ON ALGEBRAIC RADICALS IN MOBS

BY

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We recall that a mob is a non-empty Hausdorff space $S$ together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \rightarrow xy$. Let $A$ be any subset of the mob $S$. The algebraic radical of $A$ is defined to be the set $\{x \in S | x^k \in A \text{ for some integer } k \geq 1\}$ and is denoted by $\mathcal{R}(A)$. This set $A$ is said to be radically stable if and only if $\mathcal{R}(A) = \mathcal{R}(A)$ holds. Obviously for any open subset $A$ of $S$, $A$ need not be radically stable. The purpose of this paper is to study some properties of the algebraic radicals of ideals in $S$. Our main result is:

Under some special conditions, any open ideal $A$ of $S$ can be radically stable without requiring that $\mathcal{R}(A)$ be closed.

Moreover, we will demonstrate that the notion of radical stability of an ideal in abelian mobs is useful: it gives a necessary and sufficient condition for the closure of a primary (prime) ideal to be primary (prime).

Throughout this paper, we use $\overline{C}$ to denote the closure of the set $C$ and $C'$ for the complement of $C$. Unless otherwise stated, $S$ will be regarded as a compact abelian mob with zero. The reader is referred to [4] for terminology and notations.

1. Preliminaries. In this section, pertinent notations, definitions and properties of algebraic radicals of an abelian mob $S$ (not necessarily compact) will be given. Most of them are well known results from ring theory which will be used later.

Notation. Let $A$ be a subset of $S$.

$J(A) = A \cup AS$, that is, the smallest ideal containing $A$.

$J_0(A) = \text{the union of all ideals contained in } A$, that is, the largest ideal contained in $A$ if there are any.

Definition 1.1. (1) A mob $S$ with zero is said to be 0-prime if and only if whenever $a, b \in S$, $ab = 0$, then $a = 0$ or $b = 0$.

(2) A mob $S$ is said to be an $\Omega$-mob if and only if for any two ideals $I_1$ and $I_2$ such that $I_1 \cap I_2 \neq \emptyset$, either $I_1 \subset I_2$ or $I_2 \subset I_1$.

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Definition 1.2. (1) An ideal $P$ of $S$ is said to be prime if and only if $ab \in P$ implies that $a \in P$ or $b \in P$.

(2) An ideal $Q$ of $S$ is said to be primary if and only if $ab \in Q$ implies that $a \in Q$ or there exists an integer $k \geq 1$ such that $b^k \in Q$.

(3) An ideal $R$ of $S$ is said to be semi-prime if and only if $a^2 \in R$ implies that $a \in R$.

(4) Let $A$, $B$ be ideals of $S$. Define $A : B = \{x \in S | xB \subseteq A\}$ and call it the ideal quotient of $A$ and $B$.

It is easy to see that $A : B$ is an ideal of $S$.

Definition 1.3. (1) An ideal $A$ is completely irreducible (irreducible) if and only if whenever $A$ is the intersection of a family (finite family) of ideals, then $A$ is a member of the family.

(2) An ideal $A$ is $w$-reducible if and only if $A$ is the intersection of a family of open prime ideals containing $A$ properly.

(3) An ideal $A$ is strongly reducible (weakly reducible) if and only if $A$ is the intersection of a finite family (infinite family) of ideals containing $A$ properly.

Facts 1.4. The algebraic radicals of $S$ have the following properties:

Let $A$, $B$ be any subsets of $S$. Then

(1) $A \subseteq \mathfrak{r}(A)$.

(2) $A^k \subseteq B$ implies that $\mathfrak{r}(A) \subseteq \mathfrak{r}(B)$ for any $k \geq 1$.

(3) $\mathfrak{r}(\mathfrak{r}(A)) = \mathfrak{r}(A)$.

If $A$, $B$ are ideals of $S$, then

(4) $\mathfrak{r}(A)$ is an ideal of $S$.

(5) $\mathfrak{r}(AB) = \mathfrak{r}(A \cap B) = \mathfrak{r}(A) \cap \mathfrak{r}(B)$.

(6) If $A$ is a primary ideal of $S$, then $\mathfrak{r}(A)$ is a prime ideal of $S$ which is the smallest prime ideal containing $A$.

(7) Let $P$, $Q$ be ideals of $S$. Then $Q$ is a primary ideal of $S$ with $\mathfrak{r}(Q) = P$ if and only if (i) $Q \subseteq P \subseteq \mathfrak{r}(Q)$ and (ii) $ab \in Q$, $a \notin Q$ imply that $b \notin P$.

