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THE SHAPE GENUS OF A SHAPE MAP

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1. Introduction. In [14] Švarc introduced the notion of the genus of a fibre map. Later Berstein and Ganea [1] extended that concept to arbitrary maps. The present paper generalizes their definition further by defining the shape genus of a shape map $f: X \to Y$ of topological spaces. Our method is modelled after the paper [3] where we made a similar extension of the notion of the category of a map from [1].

We first generalize the definition of the genus of a map $f: X \to Y$ from [1] by introducing the notion of the $\mathfrak{F}$-genus, $\mathrm{gen}_{\mathfrak{F}}(f, g)$, of a pair (f, g) of maps $f: X \to Y$ and $g: Z \to Y$, where $\mathfrak{F}$ is a class of homotopy invariant functors on the category of topological spaces. Then we investigate in what sense $\mathrm{gen}_{\mathfrak{F}}(f, g)$ depends on $\mathfrak{F}$ and f, g. Next, we define the $\mathfrak{F}$-genus $\mathrm{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g})$ of a pair of maps of inverse systems $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}: \boldsymbol{Z} \to \boldsymbol{Y}$ and prove two lemmas (see Lemmas 3.2 and 3.3) which show that we can introduce the shape $\mathfrak{F}$-genus $\mathbf{gen}_{\mathfrak{F}}(f, g)$ of a pair $f: X \to Y$ and $g: Z \to Y$ of shape maps of topological spaces [10] by taking it to be the $\mathfrak{F}$-genus of any pair $(\boldsymbol{f}, \boldsymbol{g})$ of maps of inverse ANR-systems $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}: \boldsymbol{Z} \to \boldsymbol{Y}$ associated (in the sense of Morita [13]) with f and g, respectively. In the case where $f: X \to Y$ and $g: Z \to Y$ are maps of CW-complexes, $\mathbf{gen}_{\mathfrak{F}}(f, g) = \mathrm{gen}_{\mathfrak{F}}(f, g)$, but for locally more complicated spaces these two integers are not equal in general (see Example 3.6).

The final Section 4 presents a number of estimates for the shape genus of a pair of shape maps that represent generalizations of some results in [1] and [14].

The concept of the genus of a map is little known in the homotopy theory. It is a numerical homotopy invariant closely related to the category of a map [1] (see Corollaries 2.7 and 2.8). The genus can be used to obtain lower (see Proposition 2.4 (a)) as well as upper (see Proposition 4.1) bounds for the category of a map. For the applications of the notion of the genus we refer the reader to Chapter VII of [14]. The most interesting of those applications are concerned with the problem of embedding n-dimensional polyhedra into the Euclidean space E^{2n}.

In a paper that is now in preparation we shall obtain necessary and sufficient conditions, involving the shape genus of certain maps, for an n-dimensional compactum to be embeddable up to shape into E^{2n}, generalizing recent results of Ivanšić [7] and Duvall and Husch [6].

We assume that the reader is familiar with shape theory and, in particular, with Morita's description [13] of the shape category and with inverse ANR-systems and their maps [11].

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2. $\mathfrak{F}$-genus of a map. Throughout this section and the next one, $\mathfrak{F}$ will stand for a class of functors $\mathcal{F}: \mathcal{H} \to \mathcal{K}_{\mathcal{F}}$ from the homotopy category $\mathcal{H}$ of topological spaces into any category $\mathcal{K}_{\mathcal{F}}$. The homotopy class of a map $f: X \to Y$ is denoted by $[f]$. We say that maps $f, g: X \to Y$ are *$\mathfrak{F}$-equal* provided $\mathcal{F}([f]) = \mathcal{F}([g])$ for every functor $\mathcal{F}$ in $\mathfrak{F}$.

2.1. Definition. Let $f: X \to Y$ and $g: Z \to Y$ be maps into Y. The *$\mathfrak{F}$-genus* of the pair (f, g), $\mathrm{gen}_{\mathfrak{F}}(f, g)$, is the least integer $k \geq 1$ for which there are open sets V_m and maps $h_m: V_m \to X$, $1 \leq m \leq k$, such that $Z = \bigcup V_m$ and $f \circ h_m$ and $g \circ j_m$ are $\mathfrak{F}$-equal, where $j_m: V_m \to Z$ is the inclusion map $(1 \leq m \leq k)$: if no such integer exists, we put $\mathrm{gen}_{\mathfrak{F}}(f, g) = \infty$. The $\mathfrak{F}$-genus of the pair (f, id_Y) will be called the *$\mathfrak{F}$-genus of a map f* and denoted by $\mathrm{gen}_{\mathfrak{F}} f$.

We shall first prove that the $\mathfrak{F}$-genus of a pair of maps into Y does not increase if the class $\mathfrak{F}$ is replaced by a class $\mathfrak{G}$ which is dominated by $\mathfrak{F}$. Recall [3] that the class $\mathfrak{F}$ of functors on $\mathcal{H}$ *dominates* another such class $\mathfrak{G}$ if for each functor $\mathcal{G}: \mathcal{H} \to \mathcal{K}_{\mathcal{G}}$ in $\mathfrak{G}$ there are a functor $\mathcal{F}: \mathcal{H} \to \mathcal{K}_{\mathcal{G}}$ in $\mathfrak{F}$ and natural transformations $\alpha: \mathcal{F} \to \mathcal{G}$ and $\beta: \mathcal{G} \to \mathcal{F}$ such that $\alpha \circ \beta = \mathrm{id}_{\mathcal{G}}$.

2.2. THEOREM. *Let $f: X \to Y$ and $g: Z \to Y$ be maps into Y and let the class $\mathfrak{F}$ of functors on $\mathcal{H}$ dominate the class $\mathfrak{G}$. Then* $\mathrm{gen}_{\mathfrak{F}}(f, g) \geq \mathrm{gen}_{\mathfrak{G}}(f, g)$.

Proof. Let U be an open set in Z and assume that there is a map $h: U \to X$ such that for every functor $\mathcal{F} \in \mathfrak{F}$, morphisms $\mathcal{F}([f \circ h])$ and $\mathcal{F}([g \circ j])$ agree, where $j: U \to Z$ is the inclusion. It suffices to prove, clearly, that $f \circ h$ and $g \circ j$ are $\mathfrak{G}$-equal.

Given a functor $\mathcal{G}: \mathcal{H} \to \mathcal{K}$ in $\mathfrak{G}$, select a functor $\mathcal{F}: \mathcal{H} \to \mathcal{K}$ in $\mathfrak{F}$ and natural transformations $\alpha: \mathcal{F} \to \mathcal{G}$ and $\beta: \mathcal{G} \to \mathcal{F}$ such that $\alpha \circ \beta = \mathrm{id}_{\mathcal{G}}$. Then

$$\begin{aligned}\mathcal{G}([f \circ h]) &= \alpha_Y \circ \beta_Y \circ \mathcal{G}([f \circ h]) = \alpha_Y \circ \mathcal{F}([f \circ h]) \circ \beta_U \\ &= \alpha_Y \circ \mathcal{F}([g \circ j]) \circ \beta_U = \alpha_Y \circ \beta_Y \circ \mathcal{G}([g \circ j]) = \mathcal{G}([g \circ j])\end{aligned}$$

since $\alpha_Y \circ \beta_Y = \mathrm{id}_{\mathcal{G}(Y)}$.

