Periodic solutions of $x'' + f(\mu, x) = 0$

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Abstract. An examination is made of those regions (called admissible sets) in the $(\mu, A)$-plane for which the initial value problem $x''(t) + f(\mu, x(t)) = 0$, $x(0) = A$, $x'(0) = 0$, has a non-trivial periodic solution. In particular, results obtained previously for the case that $f$ is linear in $\mu$ are generalized to the non-linear case.

In addition, the converse problem is discussed in some detail, whence it is shown how to construct differential equations having rather general sets in the $(\mu, A)$-plane as boundaries of their admissible sets.

1. Introduction. In [2], there was considered the problem of characterizing the set of points $(\mu, A)$ (the admissible set) for which the solution of the equation

$$(1) \quad x''(t) + g(x(t)) + \mu h(x(t)) = 0 \quad (\dot{} = d/dt)$$

with initial conditions $x(0) = A$, $x'(0) = 0$, is a non-trivial periodic function. In this paper, we extend some of these results to the equation

$$(2) \quad x''(t) + f(\mu, x(t)) = 0, \quad x(0) = A, \quad x'(0) = 0$$

and illustrate certain essential differences that occur when $f$ is non-linear in $\mu$. We shall also be interested in obtaining some results of a converse nature.

We shall use the following notation:

$$F(\mu, y) = \int_0^y f(\mu, x) \, dx,$$

$${\mathcal A} = \{(\mu, A) \in \mathbb{R}^2 : (2) \text{ admits a non-trivial periodic solution}\}.$$

Throughout, we shall assume that $f(\mu, x)$ is jointly continuous in both variables and that solutions of (2) are unique, although continuity in $\mu$ will be relaxed in one case involving a converse theorem.

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By a periodic solution, we shall always mean a non-constant periodic solution.

Among the many other works devoted to the study of periodic solutions to (2), we mention in particular those of Cesari [3] and Loud [5].

One could interpret $\mu$ as a non-linear control; for example a knowledge of the admissible set might allow a path in the $(\mu, A)$-plane to be found permitting a transfer from a non-periodic orbit to a periodic orbit.

In [2], a general criterion for the existence of a periodic solution to (2) was given under somewhat weaker hypotheses on $f$, than in [3] or [5]. We shall find useful a corollary of that result which we state without proof as

**Lemma 1.** Let $f(\mu, A) < 0 \ (f(\mu, A) > 0)$. Then a necessary and sufficient condition for the solution of (2) to be periodic is that there exist $B > A \ (B < A)$ such that $F(\mu, A) = F(\mu, B) > F(\mu, y)$ for $A < y < B \ (B < y < A)$.

A corresponding result has been obtained by one of the authors [1] for certain equations of the form $x'' + g(x)h(x^2) + f(\mu, x) = 0$.

2. **Admissible regions of the $(\mu, A)$-plane.** If $(\mu, A) \in \mathcal{A}$, we shall say that $(\mu, A)$ is admissible for (2), and $\mathcal{A}$ will be called the admissible set for $f$.

The following result generalizes the case that $f$ is linear in $\mu$, which was given in [2].

**Theorem 1.** $\mathcal{A}$ is open.

**Proof.** Let $(\mu_0, A_0) \in \mathcal{A}$. By Lemma 1, there exists $B_0$ such that

$$F(\mu_0, A_0) = F(\mu_0, B_0) > F(\mu_0, y),$$

for $y$ between $A_0$ and $B_0$, and we may, without loss of generality, assume that $A_0 < B_0$, so that $f(\mu_0, B_0) < 0 < f(\mu_0, A_0)$. By continuity, there exists $\delta_0 > 0 \exists f(\mu, B) < -\delta_0 < 0 < \delta_0 < f(\mu, A)$ whenever $(\mu, A)$ and $(\mu, B)$ are sufficiently close to $(\mu_0, A_0)$, $(\mu_0, B_0)$, respectively. Further, by the implicit function theorem and the fact that $f(\mu_0, B_0) \neq 0$, it follows that

$$F(\mu, A) = F(\mu, y)$$

has a unique solution $x = B = B(\mu, A)$ for $(\mu, A)$ sufficiently close to $(\mu_0, A_0)$ such that $B(\mu, A) \rightarrow B_0$ as $(\mu, A) \rightarrow (\mu_0, A_0)$ and $F(\mu, A) = F(\mu, B) > F(\mu, y)$ whenever $B < y < A$, and the proof of the theorem is complete.

3. **Description of the boundary of $\mathcal{A}$.** As in [2] we introduce the following classification of the boundary points of $\mathcal{A}$. Let $(\mu_0, A_0) \in \partial \mathcal{A}$. We shall say that $(\mu_0, A_0)$ is of type I if $f(\mu_0, A_0) = 0$, is of type II if it
is in the closure of the set \( \{ (\mu, A) \in \partial \mathcal{A} : \text{there exist } B \neq A \text{ such that } F(\mu, B) = F(\mu, A) \text{ and } f(\mu, B) = 0, \text{ and } (\mu, A) \text{ is not of type I} \} \), and is of type III if it is in the closure of the set \( \{ (\mu, A) \in \partial \mathcal{A} : F(\mu, A) > F(\mu, y) \}, \) either for all \( y > A \) or for \( y < A \), and \( (\mu, A) \) is not of type I or II.

That this gives a complete classification of the boundary points of \( \mathcal{A} \) is indicated by

**Theorem 2.** Let \( (\mu_0, A_0) \in \partial \mathcal{A} \). Then \( (\mu_0, A_0) \) is one of the types I, II or III.