The proofs of the above results are analogous to those in ring theory and we omit the proofs. The reader is referred to [6].

2. Prime and primary ideals. We are going to study, in this section, the prime and primary ideals of $S$, and, in particular, the algebraic radical of such ideals and their relationship.

Proposition 2.1. An ideal $N$ of $S$ is a compact prime ideal if and only if $S/N$ is an $0$-prime mob.

Proof. Suppose $N$ is a compact prime ideal of $S$. Then $N$ is closed in $S$. The Rees quotient $S/N$ is formed by shrinking $N$ to a single point with the quotient topology. $S/N$ is a mob. Recall that the multiplication $\ast$ of $S/N$ is defined in the following way:
\[ a \star b = ab \quad \text{if } a, b \text{ and } ab \text{ are in } S - N, \]
\[ a \star b = 0 \quad \text{if } ab \in N, \]
\[ a \star b = 0 \quad \text{if } a = 0 \text{ or } b = 0. \]

If \( a \star b = 0 \), there are two possible cases: either (i) \( a = 0 \) or \( b = 0 \), or (ii) \( ab \in N \). In case (ii), since \( N \) is prime, we have \( a \in N \) or \( b \in N \). This implies that \( a = 0 \) or \( b = 0 \) in \( S/N \). Thus in either case \( a = 0 \) or \( b = 0 \). Hence \( S/N \) is 0-prime. Conversely, assuming that \( S/N \) is an 0-prime mob, since \( S/N \) is Hausdorff, the ideal \( N \) is closed in \( S \) and hence is compact. Suppose \( x \star y = 0 \) in \( S/N \), then we have \( x = 0 \) or \( y = 0 \) in \( S/N \). This means that \( x \in N \) or \( y \in N \) in the mob \( S \). Hence \( N \) is a compact prime ideal of \( S \).

**Theorem 2.2.** Let \( A \) be an ideal of \( S \) such that \( \mathcal{R}(A) \) is proper maximal in \( S \). Then \( A \) is primary if and only if \( S/\mathcal{R}(A) \) is an abstract completely 0-simple semigroup.

**Proof.** Suppose \( A \) is a primary ideal of \( S \); then \( \mathcal{R}(A) \) is a prime ideal. As \( S \) is compact, it follows that \( \mathcal{R}(A) \) is open by [4], p. 28. By theorem 2 of [3], p. 677, \( \mathcal{R}(A) \) has the form \( J_0(S - e) \) with \( e \) being a non-minimal idempotent of \( S \). Therefore there exists \( e^2 = e \in \mathcal{R}(A) \). Now form the Rees quotient \( S/\mathcal{R}(A) \). Clearly, \( S/\mathcal{R}(A) \) is 0-simple ([4], p. 39) and contains \( e \). Hence by [1], p. 655, \( S/\mathcal{R}(A) \) is completely 0-simple. Conversely, suppose that \( S/\mathcal{R}(A) \) is completely 0-simple. Then there exists an \( e^2 = e \in \mathcal{R}(A) \). Clearly, \( e \) is non-minimal. By the maximality of \( \mathcal{R}(A) \), we have \( \mathcal{R}(A) = J_0(S - e) \). By theorem 2 of [3], p. 677 again, \( \mathcal{R}(A) \) is an open prime ideal of \( S \). Now take \( xy \in A \), then \( xy \in \mathcal{R}(A) \). Thus \( x \in \mathcal{R}(A) \) or \( y \in \mathcal{R}(A) \). This implies that \( A \) is primary.

**Corollary.** If \( E \), the set of idempotents of \( S \), is contained in a maximal proper ideal \( J \) of \( S \), then \( J \) is a primary ideal of \( S \).

**Proof.** By [1], p. 655, \( S/J \) is either the zero semigroup of order two or else completely 0-simple. Since \( E \subseteq J \), \( S/J \) contains no idempotents other than \( 0 \) and hence \( S/J \) is the zero semigroup of order 2. Suppose \( xy \in J \), \( x \not\in J \), \( y \not\in J \). Then \( x \in S - J \), \( y \in S - J \) in \( S/J \). Since \( S/J \) is the zero semigroup of order 2, we have \( y^2 = 0 \), \( x^2 = 0 \) in \( S/J \). This implies that \( x^2 \in J \), \( y^2 \in J \) in the mob \( S \). Thus \( J \) is a primary ideal of \( S \).

A. D. Wallace has proved the following result:

Let \( S \) be a compact mob (not necessarily abelian). Then each open prime ideal is completely irreducible, and each completely irreducible ideal is open by [5], p. 39.