It is clear from the definition that $\mathrm{gen}_{\mathfrak{F}}(f, g)$ and $\mathrm{gen}_{\mathfrak{F}}(f', g')$ are equal if $f \simeq f'$ and $g \simeq g'$ ("$\simeq$" means "is homotopic to"). Our next theorem provides a generalization of this statement. Recall [3] that a map $\beta\colon Y' \to Y$ is an $\mathfrak{F}$-*monomorphism* if for maps $h, k\colon W \to Y'$ the fact that $\beta \circ h$ and $\beta \circ k$ are $\mathfrak{F}$-equal implies that h and k are already $\mathfrak{F}$-equal.

2.3. THEOREM. *Let $\mathfrak{F}$ be a class of functors on $\mathscr{H}$ and let (f, g) and (f', g') be two pairs of maps $f\colon X \to Y$, $g\colon Z \to Y$, and $f'\colon X' \to Y'$, $g'\colon Z' \to Y'$. Suppose one of the following conditions holds:*

(a) *there are maps $\alpha\colon X \to X'$, $\gamma\colon Z' \to Z$, and an $\mathfrak{F}$-monomorphism $\beta\colon Y' \to Y$ such that f and $g \circ \gamma$ are $\mathfrak{F}$-equal to $\beta \circ f' \circ \alpha$ and $\beta \circ g'$, respectively;*

(b) *there are maps $\alpha\colon X \to X'$, $\gamma\colon Z' \to Z$, and $\beta\colon Y \to Y'$ such that $f' \circ \alpha$ and $\beta \circ g \circ \gamma$ are $\mathfrak{F}$-equal to $\beta \circ f$ and g', respectively.*

Then $\mathrm{gen}_{\mathfrak{F}}(f, g) \geqslant \mathrm{gen}_{\mathfrak{F}}(f', g')$.

Proof. If (a) holds, let U be an open subset of Z and assume that there is a map $h\colon U \to X$ such that $f \circ h$ and $g \circ j$ are $\mathfrak{F}$-equal, where $j\colon U \to Z$ is the inclusion. Let $U' = \gamma^{-1}(U)$, let $h^*\colon U \to X'$ be the composition $\alpha \circ h$, let $\tilde{\gamma} = \gamma | U'\colon U' \to U$, and let $j'\colon U' \to Z'$ be the inclusion. Put $h' = h^* \circ \tilde{\gamma}$. We show that $f' \circ h'$ and $g' \circ j'$ are $\mathfrak{F}$-equal.

Indeed, for every functor $\mathscr{F}$ in $\mathfrak{F}$ we have

$$\mathscr{F}([\beta \circ f' \circ h']) = \mathscr{F}([\beta \circ f' \circ \alpha \circ h \circ \tilde{\gamma}]) = \mathscr{F}([f \circ h \circ \tilde{\gamma}])$$

$$= \mathscr{F}([g \circ j \circ \tilde{\gamma}]) = \mathscr{F}([g \circ \gamma \circ j']) = \mathscr{F}([\beta \circ g' \circ j']):$$

since β is an $\mathfrak{F}$-monomorphism, we get $\mathscr{F}([f' \circ h']) = \mathscr{F}([g' \circ j'])$, $\mathscr{F} \in \mathfrak{F}$, as claimed.

If (b) holds, the proof is similar.

The simple proof of the following proposition is left to the reader. Its parts (a) and (b) are obvious while parts (c) and (d) follow from Theorem 2.3. Recall from [3] that the $\mathfrak{F}$-*category* of a map $f\colon X \to Y$, $\mathrm{cat}_{\mathfrak{F}} f$, is the least integer $k \geqslant 1$ with the property that X can be covered with k open sets $\{U_m\}_{m=1}^{k}$ such that $f \circ j_m$ and $c_m \circ j_m$ are $\mathfrak{F}$-equal, where $c_m\colon X \to Y$ is a constant map and $j_m\colon U_m \to X$ is the inclusion $(m = 1, \ldots, k)$; if no such integer exists, we put $\mathrm{cat}_{\mathfrak{F}} f = \infty$.

2.4. PROPOSITION. *Let $f\colon X \to Y$, $g\colon Z \to Y$, $h\colon Y \to Y'$, $f'\colon X' \to X$, and $g'\colon Z' \to Z$ be continuous maps. Then*

(a) $\mathrm{gen}_{\mathfrak{F}}(f, g) \leqslant \mathrm{cat}_{\mathfrak{F}}\, g$ *if Y is arcwise-connected.*

(b) $\mathrm{gen}_{\mathfrak{F}}(f, g) = 1$ *if and only if there is a map $\chi\colon Z \to X$ such that $f \circ \chi$ and g are $\mathfrak{F}$-equal. In particular, if f is a homotopy domination, then* $\mathrm{gen}_{\mathfrak{F}}(f, g) = 1$.

(c) $\mathrm{gen}_{\mathfrak{F}}(f \circ f', g) \geqslant \mathrm{gen}_{\mathfrak{F}}(f, g) \geqslant \mathrm{gen}_{\mathfrak{F}}(f, g \circ g')$, *and if both f' and g' are homotopy equivalences, then these three numbers are equal.*

(d) $\mathrm{gen}_{\mathfrak{F}}(h \circ f, h \circ g) \leqslant \mathrm{gen}_{\mathfrak{F}}(f, g)$, *and if h is a homotopy equivalence, then* $\mathrm{gen}_{\mathfrak{F}}(h \circ f, h \circ g) = \mathrm{gen}_{\mathfrak{F}}(f, g)$.

Let $\mathscr{C}$ be a class of topological spaces. The $\mathfrak{F}$-genus of a map f [of a pair (f, g)] for a class of functors $\mathfrak{F} = \{[X, -]: X \in \mathscr{C}\}$ will be denoted by $\text{gen}_{\mathscr{C}} f$ [$\text{gen}_{\mathscr{C}}(f, g)$] and, simply, by $\text{gen} f$ ($\text{gen}_n f$) [$\text{gen}(f, g)$ ($\text{gen}_n(f, g)$)] in the case where $\mathscr{C} = \mathscr{C}\mathscr{W}$ ($\mathscr{C} = \mathscr{C}\mathscr{W}_n$), i.e., the class of all spaces having homotopy type of CW-complexes (of dimension less than or equal to n). Here, $[X, -]: \mathscr{H} \to Sets$ is a functor that associates with a space Y the set $[X, Y]$ of homotopy classes of maps $X \to Y$ and the function

$$f_{\#} = f_{\#}^{X}: [X, Y] \to [X, Z],$$

given by $f_{\#}([g]) = [f \circ g]$, $[g] \in [X, Y]$, with a homotopy class $[f]: Y \to Z$.