**Proof.** This follows along the lines of the proof for the linear case. See [2].

In [2], suitable hypotheses were given to ensure that boundary points exclusively of one of the types I, II or III were interior to a continuous arc of such points. Again, for \( f \) non-linear in \( \mu \), analogous results hold true, the proofs requiring merely a straightforward adaptation of those given in [2] and we obtain

**Theorem 3.** In parts (a), (b) and (c), assume that \( (\mu_0, A_0) \) is a boundary point of \( \mathcal{A} \), exclusively of types I, II and III, respectively.

(a) Suppose that \( f_\mu \) exists as a continuous, non-vanishing function of \( (\mu, A) \) in some neighbourhood of \( (\mu_0, A_0) \). Then \( (\mu_0, A_0) \) is relatively interior to a continuous arc of boundary points exclusively of type I.

(b) Suppose that \( f_\mu \) and \( f_A \) exist as continuous, non-vanishing functions in some neighborhood of \( (\mu_0, B_0) \) (see definition of type II for meaning of \( B_0 \)). Assume, in addition, that

\[
\int_{A_0}^{B_0} \frac{f_\mu(\mu_0, y)}{y} \, dy \neq 0.
\]

Then \( (\mu_0, A_0) \) is relatively interior to a continuous arc of boundary points exclusively of type II.

(c) Define \( G(\mu) \) to be \( \limsup_{y \to \infty} F(\mu, y) \). Suppose that \( G_\mu \) and \( F_\mu \) exist as continuous functions in neighbourhoods of \( \mu_0 \) and \( (\mu_0, A_0) \), respectively, with \( G_\mu(\mu_0) \neq F_\mu(\mu_0, A_0) \). In addition, let \( F(\mu_0, A_0) > F(\mu_0, x) \) for all \( x > A_0 \). Then \( (\mu_0, A_0) \) is relatively interior to a continuous arc of boundary points exclusively of type III.

For certain cases of equation (2), the boundary curves may be explicitly parameterized. For example, for the equation

\[ x'' + x + \mu x^2 = 0, \quad x(0) = A, \quad x'(0) = 0, \]

the boundary curves of type II are the branches of the hyperbola \( \mu A = \frac{1}{2} \) [6].
In [2], this monotone behaviour of type II curves was shown to hold for more general equations of the form (2) with \( f \) linear in \( \mu \). Here we shall further extend this result for \( f \) non-linear in \( \mu \). The proof is based on that given for the linear case. We shall sketch the proof, giving the necessary modifications.

**Theorem 4.** Assume that \( f(\mu, x) \) is continuously differentiable with respect to \( \mu \) for each \( x \), with \( f_\mu(\mu, x) > 0 \) for all \( (\mu, x) \) and that there exists \( \overline{\mu}, \overline{A} \) such that \( (x - \overline{A})f(\overline{\mu}, x) > 0 \) for \( x \neq \overline{A} \). Let \( (\mu_0, A_0) \) be a boundary point of type II but not of type III, with \( \mu_0 > \overline{\mu}, A_0 > \overline{A} \). Then there exists a continuous strictly decreasing arc

\[
\Gamma = \{ (\mu, A(\mu)) : \mu_0 \leq \mu < \mu^* \}
\]

of such points, with \( \lim_{\mu \to \mu^*} A(\mu) = 0 \) if the maximal interval \([\mu_0, \mu^*]\) of definition of the arc is finite.

**Proof.** Since \( (\mu_0, A_0) \) is of type II, but not of type III, there exists \( B_0 < \overline{A} \) such that \( F(\mu_0, B_0) = F(\mu_0, A_0) > F(\mu_0, y) \) for \( B_0 < y < A_0 \), and \( f(\mu_0, B_0) = 0 \). The conditions of the theorem imply that \( f(\mu, B_0) > 0 \), for \( \mu > \mu_0 \), and so, provided there exists \( B \) with \( B_0 < B < \overline{A} \) such that \( F(\mu, B) > F(\mu, \overline{A}) \), we may define \( B(\mu) \) (for \( \mu_0 \leq \mu < \mu^* \), say) to be

\[
\sup \{ y : B_0 < y < \overline{A} \text{ and } F(\mu, y) = \sup_{B_0 < y < \overline{A}} F(\mu, B) \}.
\]

We have \( B_0 < B(\mu) < 0 \) and \( f(\mu, B(\mu)) = 0 \). \( B(\mu) \) is equal to \( B_0 \). \( B(\mu) \) is non-decreasing; for let \( \mu_0 \leq \mu_1 < \mu_2 < \mu^* \) and let \( B(\mu_i) = B_i \). Suppose that \( B_1 > B_2 \). Then \( F(\mu_2, B_2) > F(\mu_2, B_1) \) (definition of \( B_2 = B(\mu_2) \)), which inequality may be written

\[
\int_{B_1}^{B_2} f(0, x)dx > \int_{B_1}^{B_1} (f(\mu_2, x) - f(0, x))dx.
\]

However, using the condition on \( f_\mu \),

\[
f(\mu_1, B) < f(\mu_2, B) = 0
\]

and so

\[
F(\mu_1, B_2) < \sup_{B_0 < y < \overline{A}} F(\mu_1, y) = F(\mu_1, B_1).
\]

Therefore

\[
\int_{B_1}^{B_2} f(0, x)dx
\]

\[
< \int_{B_1}^{B_1} (f(\mu_1, x) - f(0, x))dx < \int_{B_2}^{B_1} (f(\mu_2, x) - f(0, x))dx
\]

again using the condition on \( f_\mu \). This contradicts (3). Thus \( B_1 \leq B_2 \) and \( B(\mu) \) is non-decreasing.
The next step is to show that $F(\mu, B(\mu))$ is continuous as a function of $\mu$. This is a straightforward copy of the argument given in the linear case and we omit the details. That the equation $F(\mu, A) = F(\mu, B(\mu))$ has a unique solution also follows straightforwardly using the argument for the linear case, as does the continuity of $A = A(\mu)$.

