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Periodic solutions of $x''+f(\mu, x) = 0$

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Abstract. An examination is made of those regions (called ***admissible sets***) in the (μ, A)-plane for which the initial value problem $x''(t)+f(\mu, x(t)) = 0$, $x(0) = A$, $x'(0) = 0$, has a non-trivial periodic solution. In particular, results obtained previously for the case that f is linear in μ are generalized to the non-linear case.

In addition, the converse problem is discussed in some detail, whence it is shown how to construct differential equations having rather general sets in the (μ, A)-plane as boundaries of their admissible sets.

1. Introduction. In [2], there was considered the problem of characterizing the set of points (μ, A) (the admissible set) for which the solution of the equation

$$x''(t)+g(x(t))+\mu h(x(t)) = 0 \qquad (' = d/dt) \tag{1}$$

with initial conditions $x(0) = A$, $x'(0) = 0$, is a non-trivial periodic function. In this paper, we extend some of these results to the equation

$$x''(t)+f(\mu, x(t)) = 0, \quad x(0) = A, \quad x'(0) = 0 \tag{2}$$

and illustrate certain essential differences that occur when f is non-linear in μ. We shall also be interested in obtaining some results of a converse nature.

We shall use the following notation:

$$F(\mu, y) = \int_0^y f(\mu, x)\,dx,$$

$$\mathscr{A} = \{(\mu, A) \in R^2\colon (2) \text{ admits a non-trivial periodic solution}\}.$$

Throughout, we shall assume that $f(\mu, x)$ is jointly continuous in both variables and that solutions of (2) are unique, although continuity in μ will be relaxed in one case involving a converse theorem.

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By a periodic solution, we shall always mean a non-constant periodic solution.

Among the many other works devoted to the study of periodic solutions to (2), we mention in particular those of Cesari [3] and Loud [5].

One could interpret μ as a non-linear control; for example a knowledge of the admissible set might allow a path in the (μ, A)-plane to be found permitting a transfer from a non-periodic orbit to a periodic orbit.

In [2], a general criterion for the existence of a periodic solution to (2) was given under somewhat weaker hypotheses on f, than in [3] or [5]. We shall find useful a corollary of that result which we state without proof as

LEMMA 1. *Let $f(\mu, A) < 0$ $(f(\mu, A) > 0)$. Then a necessary and sufficient condition for the solution of* (2) *to be periodic is that there exist $B > A$ $(B < A)$ such that $F(\mu, A) = F(\mu, B) > F(\mu, y)$ for $A < y < B$ $(B < y < A)$.*

A corresponding result has been obtained by one of the authors [1] for certain equations of the form $x'' + g(x)h(x'^2) + f(\mu, x) = 0$.

2. Admissible regions of the (μ, A)-plane. If $(\mu, A) \in \mathscr{A}$, we shall say that (μ, A) is *admissible for* (2), and $\mathscr{A}$ will be called the *admissible set for f*.

The following result generalizes the case that f is linear in μ, which was given in [2].

THEOREM 1. *$\mathscr{A}$ is open.*

Proof. Let $(\mu_0, A_0) \in \mathscr{A}$. By Lemma 1, there exists B_0 such that

$$F(\mu_0, A_0) = F(\mu_0, B_0) > F(\mu_0, y),$$

for y between A_0 and B_0, and we may, without loss of generality, assume that $B_0 < A_0$, so that $f(\mu_0, B_0) < 0 < f(\mu_0, A_0)$. By continuity, there exists $\delta_0 > 0 \ni f(\mu, B) < -\delta_0 < 0 < \delta_0 < f(\mu, A)$ whenever (μ, A) and (μ, B) are sufficiently close to (μ_0, A_0), (μ_0, B_0), respectively. Further, by the implicit function theorem and the fact that $f(\mu_0, B_0) \neq 0$, it follows that

$$F(\mu, A) = F(\mu, x)$$

has a unique solution $x = B = B(\mu, A)$ for (μ, A) sufficiently close to (μ_0, A_0) such that $B(\mu, A) \to B_0$ as $(\mu, A) \to (\mu_0, A_0)$ and $F(\mu, A) = F(\mu, B) > F(\mu, y)$ whenever $B < y < A$, and the proof of the theorem is complete.

