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On the inverse problem of the Sturm-Liouville type for a linear elliptic partial differential equation with constant coefficients of the second order

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Introduction. The purpose of the present paper is to transfer some theorems on the so-called inverse problem of the Sturm–Liouville type known for the ordinary differential equations (cf. [10]) and a for special class of partial differential equations (cf. [9], [4]), to a litle more general equations. In particular, we deal in this paper with a certain theorem of Ambarzumian [1] and with its generalizations (see [9], [4]).

Let D be a bounded domain in the space E^m with a sufficiently regular boundary ∂D. We assume that the boundary ∂D of D is so regular that there exist a Green function and sequences of eigenvalues and eigenfunctions for the value problem which is treated below. In the sequel we denote by $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_m)$ the points of E^m.

We shall consider the problem of eigenvalues and eigenfunctions for the differential equation of the form

$$L(u)+[\lambda-q(X)]u = 0, \tag{1}$$

with the boundary condition

$$\frac{du}{d\nu}-h(X)u = 0 \text{ on } \partial D-\Gamma, \quad u = 0 \text{ on } \Gamma, \tag{2}$$

where

$$L(u) = \sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{3}$$

$$\frac{du}{d\nu} = \sum_{i,j=1}^{m} a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i), \tag{4}$$

n — being the interior normal to ∂D, and Γ denotes an $(m-1)$-dimensional part of ∂D (Γ being connected or not). We assume that $a_{ij} = a_{ji}$

$(i,j = 1, \ldots, m)$ are constant, the quadratic form $\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j$ is positive definite and $h(X) \geqslant 0$ is a continuous function on ∂D. The boundary condition (2) may be taken in the sense of generalization (cf. [3]).

1. The Green function and eigenvalues and eigenfunctions of problem (1) (2). Let us denote by $G(X, Y, \lambda)$ the Green function and by $\{\lambda_n\}$, $\{\varphi_n(X)\}$ the sequences of eigenvalues and eigenfunctions of problem (1) (2), respectively. We assume that $\lambda \neq \lambda_n$ $(n = 1, 2, 3, \ldots)$ and that $\{\varphi_n(X)\}$ form an orthonormal set of functions in D. We denote by $G^{(n)}(X, Y, \lambda)$ the iterated Green functions. For a definition and fundamental properties of the functions $G^{(n)}(X, Y, \lambda)$, see [12] or [4]. In the sequel by $G_0(X, Y, \lambda)$ we shall mean the Green function of the whole E^m with respect to equation (1) when $q(X) \equiv 0$.

Let us assume that the parameter λ in equation (1) is a real negative number and write $\varrho = -\lambda$, where $\varrho > 0$. It is known (see [2] and [12]) that

$$G_0(X, Y, -\varrho) = a^{\frac{1}{2}}(2\pi)^{-\frac{m}{2}} r^{-\left(\frac{m}{2}-1\right)} \varrho^{\frac{m}{4}-\frac{1}{2}} K_{\frac{m}{2}-1}\left(r\sqrt{\varrho}\right),$$

where

$$r^2 = \sum_{i,j=1}^{m} A_{ij}(x_i - x_j)(y_i - y_j), \qquad a = \det\|a_{ij}\|,$$

A_{ij} are the elements of the inverse matrix to the matrix $\|a_{ij}\|$ and $K_{\frac{m}{2}-1}(t)$ is the modified Bessel function of the second kind of $(\frac{1}{2}m-1)$-th order.