Recall from [9] (see also [2]) that the class of spaces $\mathscr{C}$ *shape dominates* another such class $\mathscr{D}$ provided the class of functors $\{[X, -]: \mathscr{C}\mathscr{W} \to Sets \mid X \in \mathscr{C}\}$ dominates the class of functors $\{[X, -]: \mathscr{C}\mathscr{W} \to Sets \mid X \in \mathscr{D}\}$. Hence, Theorem 2.2 implies the following

2.5. Corollary. *Let $f: X \to Y$ and $g: Z \to Y$ be maps of ANR's. If a class of topological spaces $\mathscr{C}$ shape dominates another such class $\mathscr{D}$, then $\text{gen}_{\mathscr{C}}(f, g) \geq \text{gen}_{\mathscr{D}}(f, g)$. In particular, $\text{gen}_{\mathscr{C}}(f, g)$ depends only on the shape properties of spaces in $\mathscr{C}$.*

The computation of the category $\text{cat} f$ of a map f (i.e., the category of f with respect to the class of functors $\{[X, -]: X \in \mathscr{C}\mathscr{W}\}$) can always be reduced to the computation of the genus of a related (fibre) map as the first corollary to the next proposition shows.

2.6. Proposition. *Let $f: X \to Y$ and $h: Y \to W$ be continuous maps.*

(a) *If* $\ker h_{\#}^{A} \subset \text{Im} f_{\#}^{A}$ *for every space A, then* $\text{gen} f \leq \text{cat}\, h$.

(b) *If* $\text{Im} f_{\#}^{A} \subset \ker h_{\#}^{A}$ *for every space A, then* $\text{cat}\, h \leq \text{gen} f$.

Proof. We prove only (a) because the proof of (b) is similar.

Let U be an open subset of Y and assume that the composition $h \circ j$ of the inclusion $j: U \to Y$ with h is homotopic to some constant map $c: U \to W$. In other words, assume

$$[j] \in \ker h_{\#}^{U} = \{[g]: U \to Y \mid h \circ g \text{ is homotopic to a constant map } U \to W\}.$$

By assumption, there is a map $k: U \to X$ such that $[j] = f_{\#}^{U}([k]) = [f \circ k]$. This clearly implies that $\text{gen} f \leq \text{cat}\, h$.

2.7. Corollary. *For any map $f: X \to Y$, there is a map $f': Z \to X$ such that* $\text{gen} f' = \text{cat} f$.

Proof. Let LY denote the path space of Y (see [12], Chapter VI) and let Z be the subspace of $X \times LY$ of pairs (x, λ) such that $f(x) = \lambda(0)$. Let $f': Z \to X$ be the map defined by $f'(x, \lambda) = x$. By [12], Proposition 6.4.10, for any space A and any choice of base points in X and Y, the sequence

$$[A, Z] \xrightarrow{f'_{\#}} [A, X] \xrightarrow{f_{\#}} [A, Y]$$

is an exact sequence of based sets. Hence, conditions (a) and (b) in Proposition 2.6 hold for maps f and f' and the claim follows.

In the same way, from [12], Corollary 6.5.8, we get the proof of the following result stated in [1] as Proposition 2.6:

2.8. COROLLARY. *If $p\colon E \to B$ is a (Hurewicz) fibration with the fibre F and a path-connected base B, then* $\operatorname{cat} p = \operatorname{gen} i$, *where $i\colon F \to E$ is the inclusion map.*

3. Shape $\mathfrak{F}$-genus. In this section we shall define the notion of the shape $\mathfrak{F}$-genus of a pair (f, g) of shape maps $f\colon X \to Y$ and $g\colon Z \to Y$ of arbitrary topological spaces. The idea, like that in Section 3 of [3], is to extend Definition 2.1 to morphisms in the pro-category pro-$\mathscr{H}$ of the homotopy category $\mathscr{H}$ of topological spaces and then use the isomorphism of the shape category and the category pro-$\mathscr{CW}$ [13].

3.1. Definition. Let $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $Y = (Y_\mu, q_{\mu\mu'}, M)$, and $Z = (Z_\nu, r_{\nu\nu'}, N)$ be objects in pro-$\mathscr{H}$. Let $\boldsymbol{f} = (f, f_\mu)\colon \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} = (g, g_\mu)\colon \boldsymbol{Z} \to \boldsymbol{Y}$ be maps of inverse systems [11] and let $\mathfrak{F}$ be a class of functors on $\mathscr{H}$. We define $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g})$ to be the least integer $k \geqslant 1$ such that for each $\mu \in M$ there is $\nu \geqslant g(\mu)$ satisfying

$$\operatorname{gen}_{\mathfrak{F}}(f_\mu \circ p_{f(\mu)\lambda}, g_\mu \circ r_{g(\mu)\nu}) \leqslant k \quad \text{for each } \lambda \geqslant f(\mu).$$

If no such integer exists, we put $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g}) = \infty$.

The definitions of $\operatorname{gen}_{\mathfrak{F}}\boldsymbol{f}$, $\operatorname{gen}_{\mathscr{C}}(\boldsymbol{f}, \boldsymbol{g})$, $\operatorname{gen}(\boldsymbol{f}, \boldsymbol{g})$, $\operatorname{gen}_n(\boldsymbol{f}, \boldsymbol{g})$, $\operatorname{gen}_{\mathscr{C}}\boldsymbol{f}$, $\operatorname{gen}\boldsymbol{f}$, and $\operatorname{gen}_n\boldsymbol{f}$ are obvious modifications of the corresponding definitions from Section 2.

3.2. LEMMA. *If $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}, \boldsymbol{g}'\colon \boldsymbol{Z} \to \boldsymbol{Y}$ are equivalent (homotopic) pairs of maps of inverse systems* [11], *then* $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g}) = \operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}', \boldsymbol{g}')$.