3. Description of the boundary of $\mathscr{A}$. As in [2] we introduce the following classification of the boundary points of $\mathscr{A}$. Let $(\mu_0, A_0) \in \partial\mathscr{A}$. We shall say that (μ_0, A_0) is of *type* I if $f(\mu_0, A_0) = 0$, is of *type* II if it is in the closure of the set $\{(\mu, A) \in \partial \mathscr{A}$: there exist $B \neq A$ such that $F(\mu, B) = F(\mu, A)$ and $f(\mu, B) = 0$, and (μ, A) is not of type I$\}$, and is of *type* III if it is in the closure of the set $\{(\mu, A) \in \partial \mathscr{A}$: $F(\mu, A) > F(\mu, y)$, either for all $y > A$ or for $y < A$, and (μ, A) is not of type I or II$\}$.

That this gives a complete classification of the boundary points of $\mathscr{A}$ is indicated by

THEOREM 2. *Let* $(\mu_0, A_0) \in \partial \mathscr{A}$. *Then* (μ_0, A_0) *is one of the types* I, II *or* III.

Proof. This follows along the lines of the proof for the linear case. See [2].

In [2], suitable hypotheses were given to ensure that boundary points exclusively of one of the types I, II or III were interior to a continuous arc of such points. Again, for f non-linear in μ, analogous results hold true, the proofs requiring merely a straightforward adaptation of those given in [2] and we obtain

THEOREM 3. *In parts* (a), (b) *and* (c), *assume that* (μ_0, A_0) *is a boundary point of* $\mathscr{A}$, *exclusively of types* I, II *and* III, *respectively.*

(a) *Suppose that* f_μ *exists as a continuous, non-vanishing function of* (μ, A) *in some neighbourhood of* (μ_0, A_0). *Then* (μ_0, A_0) *is relatively interior to a continuous arc of boundary points exclusively of type* I.

(b) *Suppose that* f_μ *and* f_A *exist as continuous, non-vanishing functions in some neighborhood of* (μ_0, B_0) (*see definition of type* II *for meaning of* B_0). *Assume, in addition, that*

$$\int_{A_0}^{B_0} f_\mu(\mu_0, y)\,dy \neq 0.$$

Then (μ_0, A_0) *is relatively interior to a continuous arc of boundary points exclusively of type* II.

(c) *Define* $G(\mu)$ *to be* $\limsup_{y\to\infty} F(\mu, y)$. *Suppose that* G_μ *and* F_μ *exist as continuous functions in neighbourhoods of* μ_0 *and* (μ_0, A_0), *respectively, with* $G_\mu(\mu_0) \neq F_\mu(\mu_0, A_0)$. *In addition, let* $F(\mu_0, A_0) > F(\mu_0, x)$ *for all* $x > A_0$. *Then* (μ_0, A_0) *is relatively interior to a continuous arc of boundary points exclusively of type* III.

For certain cases of equation (2), the boundary curves may be explicitly parameterized. For example, for the equation

$$x''+x+\mu x^2 = 0, \quad x(0) = A, \quad x'(0) = 0,$$

the boundary curves of type II are the branches of the hyperbola $\mu A = \frac{1}{2}$ [6].

In [2], this monotone behaviour of type II curves was shown to hold for more general equations of the form (2) with f linear in μ. Here we shall further extend this result for f non-linear in μ. The proof is based on that given for the linear case. We shall sketch the proof, giving the necessary modifications.

THEOREM 4. *Assume that $f(\mu, x)$ is continuously differentiable with respect to μ for each x, with $f_\mu(\mu, x) > 0$ for all (μ, x) and that there exists $\bar{\mu}, \bar{A}$ such that $(x-\bar{A})f(\bar{\mu}, x) > 0$ for $x \neq \bar{A}$. Let (μ_0, A_0) be a boundary point of type* II *but not of type* III, *with $\mu_0 > \bar{\mu}$, $A_0 > \bar{A}$. Then there exists a continuous strictly decreasing arc*

$$\Gamma = \{(\mu, A(\mu)) \colon \mu_0 \leqslant \mu < \mu^*\}$$

of such points, with $\lim_{\mu\to\mu^*} A(\mu) = 0$ if the maximal interval $[\mu_0, \mu^*)$ of definition of the arc is finite.

