1932), p. 93–121.

[12] E. M. Silvia, *On a subclass of spiral-like functions*, Proc. Amer. Math. Soc. 44, 2 (1974), p. 411–420.

[13] K. Skalska, *Certain subclasses of the class of typically real functions*, Ann. Polon. Math. 38 (1980), p. 141–152.

[14] G. Schober, *Univalent functions-selected topics*, Lecture Notes Math. 478 (1975).

[15] S. Walczak, *The radius of conformity of some classes of regular functions*, Ann. Polon. Math. 27 (1973), p. 189–195.

Reçu par la Rédaction le 3. 1. 1978