Proof. Suppose $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g}) = k < \infty$. For a $\mu \in M$ pick $\nu \geqslant g(\mu)$ as in the above definition. Now, use homotopies $\boldsymbol{f} \simeq \boldsymbol{f}'$ and $\boldsymbol{g} \simeq \boldsymbol{g}'$ and choose indices $\nu' \geqslant \nu$, $g'(\mu)$ in N and $\lambda \geqslant f(\mu)$, $f'(\mu)$ in Λ such that

$$g_\mu \circ r_{g(\mu)\nu'} \simeq g'_\mu \circ r_{g'(\mu)\nu'} \quad \text{and} \quad f_\mu \circ p_{f(\mu)\lambda} \simeq f'_\mu \circ p_{f'(\mu)\lambda}.$$

Consider an arbitrary index $\lambda' \geqslant f'(\mu)$ in Λ. Let $\lambda'' \geqslant \lambda, \lambda'$. Since

$$\operatorname{gen}_{\mathfrak{F}}(f'_\mu \circ p_{f'(\mu)\lambda'}, g'_\mu \circ r_{g'(\mu)\nu'}) \leqslant \operatorname{gen}_{\mathfrak{F}}(f'_\mu \circ p_{f'(\mu)\lambda'} \circ p_{\lambda'\lambda''}, g'_\mu \circ r_{g'(\mu)\nu'})$$
$$= \operatorname{gen}_{\mathfrak{F}}(f_\mu \circ p_{f(\mu)\lambda} \circ p_{\lambda\lambda''}, g_\mu \circ r_{g(\mu)\nu} \circ r_{\nu\nu'}) \leqslant \operatorname{gen}_{\mathfrak{F}}(f_\mu \circ p_{f(\mu)\lambda''}, g_\mu \circ r_{g(\mu)\nu}) \leqslant k,$$

we have $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}', \boldsymbol{g}') \leqslant \operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g})$. In a similar way one proves the inverse inequality and the case where either $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g})$ or $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}', \boldsymbol{g}')$ is infinite.

Since morphisms in pro-$\mathscr{H}$ are equivalence classes of maps of inverse systems, Lemma 3.2 allows us to define the $\mathfrak{F}$-genus of a pair $([\boldsymbol{f}], [\boldsymbol{g}])$ of morphisms in pro-$\mathscr{H}$ by taking it to be $\operatorname{gen}_{\mathfrak{F}}(\boldsymbol{f}, \boldsymbol{g})$ of any representatives $\boldsymbol{f}$ of

$[f]$ and g of $[g]$. In order to make the transition from morphisms in pro-$\mathscr{H}$ to shape maps we need the following lemma which resembles Theorem 2.3.

3.3. LEMMA. *Let*

$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Y & \xleftarrow{\ g\ } & Z \\ {\scriptstyle i}\uparrow & & {\scriptstyle j}\uparrow & \bullet & \uparrow{\scriptstyle k} \\ X' & \xrightarrow[f']{} & Y' & \xleftarrow[g']{} & Z' \end{array}$$

be a homotopy commutative diagram of maps of inverse systems. Assume that i *has a right homotopy inverse and* j *has a left homotopy inverse. Then* $\operatorname{gen}_{\mathfrak{F}}(f', g') \leqslant \operatorname{gen}_{\mathfrak{F}}(f, g)$.

Proof. Let μ' be an arbitrary element of the index set M' of Y'. Let $j^* = (j^*, j^*_{\mu'})\colon Y \to Y'$ be a left homotopy inverse of j and let $i^* = (i^*, i^*_{\lambda'})\colon X \to X'$ be a right homotopy inverse of $i = (i, i_\lambda)$. Let $\alpha = f'(\mu')$, $\beta = j^*(\mu')$, $\gamma = f(\beta)$, $\delta = i(\gamma)$, $\varepsilon = i^*(\delta)$, $\zeta = g'(\mu')$, $\eta = g(\beta)$, and $\theta = k(\eta)$. Pick $\varkappa' \geqslant \zeta, \theta$ in N' such that

$$g'_{\mu'} \circ r'_{\zeta\varkappa'} \simeq j^*_{\mu'} \circ g_\beta \circ k_\eta \circ r'_{\theta\varkappa'}.$$

Next, select a $\lambda' \geqslant \alpha$, δ in Λ' such that

$$f'_{\mu'} \circ p'_{\alpha\lambda'} \simeq j^*_{\mu'} \circ f_\beta \circ i_\gamma \circ p'_{\delta\lambda'}.$$

Suppose $\operatorname{gen}_{\mathfrak{F}}(f, g) = m < \infty$ (in the case where $\operatorname{gen}_{\mathfrak{F}}(f, g) = \infty$ there is nothing to prove). Pick $\nu \geqslant \eta$ in N such that for every $\lambda \geqslant \gamma$ the $\mathfrak{F}$-genus of the pair $(f_\beta \circ p_{\gamma\lambda}, g_\beta \circ r_{\eta\nu})$ is less than or equal to m. Let $\sigma = k(\nu)$ and let $\nu' \geqslant \varkappa'$ (σ has the property that $r_{\eta\nu} \circ k_\nu \circ r'_{\sigma\nu'} \simeq k_\eta \circ r'_{\theta\nu'}$).

We claim that for every $\lambda'' \geqslant \alpha$ the $\mathfrak{F}$-genus of the pair $(f'_{\mu'} \circ p'_{\alpha\lambda''}, g'_{\mu'} \circ r'_{\zeta\nu'})$ is at most m, i.e., $\operatorname{gen}_{\mathfrak{F}}(f', g') \leqslant m$.

Indeed, let $\tau = i^*(\lambda'')$ and let $\xi \geqslant \gamma$. Then τ has the property that

$$p'_{\delta\lambda''} \circ i^*_{\lambda''} \circ p_{\tau\xi} \simeq i^*_\delta \circ p_{\varepsilon\xi} \quad \text{and} \quad p_{\gamma\xi} \simeq i_\gamma \circ i^*_\delta \circ p_{\varepsilon\xi}.$$

Applying Proposition 2.4, the fact that the $\mathfrak{F}$-genus of the pair (f, g) depends only on the homotopy classes of f and g, and the homotopies mentioned in the selection of the indices, we get

$$\begin{aligned} &\operatorname{gen}_{\mathfrak{F}}(f'_{\mu'} \circ p'_{\alpha\lambda''}, g'_{\mu'} \circ r'_{\zeta\nu'}) \\ &\leqslant \operatorname{gen}_{\mathfrak{F}}(f'_{\mu'} \circ p'_{\alpha\lambda'} \circ p'_{\lambda'\lambda''} \circ i^*_{\lambda''} \circ p_{\tau\xi}, g'_{\mu'} \circ r'_{\zeta\varkappa'} \circ r'_{\varkappa'\nu'}) \\ &= \operatorname{gen}_{\mathfrak{F}}(j^*_{\mu'} \circ f_\beta \circ i_\gamma \circ p'_{\delta\lambda'} \circ p'_{\lambda'\lambda''} \circ i^*_{\lambda''} \circ p_{\tau\xi}, j^*_{\mu'} \circ g_\beta \circ k_\eta \circ r'_{\theta\varkappa'} \circ r'_{\varkappa'\nu'}) \\ &= \operatorname{gen}_{\mathfrak{F}}(j^*_{\mu'} \circ f_\beta \circ i_\gamma \circ i^*_\delta \circ p_{\varepsilon\xi}, j^*_{\mu'} \circ g_\beta \circ r_{\eta\nu} \circ k_\nu \circ r'_{\sigma\nu'}) \\ &\leqslant \operatorname{gen}_{\mathfrak{F}}(j^*_{\mu'} \circ f_\beta \circ p_{\gamma\xi}, j^*_{\mu'} \circ g_\beta \circ r_{\eta\nu}) \leqslant \operatorname{gen}_{\mathfrak{F}}(f_\beta \circ p_{\gamma\xi}, g_\beta \circ r_{\eta\nu}) \leqslant m. \end{aligned}$$

Now we can define the notion of the $\mathfrak{F}$-genus in the shape category as follows.