One would naturally ask whether the irreducibility of an ideal \( Q \) in an abelian semigroup is a necessary and sufficient condition for \( Q \) to be primary. (This question was asked by A. D. Wallace in his lecture
notes on topological semigroups, problem J6, p. 39 of [5]. We show here, by giving a counterexample, that the answer is negative.)

Example 2.3. Let $S$ be an abelian semigroup consisting of four elements $\{0, a, b, c\}$ with multiplication table.

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The sets $\{0, b\}$, $\{0, c\}$, $\{0, a, b\}$ are ideals of $S$. Now $\{0, b\} = \{0, b, c\} \cap \{0, a, b\}$. It is easily seen that $\{0, b\}$ is a primary ideal of $S$, but it is not irreducible. Thus we have shown that primary ideals in abelian mobs are not necessarily irreducible.

Theorem 2.4. If $Q$ is an open semi-prime ideal of $S$, then $Q$ is w-reducible.

Before proving this theorem, we need the following two lemmas.

Lemma 2.5. $Q$ is a semi-prime ideal if and only if $\mathcal{A}(Q) = Q$.

Proof. If $\mathcal{A}(Q) = Q$, then it is easily seen that $Q$ is semi-prime. Conversely, suppose that $Q \subseteq \mathcal{A}(Q)$, then there exists $a \in \mathcal{A}(Q)$ with $a \not\in Q$. Let $k > 1$ be the minimal integer such that $a^k \not\in Q$. Suppose $Q$ is semi-prime. Then $k$ must be odd. Write $k = 2n + 1 (n > 0)$. Since $Q$ is an ideal, we infer that $a^{k+1} = a^{k} \cdot a \in Q$. Thus $a^{k+1} = a^{2n+2} = (a^{n+1})^2 \in Q$. Since $Q$ is semi-prime, it follows that $a^{n+1} \not\in Q$. This contradicts the minimality of $k$. Hence $\mathcal{A}(Q) = Q$.

Lemma 2.6. Let $Q$ be an open ideal of $S$, then $\mathcal{A}(Q) = \bigcap P_a$, where $\{P_a\}$ are all the open prime ideals of $S$ containing $Q$.

Proof. Take $x \in \mathcal{A}(Q)$. Then there exists integer $k \geq 1$ such that $x^k \not\in Q \subseteq P_a$ for all $a$. Since $P_a$ is prime, $x \not\in P_a$ for all $a$, that is, $x \not\in \bigcap P_a$. Hence $\mathcal{A}(Q) \subseteq \bigcap P_a$. Conversely, suppose that $\bigcap P_a \not\in \mathcal{A}(Q)$. Then we can find an element $y$ of $\bigcap P_a$ such that $y \not\in \mathcal{A}(Q)$. We have $\{y, y^2, \ldots\} = \Gamma(y) \subseteq \bigcap P_a$.

Since $\Gamma(y)$ is compact, there exists an idempotent $e$ such that $e \not\in \Gamma(y) \subseteq P_a$ and $e \not\in Q$. (For if $e \not\in Q$, then $\Gamma(y) \subseteq Q$. But $y \not\in \mathcal{A}(Q)$.) Thus $J_e(S - e) \supset Q$. By theorem 2 of [3], p. 677, $J_e(S - e)$ is an open prime ideal of $S$. Therefore $J_e(S - e) \supset \bigcap P_a$. This implies that $e \not\in \bigcap P_a$, a contradiction. Thus $\bigcap P_a \in \mathcal{A}(Q)$.

By now, one can easily see that Theorem 2.4 is an immediate consequence of these two lemmas.
COROLLARY 1. If $|E| < \infty$, any open ideal of $S$ is semi-prime if and only if it is $w$-reducible.

COROLLARY 2. $\mathcal{R}(Q)$ is the smallest semi-prime ideal of $S$ containing the ideal $Q$.

COROLLARY 3. Let $Q$ be an open semi-prime ideal of $S$. If $B$ is an ideal of $S$ which is not contained in $Q$, then $B$ contains an idempotent $e$ with $Se \subset Q$.