Using the properties of the Green functions $G(X, Y, \lambda)$ and $G_0(X, Y, \lambda)$ and the form of $G_0(X, Y, \lambda)$ and its iterated Green functions and using also the asymptotic distribution of eigenvalues of problem (1) (2), we can prove the following formula:

$$\sum_{n=1}^{\infty} \frac{\psi_n^2(X)}{(\mu_n + \varrho)^{k+1}} = \frac{1}{k!}\left\{a^{\frac{1}{2}}(2\pi)^{-m}\Gamma(k+1-m/2)\,\varrho^{\frac{m}{2}-k-1} - \Phi_k(X, \varrho)\right\}. \tag{5}$$

The proof of formula (5) is quite similar to the proof of an analogous formula in [4]. In formula (5) $\{\mu_n\}$ and $\{\psi_n(X)\}$ denote the sequences of eigenvalues and eigenfunctions of problem (1) (2), respectively, in the case $q(X) \equiv 0$ in D and $\Phi_k(X, \varrho)$ is a continuous function in D satisfying the condition

$$\int_D |\Phi_k(X, \varrho)|\,dX = o\left(\varrho^{\frac{m}{2}-k-1}\right) \quad \text{for } \varrho \to +\infty, \tag{6}$$

where k in (5) and (6) is a positive integer $k = [m/2]$ (here $[m/2]$ denotes the entire part of $m/2$).

It follows from (5) by Dini's well-known theorem that the series on the left-hand of (5) is uniformly convergent in the domain D. Therefore, integrating over the domain D, we have

$$(7)\qquad \sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}} = \frac{|D|\, a^{1}\Gamma\left(k+1-\frac{m}{2}\right)}{k!\,(2\pi)^m \varrho^{k+1-m/2}} - \frac{1}{k!}\int_D \Phi_k(X,\varrho)\,dX,$$

where $|D|$ is a measure of D.

2. Some properties of the trace of a linear operator. In this section we shall give a definition of the trace of a linear operator and we shall prove some lemmas, which will be used in the sequel. The trace of a linear operator A on a Hilbert space is the series

$$(8)\qquad \sum_{n=1}^{\infty} (Ax_n, x_n),$$

where $\{x_n\}$ denotes any orthonormal set of vectors in H (cf. [7], p. 125).

If series (8) is convergent for any $\{x_n\}$, we say that A has a finite trace. In the sequel the trace of the operator A will be denoted by $S(A)$. If the operator A has a finite trace, then the sum (8) is independent of $\{x_n\}$ (cf. [7], p. 127).

If the operator A is positive, then the sum (8) has the same value (finite or not) for any $\{x_n\}$ (cf. [7], p. 126).

Let B be a completely continuous operator on H. Let us write $C = (B^*B)^{1/2}$, where B^* is the adjoint operator to B. Evidently the operator C is a completely continuous and self-adjoint operator on H. We shall denote by $\{s_n(B)\}$ the decreasing sequence of all eigenvalues of C with all multiplicities listed (cf. [7], p. 46). By σ_p $(p > 0)$ we denote the class of completely continuous operators B such that

$$|B|_p = \left\{\sum_{n=1}^{\infty} [s_n(B)]^p\right\}^{1/p} < \infty.$$

If $B \in \sigma_p$ and K is a bounded operator on H, then BK and KB belong to σ_p, and $|BK|_p \leqslant \|K\|\,|B|_p$ and $|KB|_p \leqslant \|K\|\,|B|_p$.

In the sequel we shall make use of the following two lemmas (see [7], p. 121 and p. 127).

LEMMA 1. *If the operators $B_j \in \sigma_{p_j}$ $(j = 1, \ldots, l)$ and if $\sum_{j=1}^{l} p_j^{-1} = p^{-1} \leqslant 1$, then operator $B = B_1 B_2 \ldots B_l$ belongs to σ_p and $|B|_p \leqslant |B_1|_{p_1} \ldots |B_l|_{p_l}$.*

LEMMA 2. *The necessary and sufficient condition for the existence of a finite trace of B is that $B \in \sigma_1$ and then*

$$|S(B)| \leqslant |B|_1 = \sum_{n=1}^{\infty} s_n(B).$$

Let us denote by T a self-adjoint and positive operator and by V a self-adjoint and bounded operator on H. By T_ϱ we denote the operator $T+\varrho I$, where I is the identity operator on H, and ϱ is a real positive number such that the operators T_ϱ and $T_\varrho+V$ are the positively defined operators on H. We assume that the operators T_ϱ^{-1} and $(T_\varrho+V)^{-1}$ are completely continuous operators. We shall denote by $\{\mu_n\}$ and $\{\lambda_n\}$ the increasing sequences of all eigenvalues of T and $T+V$, respectively, and by $\{x_n\}$ and $\{y_n\}$ the corresponding orthonormal sequences of eigenvectors. It follows from these assumptions that $\lim \mu_n = \lim \lambda_n = +\infty$, and $\{x_n\}$ and $\{y_n\}$ form the complete orthonormal systems in H (cf. [11], p. 579).