Let X, Y, and Z be topological spaces and let $f: X \to Y$ and $g: Z \to Y$ be shape maps. By [13], there are morphisms $[\mathbf{f}]: \mathbf{X} \to \mathbf{Y}$ and $[\mathbf{g}]: \mathbf{Z} \to \mathbf{Y}$ in pro-$\mathscr{CW}$ naturally associated with f and g, respectively. We define the *shape $\mathfrak{F}$-genus of the pair* (f, g), $\mathbf{gen}_{\mathfrak{F}}(f, g)$, to be equal to $\mathrm{gen}_{\mathfrak{F}}(\mathbf{f}, \mathbf{g})$. One easily checks (using the above lemmas) that this definition does not depend on the choice of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{f}$, and $\mathbf{g}$. The *shape $\mathfrak{F}$-genus of a shape map* $f: X \to Y$, $\mathbf{gen}_{\mathfrak{F}} f$, is simply the shape $\mathfrak{F}$-genus of the pair (f, id_Y). The special selections for the class $\mathfrak{F}$ (see Section 2) will give us (possibly infinite) integers $\mathbf{gen}_{\mathscr{C}}(f, g)$, $\mathbf{gen}(f, g)$, $\mathbf{gen}_n(f, g)$, $\mathbf{gen}_{\mathscr{C}} f$, $\mathbf{gen} f$, and $\mathbf{gen}_n f$.

The results of Section 2 imply the following corollaries:

3.4. Corollary. *Let $f: X \to Y$ and $g: Z \to Y$ be shape maps and let the class $\mathfrak{F}$ of functors on $\mathscr{H}$ dominate the class $\mathfrak{G}$. Then $\mathbf{gen}_{\mathfrak{F}}(f, g) \geqslant \mathbf{gen}_{\mathfrak{G}}(f, g)$. In particular, if a class of topological spaces $\mathscr{C}$ shape dominates a class $\mathscr{D}$, then $\mathbf{gen}_{\mathscr{C}}(f, g) \geqslant \mathbf{gen}_{\mathscr{D}}(f, g)$.*

3.5. Corollary. *Let $f: X \to Y$, $g: Z \to Y$, $h: Y \to Y'$, $f': X' \to X$, and $g': Z' \to Z$ be shape maps and let $\mathfrak{F}$ be a class of functors on $\mathscr{H}$ (or on $\mathscr{CW}$). Then*

(a) $\mathbf{gen}_{\mathfrak{F}}(f, g) \leqslant \mathbf{cat}_{\mathfrak{F}} g$ *(i.e., the shape $\mathfrak{F}$-category of g [3]) if Y is connected.*

(b) *If f is a shape domination, then $\mathbf{gen}_{\mathfrak{F}} f = 1$.*

(c) $\mathbf{gen}_{\mathfrak{F}}(f \circ f', g) \geqslant \mathbf{gen}_{\mathfrak{F}}(f, g) \geqslant \mathbf{gen}_{\mathfrak{F}}(f, g \circ g')$. *If both f' and g' are shape equivalences, then these three numbers are equal.*

(d) $\mathbf{gen}_{\mathfrak{F}}(h \circ f, h \circ g) \leqslant \mathbf{gen}_{\mathfrak{F}}(f, g)$. *If h is a shape equivalence, then $\mathbf{gen}_{\mathfrak{F}}(h \circ f, h \circ g) = \mathbf{gen}_{\mathfrak{F}}(f, g)$.*

3.6. Example. It is clear that, for a map $f: X \to Y$ of CW-complexes, $\mathrm{gen}_{\mathfrak{F}} f$ and $\mathbf{gen}_{\mathfrak{F}} f$ coincide. When X and Y have a more complicated local structure, the $\mathfrak{F}$-genus and the shape $\mathfrak{F}$-genus of the map f are different in general. For example, if $f: W \to S^1$ is a natural surjection of the Warsaw circle W onto the circle S^1, then $\mathrm{gen}_{\{S^1\}} f = 2$ and $\mathbf{gen}_{\{S^1\}} f = 1$ by Corollary 3.5 (b) since f is a shape equivalence.

4. Estimates for the shape genus. The purpose of the present section is to prove several estimates for the shape genus of (a pair of) shape maps. They represent generalizations of some results from [1] and [14].

4.1. Proposition. *Let $f: X \to Y$, $g: Z \to Y$, and $h: Y \to W$ be shape maps and assume that $h \circ f$ is a trivial shape map. Then $\mathbf{gen}_{\mathfrak{F}}(f, g) \geqslant \mathbf{cat}_{\mathfrak{F}} h \circ g$ for each class $\mathfrak{F}$ of functors on $\mathscr{H}$ (or on $\mathscr{CW}$).*

Proof. Let f, g, and h be represented by maps of inverse systems

$$\mathbf{f}: \mathbf{X} \to \mathbf{Y}, \quad \mathbf{g}: \mathbf{Z} \to \mathbf{Y}, \quad \text{and} \quad \mathbf{h}: \mathbf{Y} \to \mathbf{W},$$

respectively, where

$$\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda), \quad \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M),$$
$$\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N), \quad \text{and} \quad \mathbf{W} = (W_\pi, s_{\pi\pi'}, P).$$

For an index π in P, choose $\varrho \geqslant f \circ h(\pi)$ such that $h_\pi \circ f_{h(\pi)} \circ p_{f \circ h(\pi)\varrho}$ is homotopic to a constant map. Such an index exists because $h \circ f$ is a trivial shape map. If $\mathbf{gen}_{\mathfrak{F}}(f, g) = k$, take $v \geqslant g \circ h(\pi)$ such that

$$\mathbf{gen}_{\mathfrak{F}}(f_{h(\pi)} \circ p_{f \circ h(\pi)\lambda}, g_{h(\pi)} \circ r_{g \circ h(\pi)v}) \leqslant k$$

whenever $\lambda \geqslant f \circ h(\pi)$. We claim that $\mathrm{cat}_{\mathfrak{F}} h_\pi \circ g_{h(\pi)} \circ r_{g \circ h(\pi)v} \leqslant k$.

Indeed, assume U is an open set in Z_v for which there is a map $d: U \to X_\varrho$ which satisfies

$$f_{h(\pi)} \circ p_{f \circ h(\pi)\varrho} \circ d \simeq g_{h(\pi)} \circ r_{g \circ h(\pi)v} \circ j$$

(j denotes the inclusion $U \to Z_v$). Composing both sides with h_π we see that $h_\pi \circ g_{h(\pi)} \circ r_{g \circ h(\pi)v} \circ j$ is $\mathfrak{F}$-equal to a constant map.