Proof. Let $b \in B - Q$. Consider the principal ideal $J(b)$ generated by $b$. Clearly, $J(b)$ is compact, $J(b) \subset B$, $J(b) \subset Q$. Now let $\mathcal{M}$ be the collection of all compact ideals $\{J_i\}_{i \in I}$ with the properties $J_i \subset B$, $J_i \subset Q$. By the same arguments as lemma 8 ([3], p. 676) we prove that there exists a minimal member $J$ in $\mathcal{M}$ with $J \subset B$, $J \subset Q$. Now let $x \in J - Q$, and suppose $xJx \subset Q$. Since $Q$ is semi-prime, by lemma 2.5 and lemma 2.6, $Q = \bigcap_a P_a$, where $P_a$ are open prime ideals containing $Q$. As $S$ is abelian, we have $J(x) = xJx \subset Q \subset P_a$ for all $a$. This implies that $J(x) = P_a$ for all $a$. Hence $J(x) = \bigcap_a P_a = Q$, a contradiction. So we assert that $xJx \notin Q$. Since $xJx \subset J$ and $J$ is minimal, we have $xJx = J$. Consequently, $x^nJx^n = J$ for all integers $n$. Thus $xJx = eJe = J$ with $e^2 = e \in J(x) \subset J$. Since $e \in J$, we have $eSe = Se \subset J \subset Q$.

THEOREM 2.7. Let $F$ be a closed ideal of $S$ and let $\mathcal{I} = \{\text{open ideal } G_a \text{ of } S | G_a \supset F\}$. Then $F = \bigcap_a G_a, G_a \in \mathcal{I}$ for all $a$. In other words, $F$ is weakly reducible if the family $\mathcal{I}$ exists.

Proof. Trivially, $F \subset \bigcap_a G_a$. To prove the converse containment, we only need to show that for any element $x \notin F, x \notin \bigcap_a G_a$. Since $F$ is closed in $S$, it is compact. As $S$ is compact Hausdorff, it is a regular space and hence there exists an open neighbourhood $V$ containing $F$ but excluding $x$. By the compactness of $S$, we have that $J_0(V)$ is an open ideal of $S$. Obviously, $F \subset J_0(V)$. Hence $J_0(V) \in \mathcal{I}$. Clearly, $x \notin J_0(V)$. This implies that $x \notin \bigcap_a G_a$.

COROLLARY. If $S$ satisfies the second axiom of countability, then $F$ is a $\mathcal{I}_2$-ideal, that is, $F$ can be expressed as a countable intersection of open ideals containing $F$.

This is because compact and $T_2$ imply regular, and regular and second countability imply metrizable and every closed set in any metric space is $G_b$.

THEOREM 2.8. Let $S$ be an abelian mob (not necessarily compact). If the algebraic radical of an ideal $A$ is non-prime, then it is strongly reducible.
Proof. Since \( \mathfrak{R}(A) \) is not prime, we can find elements \( x, y \) in \( S \) such that \( xy \in \mathfrak{R}(A) \) but \( x \notin \mathfrak{R}(A), y \notin \mathfrak{R}(A) \). Consider \( \mathfrak{R}(A) : J(y) = \{ z \in S | zJ(y) \subseteq \mathfrak{R}(A) \} \). Then \( \mathfrak{R}(A) : J(y) \) is an ideal of \( S \) with \( \mathfrak{R}(A) \subset \mathfrak{R}(A) : J(y) \). We claim that \( \mathfrak{R}(A) \neq \mathfrak{R}(A) : J(y) \). In fact since \( xy \in \mathfrak{R}(A) \), we have that \( xJ(y) = x(\{ y \} \cup yS) = \{ xy \} \cup xyS \subseteq \mathfrak{R}(A) \). Thus \( x \notin \mathfrak{R}(A) : J(y) \) but \( x \notin \mathfrak{R}(A) \). Now clearly \( \mathfrak{R}(A) \subset (\mathfrak{R}(A) \cup J(y)) \cap (\mathfrak{R}(A) : J(y)) \). On the other hand, if \( t \in (\mathfrak{R}(A) \cup J(y)) \cap (\mathfrak{R}(A) : J(y)) \), then \( tJ(y) \in \mathfrak{R}(A) \).

If \( t \notin \mathfrak{R}(A) \), then we must have \( t \notin J(y) \). Hence \( t^2 \notin J(y) \subset \mathfrak{R}(A) \). Since \( \mathfrak{R}(A) \) is semi-prime, we have \( t \notin \mathfrak{R}(A) \). Hence we have shown that \( \mathfrak{R}(A) = (\mathfrak{R}(A) \cup J(y)) \cap (\mathfrak{R}(A) : J(y)) \) and hence \( \mathfrak{R}(A) \) is strongly reducible.

Corollary 1. Let \( Q \) be an open primary ideal of the compact mob \( S \) with \( \mathfrak{R}(Q) = P \). If \( A \) is any closed ideal of \( S \) with \( A \notin Q \), then \( Q : A \) is an open primary ideal of \( S \) with \( \mathfrak{R}(Q : A) = P \).