We then have for $\mu_0 \leq \mu_1 < \mu_2 < \mu^*$, $\int_{\mu_1}^{\mu_2} f(\mu_2, y) \, dy = \int_{\mu_1}^{\mu_2} f(\mu_1, y) \, dy$ (condition on $f_\mu \leq \int f(\mu_1, y) \, dy$ (definition of $B_1$) = $\int f(\mu_1, y) \, dy$). The conditions of the theorem imply that $\int f(\mu, y) \, dy$ is increasing in both $\mu$ and $x$ whenever $x \geq \bar{\mu}$, and so $A_2 < A_1$. Thus $A(\mu)$ is strictly decreasing in $(\mu_0, \mu^*)$. Finally, the behavior of $A(\mu)$ as $\mu \to \mu^*$ may be verified just as in the proof given for the linear case.

As an analogue to Corollary 2 of [2], we have

**Corollary.** The energy function $F(\mu, A)$ is (strictly) decreasing along the arcs of type II defined in the above theorem.

**4. Isolated boundary points for $f(\mu, x)$ linear in $\mu$.** The question occurs whether or not there can exist isolated boundary points of $\mathcal{A}$. We shall see in the next several sections that there can in general, but we show here that in the case $f(\mu, x)$ is linear in $\mu$ isolated boundary points of $\mathcal{A}$ are impossible.

**Theorem 5.** Let $f$ be continuous in $x$ and linear in $\mu$. Then there are no isolated boundary points of $\mathcal{A}$.  

**Proof.** Let $(\mu_0, A_0) \in \partial \mathcal{A}$. Write $f(\mu, x) = g(x) + \mu h(x)$. If $f(\mu_0, A_0) = 0$, either the line $A = A_0$ (in the case $h(A_0) = 0$) or the arc $\mu = -g(A)/h(A)$, $A$ in a neighbourhood of $A_0$ (in the case $h(A_0) \neq 0$) is a curve of points in the complement of $\mathcal{A}$. Clearly, then $(\mu_0, A_0)$ cannot be an isolated point of $\partial \mathcal{A}$.

Henceforth, we may assume that $f(\mu_0, A_0) \neq 0$; without loss of generality, we shall suppose that $f(\mu_0, A_0) > 0$. Consider the set

$$ S = \{ y < A_0 : F(\mu_0, y) = F(\mu_0, A_0) \}. $$

(i) If $S$ is empty, then $F(\mu_0, A_0) > F(\mu_0, y)$ for all $y < A_0$ and, therefore,

$$ F(\mu_0, A) > F(\mu_0, y) $$

for all $y < A$, for values of $A$ in some sufficiently small right neighbourhood of $A_0$. For such values of $A$, it follows that $(\mu_0, A)$ is in the complement of $\mathcal{A}$ and again $(\mu_0, A_0)$ cannot be an isolated point of $\partial \mathcal{A}$.

(ii) If $S$ is not empty, let $B_0 = \sup S$. Then $B_0 < A_0$ and

$$ F(\mu_0, B_0) = F(\mu_0, A_0) > F(\mu_0, y) $$

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for \( B_0 < y < A_0 \). Since \((\mu_0, B_0) \notin \mathcal{A}\), we conclude that \( f(\mu_0, B_0) = 0 \) and arguing as at the beginning of this proof, there exists a non-trivial arc \( \mathcal{C} \) which contains \((\mu_0, B_0)\) such that \( f(\mu, B) = 0 \) for \((\mu, B) \in \mathcal{C}\). If the perpendicular projection of \( \mathcal{C} \) on to the \( \mu \)-axis is not the singleton \( \{\mu_0\} \), then using continuity considerations and the fact that \( f(\mu, A) \) is bounded away from zero in a suitable neighbourhood of \((\mu_0, A_0)\), we deduce that there is a non-trivial arc \( \mathcal{C}' \) through \((\mu_0, A_0)\) such that for each \((\mu, A) \in \mathcal{C}'\), there exists \((\mu, B) \in \mathcal{C} \) (same \( \mu \)) such that \( F(\mu, B) = F(\mu, A) \). Choosing \( y = B^* = B^*(\mu) \), as large as possible in the interval \( I = [B, \frac{1}{2} (A_0 + B)]\), to maximize \( F(\mu, y), y \in I \), we see that \( B^* \to B_0 \) as \((\mu, A) \to (\mu_0, A_0)\) along the arc \( \mathcal{C}' \), and \( f(\mu, B^*) = 0 \). Since \( F(\mu, B^*) \to F(\mu_0, B_0) = F(\mu_0, A_0) \) as \((\mu, A) \to (\mu_0, A_0)\) along \( \mathcal{C}' \), we may find an arc \( \mathcal{C}'' \) through \((\mu_0, A_0)\) with \( F(\mu, B^*) = F(\mu, A) \), for all \( \mu \) with \((\mu, A) \in \mathcal{C}''\). (Here we are using the fact that \( f(\mu_0, A_0) > 0 \) to assert the existence of \( \mathcal{C}'' \).)

The behaviour of \( f(\mu, A) \) near \((\mu_0, A_0)\) indicates that for \((\mu, A) \in \mathcal{C}''\) sufficiently close to \((\mu_0, A_0)\), we have

\[
F(\mu, B^*) = F(\mu, A) > F(\mu, y)
\]

for \( B^* < y < A \). Since \( f(\mu, B^*) = 0 \), it follows that these points of \( \mathcal{C}'' \) are in the complement of \( \mathcal{A} \), and \((\mu_0, A_0)\) is not an isolated point of \( \partial \mathcal{A} \).