Proof. Since (μ_0, A_0) is of type II, but not of type III, there exists $B_0 < \bar{A}$ such that $F(\mu_0, B_0) = F(\mu_0, A_0) > F(\mu_0, y)$ for $B_0 < y < A_0$, and $f(\mu_0, B_0) = 0$. The conditions of the theorem imply that $f(\mu, B_0) > 0$, for $\mu > \mu_0$, and so, provided there exists B with $B_0 < B < \bar{A}$ such that $F(\mu, B) > F(\mu, \bar{A})$, we may define $B(\mu)$ (for $\mu_0 \leqslant \mu < \mu^*$, say) to be

$$\sup\{y\colon B_0 < y < \bar{A} \text{ and } F(\mu, y) = \sup_{B_0<B<\bar{A}} F(\mu, B)\}.$$

We have $B_0 < B(\mu) < 0$ and $f(\mu, B(\mu)) = 0$. $B(\mu_0)$ is equal to B_0. $B(\mu)$ is non-decreasing; for let $\mu_0 \leqslant \mu_1 < \mu_2 < \mu^*$ and let $B(\mu_i) = B_i$. Suppose that $B_1 > B_2$. Then $F(\mu_2, B_2) > F(\mu_2, B_1)$ (definition of $B_2 = B(\mu_2)$), which inequality may be written

$$\int_{B_1}^{B_2} f(0, x)\,dx > \int_{B_2}^{B_1} \big(f(\mu_2, x) - f(0, x)\big)\,dx. \tag{3}$$

However, using the condition on f_μ,

$$f(\mu_1, B_2) < f(\mu_2, B_2) = 0$$

and so

$$F(\mu_1, B_2) < \sup_{B_0\leqslant y\leqslant \bar{A}} F(\mu_1, y) = F(\mu_1, B_1).$$

Therefore

$$\int_{B_1}^{B_2} f(0, x)\,dx \tag{4}$$
$$< \int_{B_2}^{B_1} \big(f(\mu_1, x) - f(0, x)\big)\,dx < \int_{B_2}^{B_1} \big(f(\mu_2, x) - f(0, x)\big)\,dx$$

again using the condition on f_μ. This contradicts (3). Thus $B_1 \leqslant B_2$ and $B(\mu)$ is non-decreasing.

The next step is to show that $F(\mu, B(\mu))$ is continuous as a function of μ. This is a straightforward copy of the argument given in the linear case and we omit the details. That the equation $F(\mu, A) = F(\mu, B(\mu))$ has a unique solution also follows straightforwardly using the argument for the linear case, as does the continuity of $A = A(\mu)$.

We then have for $\mu_0 \leqslant \mu_1 < \mu_2 < \mu^*$, $\int_{\bar{A}}^{A_2} f(\mu_2, y)dy = \int_{\bar{A}}^{B_2} f(\mu_2, y)dy < \int_{\bar{A}}^{B_2} f(\mu_1, y)dy$ (condition on f_μ) $\leqslant \int_{\bar{A}}^{B_1} f(\mu_1, y)dy$ (definition of B_1) $= \int_{\bar{A}}^{A_1} f(\mu_1, y)dy$. The conditions of the theorem imply that $\int_{\bar{A}}^{x} f(\mu, y)dy$ is increasing in both μ and x whenever $x \geqslant \bar{A}$, and so $A_2 < A_1$. Thus $A(\mu)$ is strictly decreasing in (μ_0, μ^*). Finally, the behaviour of $A(\mu)$ as $\mu \to \mu^*$ may be verified just as in the proof given for the linear case.

As an analogue to Corollary 2 of [2], we have

COROLLARY. *The energy function $F(\mu, A)$ is (strictly) decreasing along the arcs of type* II *defined in the above theorem.*

4. Isolated boundary points for $f(\mu, x)$ linear in μ. The question occurs whether or not there can exist isolated boundary points of $\mathscr{A}$. We shall see in the next several sections that there can in general, but we show here that in the case $f(\mu, x)$ is linear in μ isolated boundary points of $\mathscr{A}$ are impossible.