Proof. Since \( Q \) is open, \( Q' \) is closed and hence compact. \( A \) is also compact. If \( x \in Q : A \), then \( xA \cap Q' = \emptyset \). By the continuity of multiplication and the compactness of \( A \), there exists a neighbourhood \( V \) of \( x \) such that \( VA \cap Q' = \emptyset \). That is \( VA \subseteq Q \). Hence \( x \in V \subseteq Q : A \), that is, \( Q : A \) is open. By 1.4 (7) and the fact that \( (Q : A) \subseteq Q \), we can obtain that (i) \( Q : A \subset P \subset \mathfrak{R}(Q : A) \) and (ii) \( ab \in Q : A \), \( a \notin Q : A \) imply that \( b \in \mathfrak{R}(Q : A) \). Hence, by 1.4 (7) again, \( Q : A \) is an open primary ideal of \( S \) with \( \mathfrak{R}(Q : A) = P \).

Corollary 2. If \( Q \) is a compact primary ideal of the compact mob \( S \) with \( \mathfrak{R}(Q) = P \) and if \( A \) is any ideal \( \notin Q \), then \( Q : A \) is a compact primary ideal of \( S \) with \( \mathfrak{R}(Q : A) = P \).

In what follows, if the algebraic radical of an ideal \( A \) is an open prime ideal, then \( A \) is called a \( P \)-ideal of \( S \).

Proposition 2.9. The set of all \( P \)-ideals of \( S \) forms a filter on \( S \).

This proposition follows by observing that (1) Any finite intersection of \( P \)-ideals of \( S \) is a \( P \)-ideal. (2) Any arbitrary union of \( P \)-ideals of \( S \) is still a \( P \)-ideal.

Moreover, we remark that this union is a submob of \( S \) and is an open prime ideal of \( S \).

Now, let \( e \) be an idempotent of a compact mob \( S \). We say that an element \( x \in S \) belongs to the idempotent \( e \) if \( e \) is the unique idempotent of \( \Gamma(x) = \{ x, x^2, \ldots \} \). Let \( B = \{ x \in S | e_x \in \Gamma(x) \} \). We shall call it a \( B \)-class. Schwarz [4], p. 119, has proved that any compact abelian mob \( S \) can be written as the union of disjoint \( B \)-classes.

Theorem 2.10. Let \( A \) be a \( P \)-ideal of \( S \). Then there exists at least one \( B \)-class which meets \( A \) but is disjoint from \( S - \mathfrak{R}(A) \).

Proof. We may assume that there exists a \( B \)-class \( B_{a_0} \) such that \( B_{a_0} \cap A \neq \emptyset \). Let \( x \in B_{a_0} \cap A \). Then \( x \in A \) and \( x \in B_{a_0} \). Consider the principal
ideal \( J(x) \) generated by \( x \). Clearly \( J(x) \) is compact and \( \{x, x^2, \ldots\} \subseteq J(x) \subseteq A \). Thus \( \Gamma(x) \subseteq J(x) \). \( \Gamma(x) \) has a unique idempotent which must be \( e_{q_0} \) since \( x \in B_{q_0} \). Now, suppose there exists an element \( y \in B_{q_0} \cap (S - A(A)) \). The element \( y \) also belongs to the idempotent \( e_{q_0} \). But, since \( y \in S - A(A) \), and \( A(A) \) is prime, we have \( \{y, y^2, \ldots\} \subseteq S - A(A) \). As \( A(A) \) is open, \( S - A(A) \) is compact in \( S \). It follows that \( \{y, y^2, \ldots\} = \Gamma(y) \subseteq S - A(A) \). Therefore \( e_{q_0} \epsilon \Gamma(y) \subseteq S - A(A) \). Therefore, \( e_{q_0} \epsilon \Gamma(y) \subseteq S - A(A) \). This is impossible since \( A \) and \( S - A(A) \) are disjoint. Hence \( B_{q_0} \cap (S - A(A)) = \emptyset \).

**Corollary.** Any \( P \)-ideal \( A \) contains exactly the same number of disjoint \( B \)-classes as \( A(A) \). More precisely, \( A \cap \bigcup_{a} B_a = \bigcup_{a} (A \cap B_a) \) with \( B_a \subseteq P \).

### 3. Stability of algebraic radicals.

**Proposition 3.1.** If \( A \) is a subset of \( S \) with \( A(A) \) closed, and \( x \in S \) is such that \( A \subseteq xA \subseteq A \). Hence \( A(A) \subseteq A(xA) \subseteq A(A) \). We only need to prove that \( A(A) \subseteq A(A) \).