Finally, we must deal with the case that \( \mathcal{C} \) projects on to \((\mu_0)\) on the \( \mu \)-axis. Then \( \mu_0 = -g(B)/h(B) \) for \( B \) in some neighbourhood of \( B_0 \). For such \( B \) and either for all \( \mu > \mu_0 \) or for all \( \mu < \mu_0 \), we have \( f(\mu, B) = g(B) + + \mu h(B) > 0 \). For such values of \( \mu \), choose \( y = B^* \) as above and construct \( \mathcal{C}'' \) as before.

This completes the proof of the theorem.

In the remaining sections of this paper we examine the conditions under which an equation of the form (2) can be constructed having a given set imbedded in the boundary set of \( \mathcal{A} \).

5. Converse theorems for type I boundary points. We are interested here in discussing the problem of when a given set in \( R^2 \) can be considered the type I boundary of the admissible set of some equation of the form (2). We first consider sets which do not contain any vertical segments, that is line segments parallel to the \( A \)-axis.

**Theorem 6.** Let \( \Gamma \) be a closed set in \( R^2 \) with empty interior and no vertical line segments. Then there is a function \( f(\mu, x) \), locally Lipschitzian in \( \mu \) and in \( x \) such that the boundary of \( \mathcal{A} \) is \( Z = \Gamma \cup (\mu \text{-axis}) \).

**Proof.** Define \( \varphi(\mu, x) \) to be \( \varphi(\mu, x, \Gamma) \), where \( \varphi \) is the Euclidean distance function in \( R^2 \). Then \( \varphi(\mu, x) \) is Lipschitzian in \( R^2 \) and its zero set is \( \Gamma \), whereas \( \varphi(\mu, x) > 0 \) for \((\mu, x) \notin \Gamma \).

Let

\[
\Phi(\mu, x) = \int_0^x \varphi(\mu, y) \, dy,
\]

for \( B_0 < y < A_0 \). Since \((\mu_0, B_0) \notin \mathcal{A}\), we conclude that \( f(\mu_0, B_0) = 0 \) and arguing as at the beginning of this proof, there exists a non-trivial arc \( \mathcal{C} \) which contains \((\mu_0, B_0)\) such that \( f(\mu, B) = 0 \) for \((\mu, B) \in \mathcal{C}\). If the perpendicular projection of \( \mathcal{C} \) on to the \( \mu \)-axis is not the singleton \( \{\mu_0\} \), then using continuity considerations and the fact that \( f(\mu, A) \) is bounded away from zero in a suitable neighbourhood of \((\mu_0, A_0)\), we deduce that there is a non-trivial arc \( \mathcal{C}' \) through \((\mu_0, A_0)\) such that for each \((\mu, A) \in \mathcal{C}'\), there exists \((\mu, B) \in \mathcal{C} \) (same \( \mu \)) such that \( F(\mu, B) = F(\mu, A) \). Choosing \( y = B^* = B^*(\mu) \), as large as possible in the interval \( I = [B, \frac{1}{2} (A_0 + B)]\), to maximize \( F(\mu, y), y \in I \), we see that \( B^* \to B_0 \) as \((\mu, A) \to (\mu_0, A_0)\) along the arc \( \mathcal{C}' \), and \( f(\mu, B^*) = 0 \). Since \( F(\mu, B^*) \to F(\mu_0, B_0) = F(\mu_0, A_0) \) as \((\mu, A) \to (\mu_0, A_0)\) along \( \mathcal{C}' \), we may find an arc \( \mathcal{C}'' \) through \((\mu_0, A_0)\) with \( F(\mu, B^*) = F(\mu, A) \), for all \( \mu \) with \((\mu, A) \in \mathcal{C}''\). (Here we are using the fact that \( f(\mu_0, A_0) > 0 \) to assert the existence of \( \mathcal{C}'' \).)

The behaviour of \( f(\mu, A) \) near \((\mu_0, A_0)\) indicates that for \((\mu, A) \in \mathcal{C}''\) sufficiently close to \((\mu_0, A_0)\), we have

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F(\mu, B^*) = F(\mu, A) > F(\mu, y)
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for \( B^* < y < A \). Since \( f(\mu, B^*) = 0 \), it follows that these points of \( \mathcal{C}'' \) are in the complement of \( \mathcal{A} \), and \((\mu_0, A_0)\) is not an isolated point of \( \partial \mathcal{A} \).

Finally, we must deal with the case that \( \mathcal{C} \) projects on to \((\mu_0)\) on the \( \mu \)-axis. Then \( \mu_0 = -g(B)/h(B) \) for \( B \) in some neighbourhood of \( B_0 \). For such \( B \) and either for all \( \mu > \mu_0 \) or for all \( \mu < \mu_0 \), we have \( f(\mu, B) = g(B) + + \mu h(B) > 0 \). For such values of \( \mu \), choose \( y = B^* \) as above and construct \( \mathcal{C}'' \) as before.

This completes the proof of the theorem.

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**Theorem 6.** Let \( \Gamma \) be a closed set in \( R^2 \) with empty interior and no vertical line segments. Then there is a function \( f(\mu, x) \), locally Lipschitzian in \( \mu \) and in \( x \) such that the boundary of \( \mathcal{A} \) is \( Z = \Gamma \cup (\mu \text{-axis}) \).

**Proof.** Define \( \varphi(\mu, x) \) to be \( \varphi(\mu, x, \Gamma) \), where \( \varphi \) is the Euclidean distance function in \( R^2 \). Then \( \varphi(\mu, x) \) is Lipschitzian in \( R^2 \) and its zero set is \( \Gamma \), whereas \( \varphi(\mu, x) > 0 \) for \((\mu, x) \notin \Gamma \).