In the special case where h is the identity map on Y, combining Proposition 4.1 and Corollary 3.5 (a) we get

4.2. Corollary. *Let $f: X \to Y$ be a trivial shape map of X into a connected space Y and let $g: Z \to Y$ be an arbitrary shape map. Then* $\mathbf{gen}_{\mathfrak{F}}(f, g) = \mathbf{cat}_{\mathfrak{F}} g$ *for each class $\mathfrak{F}$ of functors on $\mathscr{H}$ (or on $\mathscr{CH}$).*

The shape theoretic version of Corollary 2.8 that we can now prove is the following

4.3. Theorem. *Let $p: E \to B$ be an approximate fibration in the sense of Coram and Duvall [5] of locally compact metric ANR's with B connected. Let $b \in B$ and let $i: F \to E$ be the inclusion of the fiber $F = p^{-1}(b)$ into E. Then* $\mathrm{cat}\, p = \mathbf{gen}\, i$.

Proof. We know from Proposition 4.1 that $\mathbf{cat}\, p = \mathrm{cat}\, p \leqslant \mathbf{gen}\, i$. Hence, it remains to prove that $\mathbf{gen}\, i \leqslant \mathrm{cat}\, p$. It follows from (2.6) in [3] and Corollary 3.5 (after crossing everything with the Hilbert cube and using R. D. Edwards' theorem [4]) that both E and B can be assumed to be compact Q-manifolds. This implies that F can be represented as the intersection of a decreasing sequence $F_1 \supset F_2 \supset \dots$ of compact Q-manifold neighborhoods in E.

Let V be an open set in E such that the restriction $p|V$ is null-homotopic. We claim that for each index $k > 0$ there is a map $\varphi_k: V \to F_k$ such that $i_1 \circ f_k^1 \circ \varphi_k \simeq j$, where $j: V \to E$, $i_n: F_n \to E$, and $f_m^n: F_m \to F_n$ are inclusions for all $m \geqslant n > 0$.

Indeed, choose $\varepsilon_k > 0$ such that the pre-image under p of the ε_k-neighborhood of b is contained in F_k and let the homotopy $H: V \times I \to B$ join $p \circ j$ with the constant map of V into the point b. Let $\tilde{H}: V \times I \to E$ be the ε_k-lifting of H (see [5]). Then $p \circ \tilde{H}_1(V)$ is contained in the ε_k-neighborhood of b. Hence $\tilde{H}_1(V) \subset F_k$. Clearly, $\varphi_k = \tilde{H}_1$ has the required property.

We shall close with an extension of the first half of Theorem 2.9 in [1].

The shape theoretic form of that result reads: Let $f: X \to Y$ be a shape map of connected topological spaces. Let $n \geq 1$ and suppose X is shape n-connected and the map

$$\text{pro-}f_q: \text{pro-}\pi_q(X) \to \text{pro-}\pi_q(Y)$$

is an isomorphism for $q > n$. Then $\mathbf{gen} f = \mathbf{cat}_n Y$.

However, this statement is not true as the following example shows:

4.4. Example. Let $f: Q \to Y$ be a CE map of the Hilbert cube onto a non-movable connected space Y (see [8]). If the above statement were true, by Corollary 4.2 we would have $\mathbf{cat}\, Y = \mathbf{cat}_n Y$ for every $n > 0$. But $\mathbf{cat}\, Y = 1$ and $\mathbf{cat}\, Y \geq 2$ (since Y has a non-trivial shape).

In order to get a valid theorem generalizing Theorem 2.9 in [1] we make an additional assumption about the shape map f. More precisely, f will be assumed to be an n-conditioned shape map as defined below. Of course, we begin with maps of inverse systems.

4.5. Definition. A map of inverse systems $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is *n-conditioned* provided that for each μ in the index set M of $\boldsymbol{Y}$ and each $\lambda \geq f(\mu)$ in the index set Λ of $\boldsymbol{X}$ there is $\mu' \geq \mu$ such that there are $\mu'' \geq \mu'$ and $\lambda' \geq \lambda, f(\mu')$ such that if maps $\varphi: L \to Y_{\mu''}$ and $\bar{\varphi}: L^n \to X_{\lambda'}$ of an arbitrary CW-complex L and its n-skeleton L^n, respectively, satisfy

$$q_{\mu'\mu''} \circ \varphi | L^n \simeq f_{\mu'} \circ p_{f(\mu')\lambda'} \circ \bar{\varphi},$$

then there is a map

$$\Phi: L \to X_\lambda \quad \text{with } q_{\mu\mu''} \circ \varphi \simeq f_\mu \circ p_{f(\mu)\lambda} \circ \Phi.$$

4.6. Lemma. *Let $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ be two equivalent (homotopic) maps of inverse systems. If $\boldsymbol{f}$ is n-conditioned, then so is $\boldsymbol{g}$.*

Proof. Let μ and $\nu \geq g(\mu)$ be given. Choose $\lambda \geq \nu, f(\mu)$ such that $f_\mu \circ p_{f(\mu)\lambda} \simeq g_\mu \circ p_{g(\mu)\lambda}$. Now pick $\mu' \geq \mu$ (with respect to μ and λ) using the fact that $\boldsymbol{f}$ is an n-conditioned map of inverse systems. Next, take $\mu'' \geq \mu'$ and $\lambda' \geq \lambda, f(\mu')$ (again with respect to $\boldsymbol{f}$). Let $\nu' \geq \lambda', g(\mu')$ be such that

$$f_{\mu'} \circ p_{f(\mu')\nu} \simeq g_{\mu'} \circ p_{g(\mu')\nu}.$$

Consider maps $\varphi: L \to Y_{\mu''}$ (L is an arbitrary CW-complex) and $\bar{\varphi}: L^n \to X_{\nu'}$ satisfying

$$q_{\mu'\mu''} \circ \varphi | L^n \simeq g_{\mu'} \circ p_{g(\mu')\nu'} \circ \bar{\varphi}.$$

The choice of ν' gives

$$q_{\mu'\mu''} \circ \varphi | L^n \simeq f_{\mu'} \circ p_{f(\mu')\nu'} \circ \bar{\varphi}.$$

The property of $\boldsymbol{f}$ implies that there is a map $\Phi^*: L \to X_\lambda$ such that

$$q_{\mu\mu''} \circ \varphi \simeq f_\mu \circ p_{f(\mu)\lambda} \circ \Phi^*.$$

The way λ was chosen gives us $q_{\mu\mu''}\circ\varphi \simeq g_\mu \circ p_{g(\mu)\lambda}\circ\Phi^*$. Hence, for a map $\Phi = p_{\nu\lambda}\circ\Phi^*$, we have

$$q_{\mu\mu''}\circ\varphi \simeq g_\mu \circ p_{g(\mu)\nu}\circ\Phi.$$

4.7. LEMMA. *Let $f\colon X \to Y$ be an n-conditioned map of inverse systems and let $h\colon Y \to Z$ be an equivalence. Then the composition $h\circ f$ is also n-conditioned.*

Proof. The proof is notationally quite complicated because we must take a careful control of many indices.