**Proof.** By "Swelling lemma" ([2], p. 15), \( A \subseteq xA \subseteq A \). Hence \( A(A) \subseteq A(xA) \subseteq A(A) \). We only need to prove that \( A(A) \subseteq A(A) \).

Since \( A \subseteq A(A) \), we have \( A \subseteq A(A) = A(A) \). Consequently, \( A(A) \subseteq A(A) \) and \( A(A) = A(A) \). Thus we have obtained that \( A(xA) = A(A) \).

**Theorem 3.2 (Main theorem).** Let \( A \) be an open ideal of \( S \). Then \( A \) is radically stable if and only if \( A(A) \) does not contain any idempotent lying outside of \( A \).

In order to prove this theorem, the following lemma is crucial:

**Lemma 3.3.** Let \( A \) be any open ideal of \( S \). If \( B \) is an ideal which is not contained in \( A(A) \), then \( B \) has an idempotent not in \( A \).

**Proof.** Since \( A \) is an ideal of \( S \), so is \( A(A) \). As \( B \subseteq A(A) \), there exists an element \( b \in B \) such that \( b \in A(A) \). By the same method as theorem 2.10, we prove that there exists an idempotent \( e = e \epsilon \Gamma(b) \subseteq J(b) \subseteq B \).

Suppose on the contrary that \( e \in A \). Then \( K(b) = e \Gamma(b) \subseteq A \), where \( K(b) = \bigcap_{n=1}^{\infty} \{b^i | i \geq n\} \) ([3], p. 25). Since \( A \) is open, we have \( b^n \in A \) for some integer \( n \geq 1 \). Thus \( b \in A(A) \) which is impossible.

**Remark.** For any compact abelian mob \( S \) and \( A \) a non-empty open subset of \( S \), if \( B \) is a submob of \( S \) such that \( B \subseteq A(A) \), then \( B \) contains an idempotent which is not in \( A(A) \).

We are now ready to prove Theorem 3.2. As \( A \) is an ideal, so are \( A(A) \) and \( A(A) \). For the necessity, we suppose that \( A(A) \subseteq A(A) \). Then, by our lemma, there exists an idempotent \( e = e \epsilon A(A) \). But we assume that such idempotent does not exist. Hence, \( A(A) \subseteq A(A) \). As \( A(A) \subseteq A(A) \) always holds, we have \( A(A) = A(A) \), that is, \( A \) is radically stable.
For the converse part, we assume that $A$ is radically stable, that is, $\mathcal{R}(\overline{A}) = \mathcal{R}(A)$. Suppose there exists $e^2 \in e \mathcal{R}(\overline{A})$. Then $e \in \mathcal{R}(A)$, so there exists $k \geq 1$ such that $e^k \in A$. Thus $e \in A$ and hence, $\mathcal{R}(A)$ contains no idempotents which are not in $A$. Our proof is complete.

**Corollary 1.** Let $A$ be any ideal of the mob $S$ such that $\mathcal{R}(A)$ is open and properly contained in $S$. Then any ideal of $S$ containing $\mathcal{R}(A)$ contains a compact group which is disjoint from $A$. Conversely, let $G$ be a compact group in $S$ such that $G$ is disjoint from an open ideal $A$, and suppose that $A$ contains all the other idempotents of $S$. Then $\mathcal{R}(A)$ is an open ideal of $S$ disjoint from $G$.

**Proof.** By corollary 3 of lemma 2.6, we have $eSe \in \mathcal{R}(A)$ for some idempotent $e$. Now $eSe$ is a compact submob of $S$ with identity $e$. Consider $G_e = \{g \in eSe | gg^{-1} = e\}$. This is the maximal subgroup of $eSe$. It is known that $G_e$ is a compact subgroup of $eSe$ ([2], p. 13). We claim that $e \in \mathcal{R}(A)$. For if $e \in \mathcal{R}(A)$, then $eSe \in \mathcal{R}(A)$, a contradiction. Let us now suppose that $G_e \cap \mathcal{R}(A) \neq \emptyset$, then there exists $g \in G_e$ such that $g \in \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is an ideal of $S$, $gg^{-1} = e \in \mathcal{R}(A)$, which is impossible. For the converse part, suppose $G \cap A = \emptyset$. Since $G$ is a group, $g^k \in G$ for all $k \geq 1$, where $g \in G$. Hence $g^k \notin A$ for all $k \geq 1$. This implies that $g \notin \mathcal{R}(A)$. Thus $G \cap \mathcal{R}(A) = \emptyset$. As $G$ and $S$ are compact, $J_0(S-G)$ is an open ideal of $S$. Clearly, $\mathcal{R}(A) \cap J_0(S-G)$. Suppose that $J_0(S-G) \cap \mathcal{R}(A)$. Then by our lemma 3.3, there exists $e^2 = e \in J_0(S-G)$, $e \notin A$. This contradicts our assumption on $A$. Hence $\mathcal{R}(A) = J_0(S-G)$ and hence $\mathcal{R}(A)$ is an open ideal of $S$.