THEOREM 5. *Let f be continuous in x and linear in μ. Then there are no isolated boundary points of $\mathscr{A}$.*

Proof. Let $(\mu_0, A_0) \in \partial\mathscr{A}$. Write $f(\mu, x) = g(x)+\mu h(x)$. If $f(\mu_0, A_0) = 0$, either the line $A = A_0$ (in the case $h(A_0) = 0$) or the arc $\mu = -g(A)/h(A)$, A in a neighbourhood of A_0 (in the case $h(A_0) \neq 0$) is a curve of points in the complement of $\mathscr{A}$. Clearly, then (μ_0, A_0) cannot be an isolated point of $\partial\mathscr{A}$.

Henceforth, we may assume that $f(\mu_0, A_0) \neq 0$; without loss of generality, we shall suppose that $f(\mu_0, A_0) > 0$. Consider the set

$$S = \{y < A_0\colon F(\mu_0, y) = F(\mu_0, A_0)\}.$$

(i) If S is empty, then $F(\mu_0, A_0) > F(\mu_0, y)$ for all $y < A_0$ and, therefore,

$$F(\mu_0, A) > F(\mu_0, y)$$

for all $y < A$, for values of A in some sufficiently small right neighbourhood of A_0. For such values of A, it follows that (μ_0, A) is in the complement of $\mathscr{A}$ and again (μ_0, A_0) cannot be an isolated point of $\partial\mathscr{A}$.

(ii) If S is not empty, let $B_0 = \sup S$. Then $B_0 < A_0$ and

$$F(\mu_0, B_0) = F(\mu_0, A_0) > F(\mu_0, y)$$

for $B_0 < y < A_0$. Since $(\mu_0, B_0) \notin \mathscr{A}$, we conclude that $f(\mu_0, B_0) = 0$ and arguing as at the beginning of this proof, there exists a non-trivial arc $\mathscr{C}$ which contains (μ_0, B_0) such that $f(\mu, B) = 0$ for $(\mu, B) \in \mathscr{C}$. If the perpendicular projection of $\mathscr{C}$ on to the μ-axis is not the singleton $\{\mu_0\}$, then using continuity considerations and the fact that $f(\mu, A)$ is bounded away from zero in a suitable neighbourhood of (μ_0, A_0), we deduce that there is a non-trivial arc $\mathscr{C}'$ through (μ_0, A_0) such that for each $(\mu, A) \in \mathscr{C}'$, there exists $(\mu, B) \in \mathscr{C}$ (same μ) such that $F(\mu, B) = F(\mu, A)$. Choosing $y = B^* = B^*(\mu)$, as large as possible in the interval $I = [B, \frac{1}{2}(A_0 + B)]$, to maximize $F(\mu, y)$, $y \in I$, we see that $B^* \to B_0$ as $(\mu, A) \to (\mu_0, A_0)$ along the arc $\mathscr{C}'$, and $f(\mu, B^*) = 0$. Since $F(\mu, B^*) \to F(\mu_0, B_0) = F(\mu_0, A_0)$ as $(\mu, A) \to (\mu_0, A_0)$ along $\mathscr{C}'$, we may find an arc $\mathscr{C}''$ through (μ_0, A_0) with $F(\mu, B^*) = F(\mu, A)$, for all μ with $(\mu, A) \in \mathscr{C}''$. (Here we are using the fact that $f(\mu_0, A_0) > 0$ to assert the existence of $\mathscr{C}''$.)

The behaviour of $f(\mu, A)$ near (μ_0, A_0) indicates that for $(\mu, A) \in \mathscr{C}''$ sufficiently close to (μ_0, A_0), we have

$$F(\mu, B^*) = F(\mu, A) > F(\mu, y)$$

for $B^* < y < A$. Since $f(\mu, B^*) = 0$, it follows that these points of $\mathscr{C}''$ are in the complement of $\mathscr{A}$, and (μ_0, A_0) is not an isolated point of $\partial \mathscr{A}$.