LEMMA 3. *If the series* $\sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}}$ *is convergent and* $C_1, \ldots, C_k$ *are bounded operators on H, then the operator* $A = T_\varrho^{-1} C_1 T_\varrho^{-1} \ldots T_\varrho^{-1} C_k T_\varrho^{-}$ *has a finite trace and*

$$|S(A)| \leqslant \|C_1\| \ldots \|C_k\| \sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}}.$$

Proof. We see that $A = A_1 A_2 \ldots A_{k+1}$, where $A_1 = T_\varrho^{-1} C_1$, $A_2 = T_\varrho^{-1} C_2, \ldots, A_k = T_\varrho^{-1} C_k$, $A_{k+1} = T_\varrho^{-1}$. Since $\sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}}$ is convergent so the operator $T_\varrho^{-1} \in \sigma_{k+1}$, and from this we have $A_j \in \sigma_{k+1}$ $(j = 1, \ldots, k+1)$. By Lemma 1, where $p_j = k+1$ $(j = 1, \ldots, k+1)$, we have $A \in \sigma_1$. In virtue of Lemma 2 this yields Lemma 3.

LEMMA 4. *If the series* $\sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}}$ *is convergent for sufficiently large ϱ, then*

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{(\mu_n+\varrho)^k} - \frac{1}{(\lambda_n+\varrho)^k} \right\} = S\{T_\varrho^{-k} - (T_\varrho+V)^{-k}\}. \tag{9}$$

Proof. Let ϱ be a real positive number such that $\mu_1+\varrho > \|V\|$. First we shall prove that the operator

$$T_\varrho^{-k} - (T_\varrho+V)^{-k} \tag{10}$$

has a finite trace. Indeed, operator (10) is a completely continuous operator. We observe that

$$(T_\varrho+V)^{-1} = T_\varrho^{-1}(I+VT_\varrho^{-1})^{-1} = T_\varrho^{-1}\sum_{n=0}^{\infty}(-1)^i(VT_\varrho^{-1})^i$$
$$= T_\varrho^{-1}-T_\varrho^{-1}VT_\varrho^{-1}+T_\varrho^{-1}G_\varrho T_\varrho^{-1},$$

where

$$G_\varrho = \sum_{i=2}^{\infty}(-1)^i(VT_\varrho^{-1})^{i-1}V.$$

By the assumption $\mu_1+\varrho > \|V\|$, the series $\sum_{i=0}^{\infty}(-1)^i(VT_\varrho^{-1})^i$ is convergent, and we have

$$\|G_\varrho\| \leqslant \frac{K}{\mu_1+\varrho},$$

where K is a constant independent of ϱ. Continuing, we see that

$$T_\varrho^{-k}-(T_\varrho+V)^{-k} = T_\varrho^{-k}-[T_\varrho^{-1}(I+VT_\varrho^{-1})^{-1}]^k$$
$$= T_\varrho^{-k}-[T_\varrho^{-1}-T_\varrho^{-1}VT_\varrho^{-1}+T_\varrho^{-1}G_\varrho T_\varrho^{-1}]^k.$$