Let

\[
\Phi(\mu, x) = \int_0^x \varphi(\mu, y) \, dy,
\]
and
\begin{equation}
\hat{\Phi}(\mu, x) = \inf_{x \leq y \leq x} (\Phi(\mu, 2y) - \Phi(\mu, y))
\end{equation}
and also
\begin{equation}
\bar{\Phi}(\mu, x) = \min(\hat{\Phi}(\mu, x), 1).
\end{equation}
The hypothesis concerning vertical segments implies that \(\bar{\Phi}(\mu, x)\) is never zero for \(x \neq 0\). Now we define the weight function \(w(\mu, x)\) by
\begin{equation}
w(\mu, x) = \frac{x}{\psi(x) + \bar{\Phi}(\mu, x)},
\end{equation}
where
\begin{equation}
\psi(x) = \begin{cases} (x^2 - 1)^2, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}
\end{equation}

We can now take \(f(\mu, x)\) to be defined by
\begin{equation}
f(\mu, x) = w(\mu, x)\psi(\mu, x).
\end{equation}
Clearly \(xf(\mu, x) \geq 0\) for all \(x\) with equality only for \(x = 0\) or \((\mu, x) \in \Gamma\), that is, the zero set of \(f(\mu, x)\) is \(Z\). Now we show that the theorem will follow from
\begin{equation*}
\lim_{x \to +\infty} \int_{-\infty}^{x} f(\mu, x) dx = \lim_{x \to -\infty} \int_{-\infty}^{x} f(\mu, x) dx = +\infty.\end{equation*}
For since \(xf(\mu, x) \geq 0\), \((\mu, A)\) fails to belong to \(\mathcal{A}\) iff either \((\mu, A) \in Z\) or \((\mu, A) \notin Z\) and \(F(\mu, A) = F(\mu, B)\) for some \((\mu, B) \in Z\). Denote by \(Z'\) the set of \((\mu, A)\) for which the latter alternative holds. It is easily seen that the divergence of the above integrals will imply that \(Z \cup Z'\) is closed. \((Z'\) will in fact comprise the type II boundary points of \(\mathcal{A}\).)

Suppose \((\mu_0, A_0) \in Z'.\) Keeping for the moment \(\mu = \mu_0\) fixed and regarding \(f(\mu_0, x), \ F(\mu_0, x)\) as functions of \(x\), we have \(F_x(\mu_0, A_0) = f(\mu_0, x) \neq 0\) and so the image of any neighbourhood of \(A_0\) under the map \(F(\mu_0, x)\) has positive Lebesgue measure. However, \(F\) is continuously differentiable and we may apply Sard's theorem [7] to obtain the result that the image of the critical set of \(F(\mu_0, x)\), and hence of the set of \(A\) for which \(F(\mu_0, A) = F(\mu_0, B)\) for some \(B\) in the critical set, has measure zero. It follows that \(Z'\) is nowhere dense. Thus the complement of \(\mathcal{A} = Z \cup Z'\) has non-empty interior and is therefore the boundary of \(\mathcal{A}\).

To complete the proof, therefore, we need to show that
\begin{equation*}
\lim_{x \to +\infty} \int_{-\infty}^{x} f(\mu, x) dx = +\infty. \end{equation*}
But for \(x \geq 2,
\begin{align*}
\int_{0}^{x} f(\mu, y) dy + \int_{x}^{2} f(\mu, y) dy &= \int_{0}^{x} w(\mu, y)\psi(\mu, y) dy \\
&= \int_{0}^{x} \frac{y}{\bar{\Phi}(\mu, y)} \psi(\mu, y) dy \\
&= \int_{0}^{x} \frac{y}{\bar{\Phi}(\mu, y)} \psi(\mu, y) dy \\
&\geq \int_{0}^{x} \frac{y}{\bar{\Phi}(\mu, y)} \psi(\mu, y) dy.
\end{align*}
Now for \( \frac{1}{2} x \leq y \leq x \), \( \Phi(\mu, y) \leq \Phi(\mu, x) - \Phi(\mu, \frac{1}{2} x) \) and so

\[
\int_{0}^{x} \varphi(\mu, y) \, dy \geq \frac{x}{2(\Phi(\mu, x) - \Phi(\mu, \frac{1}{2} x))} \int_{0}^{x} \varphi(\mu, y) \, dy = \frac{x}{2}.
\]

Hence \( \lim_{x \to \infty} \int_{0}^{x} f(\mu, y) \, dy = +\infty \). Similarly \( \lim_{x \to -\infty} \int_{0}^{x} f(\mu, y) \, dy = +\infty \).

We note that this proof was constructive in the sense that we actually showed how to construct the \( f(\mu, x) \). We note also that for the equation so constructed, the admissible set has no boundary points of type III and its boundary points of type I consist of \( I' \) and an additional horizontal line.

We now consider the case when \( I' \) has vertical segments. We write \( I' \) as \( I'_1 \cup I'_2 \), where \( I'_1 \) is the closure of all points in \( I' \) which do not belong to vertical segments. To obtain \( I'_2 \), we proceed as follows. Let \( I''_2 \) be the set of \((\mu, x) \in I'\) such that \((\mu, x)\) belongs to a vertical segment, and let \( \Pi(I''_2) \) be the projection of \( I''_2 \) on the \( \mu \)-axis. Then \( I'_2 = \{(\mu, x) : (\mu, x) \in I' \text{ and } \mu \in \Pi(I''_2)\} \). Thus \( I'_2 \) consists of \( I''_2 \) together with all points in \( I'_1 \) whose \( \mu \) values are the same as those in \( I''_2 \).

**Theorem 7.** Let \( \Pi(I''_2) \) be nowhere dense. (a) Then there exists \( f(\mu, x) \), continuous in \( \mu \) and Lipschitzian in \( x \) such that the type I boundary points of \( I' \cup \Pi^{-1}(\text{closure of } I''_2) \cup (\mu \text{-axis}) \). (b) There exists \( f(\mu, x) \), Lipschitzian in \( x \) (but discontinuous in \( \mu \)) such that the type I boundary points of \( \mathcal{A} \) are \( I'_1 \cup (\text{closure of } I'_2) \cup (\mu \text{-axis}) \).