Finally, we must deal with the case that $\mathscr{C}$ projects on to $\{\mu_0\}$ on the μ-axis. Then $\mu_0 = -g(B)/h(B)$ for B in some neighbourhood of B_0. For such B and either for all $\mu > \mu_0$ or for all $\mu < \mu_0$, we have $f(\mu, B) = g(B) + \mu h(B) > 0$. For such values of μ, choose $y = B^*$ as above and construct $\mathscr{C}''$ as before.

This completes the proof of the theorem.

In the remaining sections of this paper we examine the conditions under which an equation of the form (2) can be constructed having a given set imbedded in the boundary set of $\mathscr{A}$.

5. Converse theorems for type I boundary points. We are interested here in discussing the problem of when a given set in R^2 can be considered the type I boundary of the admissible set of some equation of the form (2). We first consider sets which do not contain any vertical segments, that is line segments parallel to the A-axis.

THEOREM 6. *Let Γ be a closed set in R^2 with empty interior and no vertical line segments. Then there is a function $f(\mu, x)$, locally Lipschitzian in μ and in x such that the boundary of $\mathscr{A}$ is $Z = \Gamma \cup (\mu\text{-axis})$.*

Proof. Define $\varphi(\mu, x)$ to be $\varrho((\mu, x), \Gamma)$, where ϱ is the Euclidean distance function in R^2. Then $\varphi(\mu, x)$ is Lipschitzian in R^2 and its zero set is Γ, whereas $\varphi(\mu, x) > 0$ for $(\mu, x) \notin \Gamma$.

Let

$$\Phi(\mu, x) = \int_0^x \varphi(\mu, y)\, dy, \tag{5}$$

and

(6) $$\hat{\Phi}(\mu, x) = \inf_{\frac{1}{2}x \leqslant y \leqslant x} \big(\Phi(\mu, 2y) - \Phi(\mu, y)\big)$$

and also

(7) $$\tilde{\Phi}(\mu, x) = \min\{\hat{\Phi}(\mu, x), 1\}.$$

The hypothesis concerning vertical segments implies that $\tilde{\Phi}(\mu, x)$ is never zero for $x \neq 0$. Now we define the weight function $w(\mu, x)$ by

(8) $$w(\mu, x) = \frac{x}{\psi(x) + \tilde{\Phi}(\mu, x)},$$

where

(9) $$\psi(x) = \begin{cases} (x^2-1)^2, & |x| \leqslant 1, \\ 0, & |x| > 1. \end{cases}$$

We can now take $f(\mu, x)$ to be defined by

(10) $$f(\mu, x) = w(\mu, x)\varphi(\mu, x).$$

Clearly $xf(\mu, x) \geqslant 0$ for all x with equality only for $x = 0$ or $(\mu, x) \in \Gamma$, that is, the zero set of $f(\mu, x)$ is Z. Now we show that the theorem will follow from $\lim_{x\to+\infty} \int_0^x f(\mu, x)\,dx = \lim_{x\to-\infty} \int_0^x f(\mu, x)\,dx = +\infty$. For since $xf(\mu, x) \geqslant 0$, (μ, A) fails to belong to $\mathscr{A}$ iff either $(\mu, A) \in Z$ or $(\mu, A) \notin Z$ and $F(\mu, A) = F(\mu, B)$ for some $(\mu, B) \in Z$. Denote by Z' the set of (μ, A) for which the latter alternative holds. It is easily seen that the divergence of the above integrals will imply that $Z \cup Z'$ is closed. (Z' will in fact comprise the type II boundary points of $\mathscr{A}$.)

Suppose $(\mu_0, A_0) \in Z'$. Keeping for the moment $\mu = \mu_0$ fixed and regarding $f(\mu_0, x)$, $F(\mu_0, x)$ as functions of x, we have $F_x(\mu_0, A_0) = f(\mu_0, A_0) \neq 0$ and so the image of any neighbourhood of A_0 under the map $F(\mu_0, x)$ has positive Lebesgue measure. However, F is continuously differentiable and we may apply Sard's theorem [7] to obtain the result that the image of the critical set of $F(\mu_0, x)$, and hence of the set of A for which $F(\mu_0, A) = F(\mu_0, B)$ for some B in the critical set, has measure zero. It follows that Z' is nowhere dense. Thus the complement of $\mathscr{A} = Z \cup Z'$ has non-empty interior and is therefore the boundary of $\mathscr{A}$. To complete the proof, therefore, we need to show that $\lim_{x\to+\infty} \int_0^x f(\mu, x)\,dx = \lim_{x\to-\infty} \int_0^x f(\mu, x)\,dx = +\infty$. But for $x \geqslant 2$,