**Corollary 2.** Let $S$ be an $\Omega$-mob. If $A$ is an open ideal of $S$ which is not radically stable and $\mathcal{R}(A)$ is semi-prime, then $\mathcal{R}(\overline{A})$ is closed and has the form $Se$ with $e^2 = e \mathcal{R}(A)$.

**Proof.** The non-radical stability of $A$ implies that $\mathcal{R}(A) \not\subseteq \mathcal{R}(\overline{A})$. By using the same method as lemma 8 in [3], p. 676, and our lemma 3.3, we can prove that there exists a minimal closed ideal $M$ contained in $\mathcal{R}(\overline{A})$, but not contained in $\mathcal{R}(A)$. Moreover, $M$ has the form $Se$ with $e^2 = e \mathcal{R}(A)$. Since $S$ is compact, $Se \cap \mathcal{R}(A) \neq \emptyset$. As $S$ is $\Omega$, it follows that $\mathcal{R}(A) \subseteq Se \subseteq \mathcal{R}(\overline{A})$. Hence $\mathcal{R}(A) \subset Se$. Since $\mathcal{R}(A)$ is semi-prime, we have $\mathcal{R}(\mathcal{R}(A)) = \mathcal{R}(A)$. Thus $A \subset \mathcal{R}(A)$ implies that $\mathcal{R}(\overline{A}) \subset \mathcal{R}(A)$. We have, therefore, $\mathcal{R}(\overline{A}) = Se$ with $e^2 = e \mathcal{R}(A)$.

**Corollary 3.** Let $A$ be an ideal which is radically stable in $S$. Then $A$ is a primary ideal if and only if $\overline{A}$ is a primary ideal.

**Proof.** We only need to observe that an ideal $A$ is primary if and only if $\mathcal{R}(A)$ is prime.

Here we give two examples to demonstrate that, without radical stability, the closure of a prime (primary) ideal need not be prime (primary).
Example 3.3. Let $S$ be the subset of the plane defined by formula $S = ([0,1] \times 0) \cup (1 \times [-1,1])$ (see Fig. 1), where the underlined brackets denote the intervals, and define a commutative multiplication on $S$ by:

$(x, 0) \cdot (1, v) = (x, 0)$ for all points $x \in [a, b]$, $v \in [c, d]$.

$(x, 0) \cdot (y, 0) = (xy, 0)$ for all points $x, y \in [a, b]$.

$(1, x) \cdot (y, 1) = (x, xy)$ for all points $x, y \in [b, c]$.

$(1, x) \cdot (1, y) = (1, -xy)$ for all points $x, y \in [b, d]$.

$(1, x) \cdot (1, y) = (1, 0)$ if $x \in [b, d]$, $y \in [b, c]$ and vice versa.

Where $xy$ is the usual product of $x$ and $y$.

Fig. 1

Clearly, $[a, b]$ is a prime ideal of $S$. Also $(1, 1) \cdot (1, -1) = (1, 0) \in [a, b]$, but $(1, 1)$, $(1, -1)$ are not points in $[a, b]$. Hence, the closure of $[a, b]$ is not a prime ideal of $S$.

Example 3.4. Let $S$ be the subset of the plane defined by $S = ([0,1] \times (-1,1)) \cup (1 \times [1, -1])$ (see Fig. 2) where the underlined brackets denote intervals, and define multiplication on $S$ by:

$(x, y) \cdot (u, v) = (xu, yv)$ for all points $(x, y), (u, v)$ in the upper half plane.

$(x, y) \cdot (u, v) = (xu, -yv)$ for all points $(x, y), (u, v)$ in the lower half plane.

$(x, y) \cdot (u, v) = (xu, 0)$ if one of the points lies in the upper half plane and the other lies in the lower half plane.

Fig. 2

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3 — Colloquium Mathematicum XXV.1.
Clearly, the rectangle \( Q = (0, 1) \times (-\frac{1}{2}, \frac{1}{2}) \) is a primary ideal of \( S \), but the closure of \( Q \) is not primary.