From the last equality we have

(11) $$T_\varrho^{-k}-(T_\varrho+V)^{-k} = B_\varrho+F_\varrho,$$

where

$$B_\varrho = T_\varrho^{-k}VT_\varrho^{-1}+\ldots+T_\varrho^{-1}VT_\varrho^{-k},$$

but F_ϱ is a sum of a finite number of operators of the form

$$T_\varrho^{-1}C_1T_\varrho^{-1}C_2\ldots T_\varrho^{-1}C_kT_\varrho^{-1},$$

where $C_1, \ldots, C_k$ are bounded operators and at least for one C_j we have

$$\|C_j\| \leqslant \frac{L}{\mu_1+\varrho},$$

where L is a constant independent of ϱ, $1 \leqslant j \leqslant k$. It follows from this, by Lemma 3, that the operators B_ϱ and F_ϱ have finite traces and

(12) $$|S(F_\varrho)| \leqslant \frac{M}{\mu_1+\varrho}\sum_{n=1}^{\infty}\frac{1}{(\mu_n+\varrho)^{k+1}},$$

whence operator (10) has a finite trace.

Expressing the trace of operator (10) by $\{x_n\}$ and next by $\{y_n\}$, we have

(13) $$\sum_{n=1}^{\infty}\left\{\frac{1}{(\mu_n+\varrho)^k}-\left((T_\varrho+V)^{-k}x_n, x_n\right)\right\} = \sum_{n=1}^{\infty}\left\{(T_\varrho^{-k}y_n, y_n)-\frac{1}{(\lambda_n+\varrho)^k}\right\}.$$

By the evaluation of *Ky* Fan (cf. [6]) we conclude that

$$\sum_{n=1}^{N}\big((T_\varrho+V)^{-k}x_n, x_n\big) \leqslant \sum_{n=1}^{N}\frac{1}{(\lambda_n+\varrho)^k},$$

$$\sum_{n=1}^{N}(T_\varrho^{-k}y_n, y_n) \leqslant \sum_{n=1}^{N}\frac{1}{(\mu_n+\varrho)^k}.$$

From this we have

$$\sum_{n=1}^{N}\left\{(T_\varrho^{-k}y_n, y_n)-\frac{1}{(\lambda_n+\varrho)^k}\right\} \leqslant \sum_{n=1}^{N}\left\{\frac{1}{(\mu_n+\varrho)^k}-\frac{1}{(\lambda_n+\varrho)^k}\right\}$$
$$\leqslant \sum_{n=1}^{N}\left\{\frac{1}{(\mu_n+\varrho)^k}-\big((T_\varrho+V)^{-k}x_n, x_n\big)\right\}.$$

If $N\to+\infty$, then, by (13), the last inequality implies (9).

LEMMA 5. *If* 1° $\sum_{n=1}^{\infty}\frac{1}{(\mu_n+\varrho)^2}$ *is convergent for sufficiently large* ϱ, *and* 2° $\sum_{n=2}^{\infty}\left(\frac{\lambda_n}{\mu_n}-1\right)$ *is convergent, then*

$$\lim_{\varrho\to\infty}\varrho S(T_\varrho^{-1}VT_\varrho^{-1}) = 0. \tag{14}$$

Proof. Let ϱ be a real positive number such that $\mu_1+\varrho>\|V\|$. Then, as in Lemma 4, we have

$$T_\varrho^{-1}-(T_\varrho+V)^{-1} = T_\varrho^{-1}VT_\varrho^{-1}-T_\varrho^{-1}G_\varrho T_\varrho^{-1}, \tag{15}$$

where G_ϱ is the operator from Lemma 4. Hence by (15)

$$S\{T_\varrho^{-1}-(T_\varrho+V)^{-1}\} = S(T_\varrho^{-1}VT_\varrho^{-1})-S(T_\varrho^{-1}G_\varrho T_\varrho^{-1}). \tag{16}$$

Let us remark that

$$|S(T_\varrho^{-1}G_\varrho T_\varrho^{-1})| \leqslant \|G_\varrho\|\sum_{n=1}^{\infty}\frac{1}{(\mu_n+\varrho)^2} \leqslant \frac{K}{\mu_1+\varrho}\sum_{n=1}^{\infty}\frac{1}{(\mu_n+\varrho)^2}. \tag{17}$$