$$\int_0^x f(\mu, y)\,dy \geqslant \int_{\frac{1}{2}x}^x f(\mu, y)\,dy = \int_{\frac{1}{2}x}^x w(\mu, y)\varphi(\mu, y)\,dy$$

$$= \int_{\frac{1}{2}x}^x \frac{y}{\tilde{\Phi}(\mu, y)}\varphi(\mu, y)\,dy \geqslant \int_{\frac{1}{2}x}^x \frac{y}{\hat{\Phi}(\mu, y)}\varphi(\mu, y)\,dy.$$

Now for $\frac{1}{2}x \leqslant y \leqslant x$, $\hat{\Phi}(\mu, y) \leqslant \Phi(\mu, x) - \Phi(\mu, \frac{1}{2}x)$ and so

$$\int_0^x \varphi(\mu, y)\,dy \geqslant \frac{x}{2\left(\Phi(\mu, x) - \Phi(\mu, \frac{1}{2}x)\right)} \int_{\frac{1}{2}x}^x \varphi(\mu, y)\,dy = \frac{x}{2}.$$

Hence $\lim_{x\to\infty} \int_0^x f(\mu, y)\,dy = +\infty$. Similarly $\lim_{x\to-\infty} \int_0^x f(\mu, y)\,dy = +\infty$.

We note that this proof was constructive in the sense that we actually showed how to construct the $f(\mu, x)$. We note also that for the equation so constructed, the admissible set has no boundary points of type III and its boundary points of type I consist of Γ and an additional horizontal line.

We now consider the case when Γ has vertical segments. We write Γ as $\Gamma_1 \cup \Gamma_2$, where Γ_1 is the closure of all points in Γ which do not belong to vertical segments. To obtain Γ_2 we proceed as follows. Let Γ_2^* be the set of $(\mu, x) \in \Gamma$ such that (μ, x) belongs to a vertical segment, and let $\Pi(\Gamma_2^*)$ be the projection of Γ_2^* on the μ-axis. Then $\Gamma_2 = \{(\mu, x) \colon (\mu, x) \in \Gamma$ and $\mu \in \Pi(\Gamma_2^*)\}$. Thus Γ_2 consists of Γ_2^* together with all points in Γ_1 whose μ values are the same as those in Γ_2^*.

THEOREM 7. *Let $\Pi(\Gamma_2)$ be nowhere dense.* (a) *Then there exists $f(\mu, x)$, continuous in μ and Lipschitzian in x such that the type* I *boundary points of $\mathscr{A}$ are $\Gamma_1 \cup \Pi^{-1}$ (closure of Γ_2) $\cup$ (μ-axis).* (b) *There exists $f(\mu, x)$, Lipschitzian in x (but discontinuous in μ) such that the type* I *boundary points of $\mathscr{A}$ are $\Gamma_1 \cup$ (closure of Γ_2) $\cup$ (μ-axis).*

Proof. Let $\varphi(\mu, x)$ and $w(\mu, x)$ be as in Theorem 6 substituting Γ_1 for Γ. Denote the closure of a set S by $\bar{S}$.

(a) Defining $f(\mu, x) = w(\mu, x)\varphi(\mu, x)\varrho\left(\mu, \overline{\Pi(\Gamma_2)}\right)$ clearly gives the required result, since if $\mu \notin \overline{\Pi(\Gamma_2)}$, $xf(\mu, x) > 0$ as before except when $(\mu, x) \in \Gamma_1$ or $x = 0$, whereas if $\mu \in \overline{\Pi(\Gamma_2)}$, $f(\mu, x) = 0$, for all x.