**Proof.** Let \( \varphi(\mu, x) \) and \( w(\mu, x) \) be as in Theorem 6 substituting \( I'_1 \) for \( I' \). Denote the closure of a set \( S \) by \( \overline{S} \).

(a) Defining \( f(\mu, x) = w(\mu, x)\varphi(\mu, x) \Phi(\mu, \Pi(I''_2)) \) clearly gives the required result, since if \( \mu \notin \Pi(I''_2) \), \( x \varphi(\mu, x) > 0 \) as before except when \((\mu, x) \in I'_1 \) or \( x = 0 \), whereas if \( \mu \in \Pi(I''_2) \), \( f(\mu, x) = 0 \), for all \( x \).

(b) Define \( \chi(\mu, x) \) by

\[
\chi(\mu, x) = \begin{cases} 
0, & \mu \notin \Pi(I''_2), \\
\varphi((\mu, x), \Pi(I''_2)), & \mu \in \Pi(I''_2).
\end{cases}
\]

Then \( f(\mu, x) = w(\mu, x)\varphi(\mu, x) \Phi(\mu, \Pi(I''_2)) + \chi(\mu, x) \) is the required function. Clearly if \( \mu \notin \Pi(I''_2) \), then \( \chi = 0 \) and arguments analogous to those used in Theorem 6 prevail. If \( \mu \in \Pi(I''_2) \) and \( (\mu, x) \notin I'_2 \), then \( f(\mu, x) = \chi(\mu, x) > 0 \). If \( \mu \in \Pi(I''_2) \) and \( (\mu, x) \in I'_2 \), then \( f(\mu, x) = \chi(\mu, x) = 0 \). Further, since \( \Pi(I''_2) \) has empty interior, then each point in \( I'_2 \) is the limit of a sequence of points in the admissible set. Hence \( I'_2 \) forms part of the type I boundary proving the theorem.
6. Converse theorems for type II boundary points. Again we suppose that we are required to find an equation of the form (2) so that a given set \( I \) forms at least a subset of the type II boundary points of \( \mathcal{A} \). We shall assume that \( I \) is given as the zero set of a certain function \( \varphi(\mu, x) \).

Consider first the case that \((\mu, A) \in I\) iff \( A = \alpha(\mu) \) for some \( \mu \)-domain \( D \), and we assume that \( \alpha \) may be extended, if necessary, to the real line. We search for a function \( f(\mu, x) \) for which

\[
F(\mu, x) = w(\mu, x) \varphi(\mu, x) + c(\mu),
\]

where \( w \) and \( c \) will be appropriately chosen so that the energies \( F(\mu, A) \), \( F(\mu, B) \) are matched, where \( B = \beta(\mu) \), \( \mu \in D \), and \((\mu, B)\) is the point of type I associated with \((\mu, A)\) (see the definitions at the beginning of Section 3).

Since \( F(\mu, x) = \int_0^x f(\mu, y) dy \), we require \( 0 = F(\mu, 0) = w(\mu, 0) + \varphi(\mu, 0) + c(\mu) \), so that

\[
F(\mu, x) = w(\mu, x) \varphi(\mu, x) - w(\mu, 0) \varphi(\mu, 0).
\]

The energy-matching condition requires that

\[
w(\mu, B) \varphi(\mu, B) = 0.
\]

However, we must have \( \varphi(\mu, B) \neq 0 \), otherwise \((\mu, B)\) would be in \( I \). Hence

\[
w(\mu, B) = 0.
\]

Moreover, for \((\mu, B)\) to be type I and \((\mu, A)\) to be type II, we need

\[
f(\mu, B) = F_x(\mu, B) = w(\mu, B) \varphi_x(\mu, B) = 0,
\]

leading to the conditions

\[
w_x(\mu, B) = 0, \quad w(\mu, A) \varphi_x(\mu, A) \neq 0.
\]

So we shall require that for a fixed \( \mu, A \) is at most a simple root of \( \varphi(\mu, x) \) = 0 and \( B \) is at least a double root of \( w(\mu, x) = 0 \). Accordingly, we choose \( w(\mu, x) \) to be \((x-B)^2\) and we shall take \( B = \beta(\mu) < \alpha(\mu) = A \). From (12), we have

\[
f(\mu, x) = (x - \beta(\mu))^2 + 2(x - \beta(\mu))(x - \alpha(\mu)) = (x - \beta(\mu))(2x - \alpha(\mu) - \beta(\mu)).
\]

Since \( f(\mu, x) < 0 \) for \( \beta(\mu) < x < \frac{1}{2}(\alpha(\mu) + \beta(\mu)) \) and \( f(\mu, x) > 0 \) for \( \frac{1}{2}(\alpha(\mu) + \beta(\mu)) < x < \alpha(\mu) \), whereas \( 0 = f(\mu, \beta(\mu)) < f(\mu, \alpha(\mu)) \), it follows from Lemma 1 that \((\mu, \alpha(\mu))\) is a type II boundary point (and \((\mu, \beta(\mu))\) is a type I boundary point). Thus we have proved
Theorem 8. Let \( \varphi(\mu, x) = x - a(\mu), \ w(\mu, x, B) = (x - \beta(\mu))^2, \) where \( \beta(\mu) < a(\mu) \) are functions of \( \mu \) on the real line and let \( \Gamma \) be the zero set of \( \varphi \) restricted to some \( \mu \)-domain \( D \). Then for \( f(\mu, x) = F_x(x, \mu) \), where \( F(x, \mu) \) is given by (12), \( \Gamma \) is a subset of the boundary points of \( \mathcal{A} \) of type II.