By (17) and since the series $\sum_{n=1}^{\infty}\frac{1}{(\mu_n+\varrho)^2}$ is uniformly convergent in ϱ, we have

$$\lim_{\varrho\to\infty}\varrho S(T_\varrho^{-1}G_\varrho T_\varrho^{-1}) = 0. \tag{18}$$

In virtue (16), (18) and (9) for $k = 1$, we have

$$\lim_{\varrho\to\infty} \varrho \sum_{n=1}^{\infty} \left\{ \frac{1}{\mu_n+\varrho} - \frac{1}{\lambda_n+\varrho} \right\} = \lim_{\varrho\to\infty} \varrho S(\boldsymbol{T}_\varrho^{-1}\boldsymbol{V}_\varrho\boldsymbol{T}_\varrho^{-1}). \tag{19}$$

By assumption 2° of Lemma 5 and by Lemma 6 of paper [4], the left-hand side of (19) is equal to zero; this yields Lemma 5.

LEMMA 6. *If* 1° $\sum_{n=1}^{\infty} \frac{1}{(\mu_n+\varrho)^{k+1}}$ *is convergent for sufficiently large* ϱ, *and* 2° $\sum_{n=2}^{\infty} \left(\frac{1}{\mu_n} - \frac{1}{\lambda_n} \right)$ *is convergent, then*

$$\lim_{\varrho\to\infty} \varrho^{k-1} \{S(\boldsymbol{B}_\varrho)+S(\boldsymbol{F}_\varrho)\} = 0, \tag{20}$$

where the operators $\boldsymbol{B}_\varrho$ *and* $\boldsymbol{F}_\varrho$ *are defined in Lemma* 4.

Proof. Let ϱ be a real positive number such that $\mu_1+\varrho > \|\boldsymbol{V}\|$. Then by (11) we have

$$S\{\boldsymbol{T}_\varrho^{-k} - (\boldsymbol{T}_\varrho+\boldsymbol{V})^{-k}\} = S(\boldsymbol{B}_\varrho)+S(\boldsymbol{F}_\varrho). \tag{21}$$

By (9) and (21) we get

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{(\mu_n+\varrho)^k} - \frac{1}{(\lambda_n+\varrho)^k} \right\} = S(\boldsymbol{B}_\varrho)+S(\boldsymbol{F}_\varrho).$$

Therefore

$$\lim_{\varrho\to\infty} \varrho^{k-1} \sum_{n=1}^{\infty} \left\{ \frac{1}{(\mu_n+\varrho)^k} - \frac{1}{(\lambda_n+\varrho)^k} \right\} = \lim_{\varrho\to\infty} \varrho^{k-1} \{S(\boldsymbol{B}_\varrho)+S(\boldsymbol{F}_\varrho)\}. \tag{22}$$

By assumption 2° of Lemma 6 and by Lemma 7 of paper [4], the left-hand side of (22) is equal to zero, this yields Lemma 6.

3. Dependence of the function $q(X)$ on the eigenvalues of problem (1) (2). In the domain D defined in the introduction to this paper we shall consider the problem of the eigenvalues and eigenfunctions for equation (1) with the boundary condition

$$\frac{du}{d\nu} = 0 \quad \text{on } \partial D. \tag{23}$$

Let $\{\lambda_n\}$ denote the increasing sequence of eigenvalues of problem (1) (23) and let $\{\varphi_n(X)\}$ denote the corresponding orthonormal sequence of eigenfunctions of this problem. Further, by $\{\mu_n\}$ and $\{\psi_n(X)\}$ we denote the sequences of eigenvalues and eigenfunctions of problem (1) (23) in the case $q(X) \equiv 0$ in D.