(b) Define $\chi(\mu, x)$ by

$$\chi(\mu, x) = \begin{cases} 0, & \mu \notin \overline{\Pi(\Gamma_2)}, \\ \varrho\left((\mu, x), \bar{\Gamma}_2\right), & \mu \in \overline{\Pi(\Gamma_2)}. \end{cases}$$

Then $f(\mu, x) = w(\mu, x)\varphi(\mu, x)\varrho\left(\mu, \overline{\Pi(\Gamma_2)}\right) + \chi(\mu, x)$ is the required function. Clearly if $\mu \notin \overline{\Pi(\Gamma_2)}$, then $\chi = 0$ and arguments analogous to those used in Theorem 6 prevail. If $\mu \in \overline{\Pi(\Gamma_2)}$ and $(\mu, x) \notin \bar{\Gamma}_2$, then $f(\mu, x) = \chi(\mu, x) > 0$. If $\mu \in \overline{\Pi(\Gamma_2)}$ and $(\mu, x) \in \bar{\Gamma}_2$, then $f(\mu, x) = \chi(\mu, x) = 0$. Further, since $\overline{\Pi(\Gamma_2)}$ has empty interior, then each point in Γ_2 is the limit of a sequence of points in the admissible set. Hence Γ_2 forms part of the type I boundary proving the theorem.

6. Converse theorems for type II boundary points. Again we suppose that we are required to find an equation of the form (2) so that a given set Γ forms at least a subset of the type II boundary points of $\mathscr{A}$. We shall assume that Γ is given as the zero set of a certain function $\varphi(\mu, x)$.

Consider first the case that $(\mu, A) \in \Gamma$ iff $A = \alpha(\mu)$ for some μ-domain D, and we assume that α may be extended, if necessary, to the real line. We search for a function $f(\mu, x)$ for which

$$F(\mu, x) = w(\mu, x)\varphi(\mu, x)+c(\mu), \tag{11}$$

where w and c will be appropriately chosen so that the energies $F(\mu, A)$, $F(\mu, B)$ are matched, where $B = \beta(\mu)$, $\mu \in D$, and (μ, B) is the point of type I associated with (μ, A) (see the definitions at the beginning of Section 3).

Since $F(\mu, x) = \int\limits_0^x f(\mu, y)dy$, we require $0 = F(\mu, 0) = w(\mu, 0)+$ $+\varphi(\mu, 0)+c(\mu)$, so that

$$F(\mu, x) = w(\mu, x)\varphi(\mu, x)-w(\mu, 0)\varphi(\mu, 0). \tag{12}$$

The energy-matching condition requires that

$$w(\mu, B)\varphi(\mu, B) = 0. \tag{13}$$

However, we must have $\varphi(\mu, B) \neq 0$, otherwise (μ, B) would be in Γ. Hence

$$w(\mu, B) = 0. \tag{14}$$

Moreover, for (μ, B) to be type I and (μ, A) to be type II, we need

$$f(\mu, B) = F_x(\mu, B) = w(\mu, B)\varphi_x(\mu, B)+w_x(\mu, B)\varphi(\mu, B) = 0, \tag{15}$$

$$f(\mu, A) = F_x(\mu, A) = w(\mu, A)\varphi_x(\mu, A)+w_x(\mu, A)\varphi(\mu, A) \neq 0, \tag{16}$$

leading to the conditions

$$w_x(\mu, B) = 0, \quad w(\mu, A)\varphi_x(\mu, A) \neq 0. \tag{17}$$

So we shall require that for a fixed μ, A is at most a simple root of $\varphi(\mu, x) = 0$ and B is at least a double root of $w(\mu, x) = 0$. Accordingly, we choose $w(\mu, x)$ to be $(x-B)^2$ and we shall take $B = \beta(\mu) < \alpha(\mu) = A$. From (12), we have

$$f(\mu, x) = \big(x-\beta(\mu)\big)^2+2\big(x-\beta(\mu)\big)\big(x-\alpha(\mu)\big) = \big(x-\beta(\mu)\big)\big(2x-\alpha(\mu)-\beta(\mu)\big).$$