Remark 1. Every ideal of the usual thread \( I \) is a primary ideal. By a usual thread we mean a semigroup topologically isomorphic to \([0, 1]\) with its usual real multiplication. Obviously, the minimal ideal, \( \{0\} \), of \( I \) is primary. Any non-minimal ideal of \( I \) has the form \([0, x]\) or \([0, x]\) for a fixed \( x \) in \([0, 1]\) by [4], p. 84. To see that \([0, x]\) is primary, suppose \( ab \in [0, x) \), \( a \notin [0, x] \). Then \( 0 \leq ab < x \), \( x \leq a \leq 1 \). Hence, \( 0 \leq b < x/a \), \( x/a \leq 1 \). Thus \( 0 \leq b < 1 \). Since \( x \) is fixed, there exists \( k \geq 1 \) such that \( b^k < x \). As \([0, x]\) is radically stable, \([0, x]\) is also a primary ideal of \( I \).

Remark 2. Every ideal of the min-thread \( I \) is prime. By a minthread, we mean a semigroup topologically isomorphic to \([0, 1]\) with multiplication \( x*y = \min(x, y) \). This remark is clear.

4. Concluding remarks. The definitions of reducibility and irreducibility of ideals can be generalized as follows: An ideal \( A \) is said to be \( \mathcal{R} \)-irreducible if \( A \) is reducible such that if \( A = \bigcap_{a} A_{a} \), where \( A_{a} \) are ideals of \( S \), then there exists at least one \( A_{a} \) such that \( \mathcal{R}(A_{a}) = \mathcal{R}(A) \). If \( \mathcal{R}(A_{a}) \neq \mathcal{R}(A) \) for all \( a \), then \( A \) is said to be \( \mathcal{R} \)-irreducible. The following example shows that \( \mathcal{R} \)-reducible ideals exist.

Example 4.1. Let \( S \) be the semigroup consisting of four elements \( \{0, a, b, c\} \) such that \( a^2 = a \), \( c^2 = c \) and all other products are zero. Clearly, \( \{0\}, \{0, a\}, \{0, c\} \) are ideals of \( S \) with \( \{0\} = \{0, a\} \cap \{0, c\} \). But \( \mathcal{R}(\{0\}) = \{0, b\}, \mathcal{R}(\{0, a\}) = \{0, a, b\}, \mathcal{R}(\{0, c\}) = \{0, c, b\} \). Thus \( \{0\} \) is \( \mathcal{R} \)-reducible.

The following facts are easily verified:

1. Any algebraic radical of an ideal which is open and non-prime in the compact mob \( S \) is \( \mathcal{R} \)-reducible.

2. If \( A \) is strongly reducible such that \( \mathcal{R}(A) \) is a maximal proper ideal of \( S \), then \( A \) is \( \mathcal{R} \)-irreducible.

3. If a primary ideal is strongly reducible, then it is \( \mathcal{R} \)-irreducible.

It should be pointed out that, in general, a primary ideal \( Q \) and its associated prime ideal \( \mathcal{R}(Q) \) are topologically unrelated. For instance, the statement “\( Q \) is compact if and only if \( \mathcal{R}(Q) \) is compact” is not true. For \( Q \) is compact does not imply \( \mathcal{R}(Q) \) is compact (cf. Remark 1 in section 3). Also \( \mathcal{R}(Q) \) is compact does not imply \( Q \) is compact. For take \( S = [0, 1] \) with the multiplication * defined by \( x*y = \frac{1}{2}xy \), for all \( x, y \) in \( S \). Then \( Q = [0, \frac{1}{2}] \) is a primary ideal of \( S \) which is not compact while \( \mathcal{R}(Q) = [0, 1] \).

Also “\( Q \) is connected if and only if \( \mathcal{R}(Q) \) is connected” is not true. For take \( S = [0, \frac{1}{2}] \cup [1, 2] \). Define \( x*y = \frac{1}{2}xy \) for all \( x, y \) in \( S \). Then \([0, \frac{1}{2}] \) is a primary ideal of \( S \), \( \mathcal{R}([0, \frac{1}{2}]) = S \) is disconnected. On the
other hand, take \( S = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\} \). Define \((x, y) \cdot (x', y') = (0, yy')\). Let \( Q = \{(x, y) | x \in \{0, 1\}, 0 \leq y \leq 1\} \). Then it can easily be checked that \( Q \) is a primary ideal of \( S \). As \( \mathcal{R}(Q) = S \), \( \mathcal{R}(Q) \) is connected, however, \( Q \) itself is disconnected.

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