We now seek to extend this to the case where \( \varphi(\mu, x) \) is a product of linear functions.

Theorem 9. Let \( \varphi(\mu, x) = \prod_{i=1}^{k} (x - a_i(\mu)), \) where \( a_1(\mu) \leq a_2(\mu) \leq \ldots \leq a_k(\mu), \) and let \( \Gamma \) be a subset of the zero set of \( \varphi \). Define \( w(\mu, x) \) to be \( (-1)^k \prod_{j=1}^{[k+1)/2]} (x - \beta_j(\mu))^2, \) where the \( \beta_j \) are functions of \( \mu \) satisfying

\[
\begin{align*}
&a_1(\mu) \leq \beta_1(\mu) \leq a_2(\mu) \leq \ldots \leq \beta_2(\mu) \leq a_3(\mu) \leq \ldots \leq a_k(\mu) \leq \beta_3(\mu) \leq \ldots,
&\text{the sequence of inequalities terminating with}
&\ldots \leq a_{k-1}(\mu) \leq \beta_{k+1}(\mu) \leq a_k(\mu), \quad k \text{ even},
&\ldots \leq a_{k-2}(\mu) \leq \beta_{k+3}(\mu) \leq a_{k-1}(\mu) \leq \beta_{k+4}(\mu) \leq a_k(\mu) \leq \beta_{k+5}(\mu), \quad k \text{ odd},
\end{align*}
\]

and any equality holding on at most an isolated set of values of \( \mu \). Then the conclusion of Theorem 8 holds.

Proof. Again we may use (12) to write

\[
f(\mu, x) = (-1)^k \prod_{j=1}^{[k+1)/2]} (x - \beta_j(\mu)) \varphi(\mu, x),
\]

where

\[
\varphi(\mu, x) = \sum_{i=1}^{[k+1)/2]} \prod_{j \neq i}^{k+1/2} 2(x - \beta_j(\mu)) \prod_{i=1}^{k} (x - a_i(\mu)) + \]

\[
+ \sum_{i=1}^{k} \prod_{j \neq i} (x - a_i(\mu)) \prod_{j=1}^{[k+1)/2]} (x - \beta_j(\mu)).
\]

We note that \( \varphi(\mu, x) \) is a polynomial in \( x \) of degree \( k + [(k+1)/2] - 1 \), and hence the number of sign changes as a function of \( x \) is at most \( k + [(k+1)/2] - 2 \).

Next we evaluate \( \varphi(\mu, a_m(\mu)) \) and \( \varphi(\mu, \beta_n(\mu)) \), \( m = 1, \ldots, k; n = 1, \ldots, [(k+1)/2] \),

\[
\varphi(\mu, a_m(\mu)) = \prod_{i=1}^{m} (a_m(\mu) - a_i(\mu)) \prod_{j=1}^{[(k+1)/2]} (a_m(\mu) - \beta_j(\mu)).
\]

Hence, whenever strict inequality holds throughout (18), we have

\[
\text{sgn} \varphi(\mu, a_m(\mu)) = (-1)^{k-m} \cdot (-1)^{[(k+1)/2] - [m/2]} = (-1)^{(1-k/2) + [m/2]},
\]

\[
\varphi(\mu, \beta_n(\mu)) = 2 \prod_{j=1}^{[(k+1)/2]} (\beta_n(\mu) - \beta_j(\mu)) \prod_{i=1}^{k} (\beta_n(\mu) - a_i(\mu)).
\]
Thus, whenever strict inequality holds throughout (18), we have
\[ \text{sgn} \psi(\mu, \beta_n(\mu)) = (-1)^{k-2n+1} \cdot (-1)^{[(k+1)/2] - n} = (-1)^{[(n-k)/2] - n + 1}. \]

Now consider the signs of \( \psi \) evaluated at consecutive terms of the
\[ a_{2s-2} \leq a_{2s-1} \leq \beta_s \leq a_{2s} \leq a_{2s+1}. \]

Suppressing the constant factor \((-1)^{[(n-k)/2]}\), the signs are, respectively, 
\((-1)^{-s}, (-1)^{-s}, (-1)^{-s+1}, (-1)^{-s}, (-1)^{-s-1}\). Thus as we progress through the sequence, the signs alternate, giving at least \(k + [(k+1)/2] - 2\) changes of sign. Since we have noted above that this is the maximum possible number, there are precisely this many sign changes, and by virtue of the factor \((-1)^k\) occurring in \(w(\mu, x)\), the conditions of Lemma 1 are fulfilled for each value of \(x\) between each adjacent pair \(a_s(\mu), \beta_s(\mu)\), and the theorem is proved.

Remarks. We note that for the function \(f\) constructed in the proof of the theorem, the zero set of \(\varphi\), and the type II boundary set of \(\mathcal{A}\) coincide, provided that \(\varphi\) is continuous. It is not difficult to see that we may, with only slight modifications in the construction, handle certain functions \(\varphi\) of the form \(\varphi(\mu, x) = \prod_{i=1}^{k} (x - a_i(\mu)) \cdot \prod_{j=1}^{m} \left[(x - \beta_j(\mu))^2 + \gamma_j(\mu)\right] \) which will allow us to obtain examples in which the boundary points of type II include isolated points (in contrast to the situation where \(f\) is linear in \(\mu\), as in Theorem 5) or may include closed curves. We believe that it should be possible to realize the zero set of any function \(\varphi(\mu, x) = \sum_{i=1}^{n} c_i(\mu)x^i\) as the type II boundary, provided that the \(c_i\) are continuous and, except for an isolated set of values of \(\mu\), the roots of \(\varphi\) (as a function of \(x\)) are simple, but have been unable to obtain a result of this generality.