In the case of E^2 and E^3 we have the following

THEOREM 1. *If* 1° $q(X)$ *is a continuous function in the domain* D, *and* 2° $\sum_{n=2}^{\infty}\left(\frac{\lambda_n}{\mu_n}-1\right)$ *is convergent, then*

$$\int_D q(X)dX = 0. \tag{24}$$

Proof. Let $H = L_2(D)$ be a Hilbert space of functions square-summable on D, and let T be a self-adjoint operator on H, which is a Friedrichs expansion of the operator L defined by (3), while V is the multiplier operator $q(X)$. Since $k = [m/2] = 1$; then by (6) and (7) and by the assumptions of Theorem 1, the assumptions of Lemma 5 are satisfied. In virtue of Lemma 5, (14) holds. We express the trace of the operator from (14) by the sequence $\{\psi_n(X)\}$. We get

$$S(T_\varrho^{-1}VT_\varrho^{-1}) = \sum_{n=1}^{\infty}\frac{(V\psi_n, \psi_n)}{(\mu_n+\varrho)^2}. \tag{25}$$

By the definition of the operator V and by the uniform convergence of the series in formula (5) with respect to $X \epsilon D$, (25) may be written in the form

$$S(T_\varrho^{-1}VT_\varrho^{-1}) = \left(q(X), \sum_{n=1}^{\infty}\frac{\psi_n^2(X)}{(\mu_n+\varrho)^2}\right). \tag{26}$$

Using formula (5) for $k = 1$, we get by (26)

$$S(T_\varrho^{-1}VT_\varrho^{-1}) = a^{1/2}\frac{\Gamma(2-m/2)}{(2\sqrt{\pi})^m}\varrho^{\frac{m}{2}-2}(q, 1) - \int_D \Phi_1(X, \varrho)q(X)dX. \tag{27}$$

Since

$$\left|\int_D \Phi_1(X, \varrho)q(X)dX\right| \leqslant \max_{X\epsilon D}|q(X)| \int_D |\Phi_1(X, \varrho)|dX,$$

we have by (6)

$$\int_D \Phi_1(X, \varrho)q(X)dX = o(\varrho^{\frac{m}{2}-2}). \tag{28}$$

In virtue of (14), (27) and (28) we have

$$\lim_{\varrho\to\infty}\left\{a^{1/2}\frac{\Gamma(2-m/2)}{(2\sqrt{\pi})^m}\varrho^{\frac{m}{2}-1}(q, 1) - o(\varrho^{\frac{m}{2}-1})\right\} = 0. \tag{29}$$

Since $m = 2$ or $m = 3$, (29) is posible only when

$$(q, 1) = 0. \tag{30}$$

It is obvious that (30) is equivalent to (25), and this concludes the proof of Theorem 1.

In the case of E^m $(m \geqslant 4)$ we have the following

THEOREM 2. *If* 1° $q(X)$ *is a continuous function in the domain* D, *and* 2° $\sum_{n=2}^{\infty}\left(\frac{1}{\mu_n}-\frac{1}{\lambda_n}\right)$ *is convergent, then*

$$\int_D q(X)dX = 0. \tag{31}$$

Proof. Let the operators T and V be analogous operators to those in Theorem 1. By (6) and (7) and by the assumptions of Theorem 2, the assumptions of Lemma 6 are satisfied. In virtue of Lemma 6 (20) holds. We express the trace of the operator B_ϱ by the sequence $\{\psi_n(X)\}$. We get

$$S(B_\varrho) = k\sum_{n=1}^{\infty}\frac{(V\psi_n, \psi_n)}{(\mu_n+\varrho)^{k+1}}. \tag{32}$$

By (6), (7) and (12) we have

$$S(F_\varrho) = O(\varrho^{\frac{m}{2}-3}). \tag{33}$$

In virtue of (5), (6), (20), (31) and (33), as in Theorem 1, we get

$$\lim_{\varrho\to\infty}\left\{a^{1/2}\frac{\Gamma(k+1-m/2)}{k!(2\sqrt{\pi})^m}\varrho^{\frac{m}{2}-2}(q, 1)-o(\varrho^{\frac{m}{2}-2})\right\} = 0. \tag{34}$$

Since $m \geqslant 4$, (34) is posible only when $(q, 1) = 0$ and it is equivalent to (31).