Let $k\colon Z \to Y$ be a map of inverse systems such that $k\circ h \simeq \mathrm{id}_Y$ and $h\circ k \simeq \mathrm{id}_Z$.

Let $\mu \in N$ (the index set of Z) and let $\lambda \geqslant f\circ h(\mu)$ be given in Λ. Put $\varrho = h(\mu)$ and $\pi = k\circ h(\mu)$. Choose $\alpha \geqslant \mu$, π such that

$$h_\mu \circ k_\varrho \circ r_{\pi\alpha} \simeq r_{\mu\alpha}. \tag{1}$$

In the next step take $\varrho' \geqslant \varrho$ with respect to λ using the fact that f is n-conditioned. Put $\beta = k(\varrho')$, $\gamma = h\circ k(\varrho')$, and $\tau = f(\varrho')$. Let $\mu' \geqslant \alpha, \beta$. Observe that $\mu' \geqslant \mu$ since $\alpha \geqslant \mu$. Put $\delta = h(\mu')$ and $\varepsilon = f\circ h(\mu')$. Let $\zeta \geqslant \delta, \gamma$ be such that

$$h_\beta \circ q_{\gamma\zeta} \simeq r_{\beta\mu'}\circ h_{\mu'}\circ q_{\delta\zeta}. \tag{2}$$

Let $\nu \geqslant \varrho'$ and $\eta \geqslant \lambda, \tau$ be chosen by using again the fact that f is n-conditioned. Let $\varrho'' \geqslant \nu, \zeta$ satisfy

$$k_{\varrho'}\circ h_\beta \circ q_{\gamma\varrho''} \simeq q_{\varrho'\varrho''}. \tag{3}$$

Combining (2) and (3) we have

$$\begin{aligned} k_{\varrho'}\circ r_{\beta\mu'}\circ h_{\mu'}\circ q_{\delta\varrho''} &\simeq k_{\varrho'}\circ r_{\beta\mu'}\circ h_{\mu'}\circ q_{\delta\zeta}\circ q_{\zeta\varrho''} \\ &\simeq k_{\varrho'}\circ h_\beta \circ q_{\gamma\zeta}\circ q_{\zeta\varrho''} \simeq q_{\varrho'\varrho''}. \end{aligned} \tag{4}$$

Put $\xi = f(\varrho'')$ and $i = k(\varrho'')$. Next, take $\lambda' \geqslant \xi, \eta$ such that

$$f_{\varrho'}\circ p_{\tau\lambda'} \simeq q_{\varrho'\varrho''}\circ f_{\varrho''}\circ p_{\xi\lambda'} \tag{5}$$

and

$$f_\delta \circ p_{\varepsilon\lambda'} \simeq q_{\delta\varrho''}\circ f_{\varrho''}\circ p_{\xi\lambda'}. \tag{6}$$

Finally, take $\mu'' \geqslant i$ such that

$$q_{\varrho'\varrho''}\circ k_{\varrho''}\circ r_{i\mu''} \simeq k_{\varrho'}\circ r_{\beta\mu''} \tag{7}$$

and

$$q_{\varrho\varrho''}\circ k_{\varrho''}\circ r_{i\mu''} \simeq k_\varrho \circ r_{\pi\mu''}. \tag{8}$$

Let L be an arbitrary CW-complex and assume that maps $\varphi: L \to Z_{\mu''}$ and $\bar{\varphi}: L^n \to X_{\lambda'}$ satisfy

(9) $$r_{\mu'\mu''} \circ \varphi | L^n \simeq h_{\mu'} \circ f_\delta \circ p_{\varepsilon\lambda'} \circ \bar{\varphi}.$$

Consider maps $\psi = q_{\nu\varrho''} \circ k_{\varrho''} \circ r_{i\mu''} \circ \varphi$ and $\bar{\psi} = p_{\eta\lambda'} \circ \bar{\varphi}$. We claim that

(10) $$q_{\varrho'\nu} \circ \psi | L^n \simeq f_{\varrho'} \circ p_{\tau\eta} \circ \bar{\psi}.$$

Indeed, we have

$$\begin{aligned} q_{\varrho'\nu} \circ \psi | L^n &= q_{\varrho'\nu} \circ q_{\nu\varrho''} \circ k_{\varrho''} \circ r_{i\mu''} \circ \varphi | L^n \\ &= q_{\varrho'\varrho''} \circ k_{\varrho''} \circ r_{i\mu''} \circ \varphi | L^n \simeq k_{\varrho'} \circ r_{\beta\mu''} \circ \varphi | L^n \quad \text{(by (7))} \\ &\simeq k_{\varrho'} \circ r_{\beta\mu'} \circ h_{\mu'} \circ f_\delta \circ p_{\varepsilon\lambda'} \circ \bar{\varphi} \quad \text{(by (9))} \\ &\simeq k_{\varrho'} \circ r_{\beta\mu'} \circ h_{\mu'} \circ q_{\delta\varrho''} \circ f_{\varrho''} \circ p_{\xi\lambda'} \circ \bar{\varphi} \quad \text{(by (6))} \\ &\simeq q_{\varrho'\varrho''} \circ f_{\varrho''} \circ p_{\xi\lambda'} \circ \bar{\varphi} \quad \text{(by (4))} \\ &\simeq f_{\varrho'} \circ p_{\tau\lambda'} \circ \bar{\varphi} \quad \text{(by (5))} \\ &= f_{\varrho'} \circ p_{\tau\eta} \circ \bar{\psi}. \end{aligned}$$

By the choice of ϱ' and η we conclude that there is a map $\Phi: L \to X_\lambda$ with $f_\varrho \circ p_{f(\varrho)\lambda} \circ \Phi \simeq q_{\varrho\nu} \circ \psi$. Composing with h_μ we have

$$\begin{aligned} h_\mu \circ f_\varrho \circ p_{f(\varrho)\lambda} \circ \Phi &\simeq h_\mu \circ q_{\varrho\nu} \circ q_{\nu\varrho''} \circ k_{\varrho''} \circ r_{i\mu''} \circ \varphi \\ &\simeq h_\mu \circ k_\varrho \circ r_{\pi\mu''} \circ \varphi \quad \text{(by (8))} \\ &\simeq h_\mu \circ k_\varrho \circ r_{\pi\alpha} \circ r_{\alpha\mu''} \circ \varphi \simeq r_{\mu\alpha} \circ r_{\alpha\mu''} \circ \varphi \quad \text{(by (1))} \\ &= r_{\mu\mu''} \circ \varphi. \end{aligned}$$

Hence $\boldsymbol{h} \circ \boldsymbol{f}$ is n-conditioned and our lemma is proved.

