UNIFORM CONVERGENCE OF LACUNARY FOURIER SERIES

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1. A subset $I$ of the integers $Z$ is called a set of uniform convergence, or a UC set, if every Fourier series of the form

$$
\sum_{n \in I} c_n e^{i\lambda n},
$$

which represents a continuous function, converges uniformly. In [1] an example was given of a UC set which is not a Sidon set, i.e. such that there exist uniformly convergent series of form (1) which are not absolutely convergent. In this note we exhibit another example of a UC set which is not Sidon, by showing that the union of a set as in [1] and a finite number of Hadamard sets is still a UC set. To prove this fact we use convolutions with the de la Vallée Poussin kernels instead of the Riesz polynomials considered in [1]. The question of whether the union of two UC sets is again a UC set remains open in general. We do not know the answer even in the case where one of the sets is a Sidon set. For subsets of the dual of the Cantor group, some partial results are contained in [3] where general properties of UC sets are also discussed.

2. Let $n_*$ be a sequence of positive integers such that

$$\frac{n_{*+1}}{n_*} > 1 + \sqrt{3},$$

and let $E = \{n_i + n_j; i \neq j\}$. Let $E_l = \{m_{ij}^{(l)}\}_{j=1}^\infty$ be sequences of positive integers such that $m_{ij+1}^{(l)}/m_{ij}^{(l)} > q > 1$ for some $q$ and $l = 1, \ldots, M$. Finally, let

$$F = \bigcup_{l=1}^M E_l.$$

Then

**Theorem.** $E \cup F$ is a UC set which is not a Sidon set.

**Proof.** Clearly, $E \cup F$ is not a Sidon set, since it contains infinite sets which are not Sidon (see [1]). Without loss of generality we may
suppose that, for some $p \epsilon Z$, $n_{s+1}/n_s \leq p$. Let $h$ be a positive integer such that $q^h > p$. Then there are at most $hM = A$ elements of $F$ between $n_s + n_{s-1}$ and $n_{s+1} + n_s$. Indeed, let $m_j^{(0)}$ be the smallest element of $E_l$ such that $m_j^{(0)} > n_s + n_{s-1}$. Then

$$m_j^{(0)} + h > q^h m_j^{(0)} > p(n_s + n_{s-1}) \geq n_{s+1} + n_s.$$

Let now $f$ be a continuous function on the circle group $T$ such that $\hat{f}(n) = 0$ if $n \notin E \cup F$. For every positive integer $N$ let

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}.$$

The theorem will follow if we prove that for every $N$

$$\|S_N(f)\|_{\infty} \leq C\|f\|_{\infty},$$

where $C$ is a constant depending only on the set $E \cup F$. Clearly, we may suppose that $N \in E \cup F$. For every positive integer $n$, let

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right)e^{ijt}$$

be the Féjer kernel, and $V_n = 2K_{2n+1} - K_n$ the de la Vallée Poussin kernel. Then (see [2], p. 15) we get

(i) $\|V_n\|_{L^1} \leq 3$,

(ii) $\hat{V}_n(j) = 1$ if $|j| \leq n+1$, and $\hat{V}_n(j) = 0$ if $|j| > 2n+1$.

Firstly, let $N = n_s + n_{s-1}$. Then, according to (ii),

$$S_N(f) = V_{N-1} \ast f - \sum \hat{V}_{N-1}(j)\hat{f}(j)e^{ijt},$$

where the summation is over all $j \in F$ such that

$$n_s + n_{s-1} < j \leq 2n_s + 2n_{s-1}.$$

Remark that $2n_s + 2n_{s-1} < n_{s+1} + n_1$, since $n_{s+1}/n_s > 1 + \sqrt{3}$.

Since, by definition, $|\hat{V}_n(j)| \leq 1$, and the summation on the right-hand side of (3) contains at most $A$ terms, by (i) we get

$$\|S_N(f)\|_{\infty} \leq (3 + A)\|f\|_{\infty}.$$

Let $N = k \epsilon F$ with $n_{s+1} + n_1 > k > n_s + n_{s-1}$. Then

$$S_N(f) = S_{n_s + n_{s-1}}(f) + \sum f(j)e^{ijt},$$

where the summation is extended to all $j \in F$ such that $n_s + n_{s-1} < j \leq k$. By (4) we have

$$\|S_N(f)\|_{\infty} \leq (3 + 2A)\|f\|_{\infty}.$$
Let now $N = n_{s+1} + n_r$ with $1 \leq r \leq s - 1$. Since, by (ii),
\[
\exp(in_{s+1}t)V_{n_{s+1}}*f = \sum_{|j-n_{s+1}| \leq 2n_r-1} \hat{V}_{n_r-1}(j-n_{s+1})\hat{f}(j)\exp(ijt)
\]
and $\hat{V}_{n_r-1}(j-n_{s+1}) = 1$ for $|j-n_{s+1}| \leq n_r$, the following identity holds true:

(6) $S_N(f) = S_{n_s+n_{s-1}}(f) + \exp(in_{s+1}t)V_{n_r-1}*f - \sum_{2n_r-1>|j-n_{s+1}|>n_r} \hat{V}_{n_r-1}(j-n_{s+1})\hat{f}(j)\exp(ijt) + \sum_{n_s+n_{s-1}+n_r<n_{s+1}} \hat{f}(j)\exp(ijt)$.

Remark that $\hat{f}(j) \neq 0$ in the summations on the right-hand side of (6) only if $j \in F$. Hence each sum contains at most $A$ terms so that, by (4) and (i),

(7) $\|S_N(f)\|_\infty \leq (3 + A)\|f\|_\infty + 3\|f\|_\infty + 2A\|f\|_\infty = (6 + 3A)\|f\|_\infty$.

Finally, suppose that $N = k \in F$ with $n_{s+1} + 1 < k < n_{s+1} + n_s$. Then, if $r$ is the largest integer such that $n_{s+1} + n_r < k$, we have

$S_N(f) = S_{n_{s+1}+n_r}(f) + \sum f(j)e^{ijt}$,

where the summation is over all $j \in F$ such that $n_{s+1} + n_r < j \leq k$. Hence

(8) $\|S_N(f)\|_\infty \leq (6 + 4A)\|f\|_\infty$.

Therefore, by (4), (5), (7) and (8), inequality (2) holds with $C = 6 + 4A$.

REFERENCES


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