Using Theorems 1 and 2, we shall prove

THEOREM 3. *Under the assumptions* 1° *and* 2° *of Theorem* 1 *in the case of* E^2 *or* E^3, *or under the assumptions* 1° *and* 2° *of Theorem* 2 *in the case of* E^m $(m \geqslant 4)$, *if* $\lambda_1 \geqslant \mu_1$, *then* $q(X) \equiv 0$ *in* D.

Proof. It is known (see [3] or [5]) that

$$\lambda_1 = \min_{\varphi\in K} J[\varphi] = \min\int_D\left\{\sum_{i,j=1}^{m}a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}+q(X)\varphi^2\right\}dX, \tag{35}$$

where K is the class of functions φ which are of class C_σ^1 in D, satisfying the condition $\int_D \varphi^2(X)dX = 1$ (for a definition of a function of class C_σ^n $(n \geqslant 0)$ see [3]). On the other hand, $\mu_1 = 0$ (cf. [3] or [5]), whence by the assumption of Theorem 3, $\lambda_1 \geqslant 0$. Putting $\varphi_1(X) = |D|^{-1/2}$ in (35), by (25) or (31) we see that $J[\varphi_1] = 0$. It follows that $\lambda_1 = 0$ and $\varphi_1(X)$ realizes the minimum (35). Therefore the function $\varphi_1(X) = |D|^{-1/2}$ satisfies (1) for $\lambda = 0$ and the boundary condition (23) (cf. [3] or [5]). It follows that $q(X) \equiv 0$ in D.

Let us now denote by $\{\lambda_n\}$ the sequence of eigenvalues of equation (1) with the boundary condition

$$\frac{du}{d\nu} - h(X)u = 0 \quad \text{on } \partial D, \tag{36}$$

where $h(X)$ is a non-negative function defined and continuous on ∂D and let $\{\mu_n\}$ denote the sequence of eigenvalues of problem (1) (35) for $q(X) \equiv 0$ in D.

We shall prove the following

THEOREM 4. *Under the assumptions 1° and 2° of Theorem 1 or Theorem 2, if the first eigenvalue λ_1 of problem (1) (36) is equal to the first eigenvalue of problem (1) (23) and is equal to zero, then $q(X) \equiv 0$ in D and $h(X) \equiv 0$ on ∂D.*

Proof. By the assumptions 1° and 2° of Theorem 1 or Theorem 2, we infer (25) or (31). On the other hand (cf. [3] or (5]),

$$\lambda_1 = \min_{\varphi\in K} \bar{J}[\varphi] = \min_{\varphi\in K}\left\{ \int_D \left[\sum_{i,j=1}^{m} a_{ij} \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} + q(X)\varphi^2 \right] dX + \int_{\partial D} h(X)\varphi^2 dS \right\}, \tag{37}$$

where K is the class defined in Theorem 3. Let us remark that

$$\bar{J}[\varphi] = J[\varphi] + \int_{\partial D} h(X)\varphi^2 dS.$$

Since $h(X) \geqslant 0$ on ∂D, we have

$$\bar{J}[\varphi] \geqslant J[\varphi] \geqslant 0, \tag{38}$$

for all functions $\varphi(X) \in K$. Let $\varphi_1(X)$ denote the function which realizes the minimum $\bar{J}[\varphi]$, i.e., $\bar{J}[\varphi_1] = 0$. This gives by (36)

$$J[\varphi_1] = 0. \tag{39}$$

From (39) it follows that the function $\varphi_1(X)$ also realizes the minimum $J[\varphi]$. On the other hand, by Theorem 3 a function which realizes the minimum $J[\varphi]$ is a constant in D. Because the first eigenvalue of problem (1) (2) is a single eigenvalue (cf. [3]), we have $\varphi_1(X) = \text{const}$ in D. This means that the function $\varphi_1(X) = \text{const}$ satisfies (1) for $\lambda = 0$ and the boundary condition (34). It follows that $q(X) \equiv 0$ in D and $h(X) \equiv 0$ on ∂D. This yields Theorem 4.

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