4.8. LEMMA. *Let* $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ *and* $\boldsymbol{h}: \boldsymbol{Z} \to \boldsymbol{X}$ *be maps of inverse systems. If* $\boldsymbol{h}$ *is an equivalence and* $\boldsymbol{f}$ *is* n*-conditioned, then the composition* $\boldsymbol{f} \circ \boldsymbol{h}$ *is also* n*-conditioned.*

Proof. Let $\boldsymbol{k}: \boldsymbol{X} \to \boldsymbol{Z}$ be a map of inverse systems such that $\boldsymbol{k} \circ \boldsymbol{h} \simeq \mathbf{id}_Z$ and $\boldsymbol{h} \circ \boldsymbol{k} \simeq \mathbf{id}_X$.

Let $\mu \in M$ (the index set of $\boldsymbol{Y}$). Put $\alpha = f(\mu) \in \Lambda$ (the index set of $\boldsymbol{X}$), $\beta = h \circ f(\mu) \in N$ (the index set of $\boldsymbol{Z}$), and $\gamma = k \circ h \circ f(\mu) \in \Lambda$. Let a $\lambda \geqslant \beta$ in N be also given. Put $\delta = k(\lambda)$.

First pick $\nu \geqslant \alpha, \delta$ such that

(i) $h_\alpha \circ k_\beta \circ p_{\gamma\nu} \simeq p_{\alpha\nu}$,

(ii) $k_\beta \circ p_{\gamma\nu} \simeq r_{\beta\lambda} \circ k_\lambda \circ p_{\delta\nu}$.

Next, using the fact that $\boldsymbol{f}$ is n-conditioned, pick (with respect to indices μ and ν) $\mu' \geqslant \mu$, $\mu'' \geqslant \mu'$, and $\nu' \geqslant \nu, f(\mu')$. Put $\varepsilon = f(\mu')$, $\eta = h \circ f(\mu')$, and $\zeta = h(\nu')$. Finally, let $\lambda' \geqslant \lambda, \zeta$ satisfy

(iii) $p_{\varepsilon\nu'} \circ h_{\nu'} \circ r_{\zeta\lambda'} \simeq h_\varepsilon \circ r_{\eta\lambda'}$.

Consider now an arbitrary CW-complex L and maps $\varphi\colon L \to Y_{\mu''}$ and $\bar{\varphi}\colon L^n \to Z_{\lambda'}$ such that

$$q_{\mu'\mu''}\circ\varphi | L^n \simeq f_{\mu'}\circ h_\varepsilon \circ r_{\eta\lambda'}\circ\bar{\varphi}.$$

From (iii) we get

$$q_{\mu'\mu''}\circ\varphi | L^n \simeq f_{\mu'}\circ p_{\varepsilon\nu'}\circ h_{\nu'}\circ r_{\zeta\lambda'}\circ\bar{\varphi}.$$

The choice of ν', μ', and μ'' implies that there is a map $\Psi\colon L \to X_\nu$ satisfying

(iv) $q_{\mu\mu''}\circ\varphi \simeq f_\mu \circ p_{\alpha\nu}\circ\Psi$.

Define a new map $\Phi\colon L \to Z_\lambda$ by $\Phi = k_\lambda \circ p_{\delta\nu}\circ\Psi$. Then

$$\begin{aligned} f_\mu\circ h_\alpha\circ r_{\beta\lambda}\circ\Phi &= f_\mu\circ h_\alpha\circ r_{\beta\lambda}\circ k_\lambda\circ p_{\delta\nu}\circ\Psi \\ &\simeq f_\mu\circ h_\alpha\circ k_\beta\circ p_{\gamma\nu}\circ\Psi \quad \text{(by (ii))} \\ &\simeq f_\mu\circ p_{\alpha\nu}\circ\Psi \quad \text{(by (i))} \\ &\simeq q_{\mu\mu''}\circ\varphi \quad \text{(by (iv))}. \end{aligned}$$

It follows from Lemmas 4.6-4.8 that we can define the notion of an n-conditioned shape map $f\colon X \to Y$ of arbitrary topological spaces (see Section 3). The reason for introducing this class of shape maps is explained immediately before Definition 4.5.

4.9. THEOREM. *If $f\colon X \to Y$ is an n-conditioned ($n \geq 1$) shape map of connected topological spaces with X shape n-connected, then* $\mathbf{gen}\, f = \mathbf{cat}_n\, Y$ *(i.e., the shape n-category of Y [3]).*

Proof. By Lemmas 4.6-4.8 and Ungar's theorem [15] we can assume that f is represented by a simple map $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ of inverse ANR-systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that all X_λ's are n-connected and all Y_λ's are connected.

$\mathbf{cat}_n\, Y \leq \mathbf{gen}\, f$. Given an index μ choose $\nu \geq \mu$ such that

$$\text{gen}\,(f_\mu\circ p_{\mu\lambda}, q_{\mu\nu}) \leq k = \mathbf{gen}\, f \quad \text{for each } \lambda \geq \mu.$$

Let V be an open subset of Y_ν for which there is a map $g\colon V \to X_\mu$ satisfying $f_\mu\circ g \simeq q_{\mu\nu}\circ j$, where $j\colon V \to Y_\nu$ is the inclusion. Let L be an at most n-dimensional CW-complex and consider any map $\psi\colon L \to V$. Since X_μ is n-connected, the map $g\circ\psi\colon L \to X_\mu$ is null-homotopic. Hence $f_\mu\circ g\circ\psi \simeq q_{\mu\nu}\circ j\circ\psi$ is also null-homotopic.

$\mathbf{gen}\, f \leq \mathbf{cat}_n\, Y$. Let μ and $\lambda \geq \mu$ be arbitrary indices. Choose $\mu' \geq \mu$, $\mu'' \geq \mu'$, and $\lambda' \geq \mu'$ using the fact that f is n-conditioned. Finally, let $\mu_0 \geq \mu''$ be such that the n-category $\text{cat}_n q_{\mu''\mu_0}$ of the map $q_{\mu''\mu_0}$ is less than or equal to $k = \mathbf{cat}_n\, Y$. Let V be an open subset of Y_{μ_0} with the property that for any map $\psi\colon K \to V$ of an at most n-dimensional CW-complex K into V the composition $q_{\mu''\mu_0}\circ j\circ\psi$ ($j\colon V \to Y_{\mu_0}$ is again the inclusion) is null-

-homotopic. Consider now any map $\alpha: L \to V$ of a CW-complex L into V. Put $\varphi = q_{\mu''\mu_0} \circ j \circ \alpha$ and let $\bar{\varphi}: L^n \to X_{\lambda'}$ be a constant map. Since

$$q_{\mu'\mu''} \circ \varphi | L^n \simeq f_{\mu'} \circ p_{\mu'\lambda'} \circ \bar{\varphi},$$

there is a map $\Phi: L \to X_\lambda$ such that $q_{\mu\mu''} \circ \varphi \simeq f_\mu \circ p_{\mu\lambda} \circ \Phi$. Hence

$$q_{\mu\mu_0} \circ j \circ \alpha \simeq f_\mu \circ p_{\mu\lambda} \circ \Phi.$$

4.10. Remark. Note that in the above proof of the first inequality we did not use the assumption that f is n-conditioned.

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