7. Converse theorems for type III boundary points. Since a type III boundary point requires of the energy function a certain asymptotic behaviour as \(x\) approaches either \(+\infty\) or \(-\infty\) (or both) there can be at most two type III boundary points for each fixed \(\mu\).

In the next two theorems we show how to construct \(f(\mu, x)\) generating one and two such boundary curves, respectively.

Theorem 10. Let \(\beta > 0\) be the solution of \(\int_{0}^{\beta} xe^x dx = \frac{1}{2}\). Then equation (2)

\[ f(\mu, x) = \begin{cases} (x + \beta - a(\mu))e^{x+\beta-a(\mu)}, & x \leq a(\mu) - \beta, \\ 2(x + \beta - a(\mu))e^{x+\beta-a(\mu)}, & x > a(\mu) - \beta, \end{cases} \]

has \(A - a(\mu) = 0\) as the type III boundary curve.
Proof. $A - a(\mu) = 0$ is a boundary point of type III if $\int_0^{\alpha(\mu)} f(\mu, x) \, dx = \int_0^{\alpha(\mu)} f(\mu, x) \, dx$, since by (19) $f(\mu, x) < 0$ for $x < a(\mu) - \beta$ and $f(\mu, x) > 0$ for $x > a(\mu) - \beta$. But

$$\int_0^{\alpha(\mu)} f(\mu, x) \, dx = \int_0^{a(\mu) - \beta} f(\mu, x) \, dx + \int_0^{a(\mu) - \beta} (x - a(\mu) + \beta) e^{x - a(\mu) + \beta} \, dx$$

$$= F(\mu, a(\mu) - \beta) + \int_0^{a(\mu)} ye^y \, dy = F(\mu, a(\mu) - \beta) + 1,$$

$$\int_0^{\alpha(\mu)} f(\mu, x) \, dx = F(\mu, a(\mu) - \beta) + \int_0^{\alpha(\mu)} 2(x - a(\mu) + \beta) e^{x - a(\mu) + \beta} \, dx$$

$$= F(\mu, a(\mu) - \beta) + 2 \int_0^{\beta} ye^y \, dy = F(\mu, a(\mu) - \beta) + 1.$$

This proves the theorem.

Theorem 11. Given $a_1(\mu) \leq a_2(\mu)$, with equality on at most an isolated set, there exists $k > 0$, $\beta_1 > 0$, $\beta_2 > 0$ such that equation (2) with

$$f(\mu, x) = \begin{cases} \frac{1}{a_1(\mu) - \beta_1} e^{x - a_1(\mu) + \beta_1}, & x \leq a_1(\mu) - \beta_1, \\ \frac{1}{a_2(\mu) - \beta_2} e^{x - a_2(\mu) + \beta_2}, & a_1(\mu) - \beta_1 \leq x \leq a_2(\mu) + \beta_2, \\ k(\frac{x - a_2(\mu) - a_1(\mu)}{a_2(\mu) - \beta_2} \frac{x - a_2(\mu) - a_1(\mu)}{a_2(\mu) - \beta_2}), & a_2(\mu) + \beta_2 \leq x, \end{cases}$$

has the zero set of $(A - a_1(\mu))(A - a_2(\mu))$ as the type III boundary of $\mathcal{M}$.

Proof. Define $u(r, s, t)$, $v(r, s, t)$ by

$$u(r, s, t) = \int_0^r ty(y + a_1 - a_2 - r - s)(y + a_1 - a_2 - r) \, dy,$$

$$v(r, s, t) = \int_0^s ty - a_1 + a_2 + r + s(y - a_1 + a_2 + s) \, dy.$$

Then $u$ and $v$ are continuous functions from $\mathbb{R}^3$ to $\mathbb{R}$, such that $u(r, s, t) = tu(r, s, 1)$, $v(r, s, t) = tv(r, s, 1)$.

Now for any fixed positive value $\beta_2$ of $s$, we have $u(0, \beta_2, 1) = 0$, $u(r, \beta_2, 1)$ is of order $r^4$ for large positive values of $r$, and $v(0, \beta_2, 1) > 0$, $v(r, \beta_2, 1)$ is of order $r$ for large positive values of $r$. Therefore there is a value $\beta_1 > 0$ of $r$ for which $u(\beta_1, \beta_2, 1) = v(\beta_1, \beta_2, 1)$, and the homogeneity of $u$, $v$ with respect to $t$ allows us to choose a value $k$ of $t$ for which
Periodic solutions of \( x'' + f(\mu, x) = 0 \)

\( u(\beta_1, \beta_2, k) = v(\beta_1, \beta_2, k) = 1 \). Now the remainder of the proof follows along the lines of Theorem 10.

Remarks. We have already observed that for any set which is a type III boundary of an admissible set \( \mathcal{A} \), the intersection of that set with any line perpendicular to the \( \mu \)-axis is necessarily empty, a singleton set or a doubleton set. (The set must also be closed, by definition.) Conversely, given any such set \( I \), we may embed it in the image of two arcs \( a_1(\mu), a_2(\mu) \) with \( a_1(\mu) \leq a_2(\mu) \) for all \( \mu \) and apply Theorem 11 to show that \( I \) may be realized as a subset of the type III boundary points of an admissible set \( \mathcal{A} \).

We have not said much about the smoothness with respect to \( \mu \) of the functions \( f(\mu, x) \) that we have constructed in the converse theorem, but it is clear, at least for these constructions that the smoothness is linked with any regularity conditions with respect to \( \mu \) that we impose on a set \( \Gamma \) of boundary points of the appropriate type.

In conclusion, we remark that the results of Sections 5–7 make it clear that there is a great generality in those closed sets which may be realized as the set of boundary points of some admissible set of a particular type, somewhat contrasting with the situation when the dependence of the parameter \( \mu \) is linear.

References


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