Since $f(\mu, x) < 0$ for $\beta(\mu) < x < \frac{1}{2}\big(\alpha(\mu)+\beta(\mu)\big)$ and $f(\mu, x) > 0$ for $\frac{1}{2}\big(\alpha(\mu)+\beta(\mu)\big) < x < \alpha(\mu)$, whereas $0 = f\big(\mu, \beta(\mu)\big) < f\big(\mu, \alpha(\mu)\big)$, it follows from Lemma 1 that $\big(\mu, \alpha(\mu)\big)$ is a type II boundary point (and $\big(\mu, \beta(\mu)\big)$ is a type I boundary point). Thus we have proved

THEOREM 8. *Let* $\varphi(\mu, x) = x - a(\mu)$, $w(\mu, x, B) = (x-\beta(\mu))^2$, *where* $\beta(\mu) < a(\mu)$ *are functions of* μ *on the real line and let* Γ *be the zero set of* φ *restricted to some* μ*-domain* D. *Then for* $f(\mu, x) = F_x(\mu, x)$, *where* $F(\mu, x)$ *is given by* (12), Γ *is a subset of the boundary points of* $\mathscr{A}$ *of type* II.

We now seek to extend this to the case where $\varphi(\mu, x)$ is a product of linear functions.

THEOREM 9. *Let* $\varphi(\mu, x) = \prod_{i=1}^{k}(x - a_i(\mu))$, *where* $a_1(\mu) \leqslant a_2(\mu) \leqslant \ldots \ldots \leqslant a_k(\mu)$, *and let* Γ *be a subset of the zero set of* φ. *Define* $w(\mu, x)$ *to be* $(-1)^k \prod_{j=1}^{[(k+1)/2]} (x-\beta_j(\mu))^2$, *where the* β_j *are functions of* μ *satisfying*

(18) $\quad a_1(\mu) \leqslant \beta_1(\mu) \leqslant a_2(\mu) \leqslant a_3(\mu) \leqslant \beta_2(\mu) \leqslant a_4(\mu) \leqslant a_5(\mu) \leqslant \beta_3(\mu) \leqslant \ldots,$

the sequence of inequalities terminating with

$$\ldots \leqslant a_{k-1}(\mu) \leqslant \beta_{\frac{1}{2}k}(\mu) \leqslant a_k(\mu), \qquad k \text{ even},$$

$$\ldots \leqslant a_{k-2}(\mu) \leqslant \beta_{\frac{1}{2}(k-1)}(\mu) \leqslant a_{k-1}(\mu) \leqslant a_k(\mu) \leqslant \beta_{\frac{1}{2}(k+1)}(\mu), \qquad k \text{ odd},$$

and any equality holding on at most an isolated set of values of μ. *Then the conclusion of Theorem* 8 *holds.*

Proof. Again we may use (12) to write

$$f(\mu, x) = (-1)^k \prod_{j=1}^{[(k+1)/2]} (x-\beta_j(\mu))\,\psi(\mu, x),$$

where

$$\psi(\mu, x) = \sum_{l=1}^{[(k+1)/2]} \prod_{\substack{j=1\\ j\neq l}}^{[(k+1)/2]} 2\,(x-\beta_j(\mu)) \prod_{i=l}^{k}(x-a_i(\mu)) + \\ + \sum_{l=1}^{k} \prod_{\substack{i=1\\ i\neq l}}^{k} (x-a_i(\mu)) \prod_{j=1}^{[(k+1)/2]} (x-\beta_j(\mu)).$$

We note that $\psi(\mu, x)$ is a polynomial in x of degree $k+[(k+1)/2]-1$, and hence the number of sign changes as a function of x is at most $k+[(k+1)/2]-2$.

Next we evaluate $\psi(\mu, a_m(\mu))$ and $\psi(\mu, \beta_n(\mu))$, $m = 1, \ldots, k$; $n = 1, \ldots \ldots, [(k+1)/2]$,

$$\psi(\mu, a_m(\mu)) = \prod_{\substack{i=1\\ i\neq m}}^{m} (a_m(\mu)-a_i(\mu)) \cdot \prod_{j=1}^{[(k+1)/2]} (a_m(\mu)-\beta_j(\mu)).$$

Hence, whenever strict inequality holds throughout (18), we have

$$\operatorname{sgn}\psi(\mu, a_m(\mu)) = (-1)^{k-m}\cdot(-1)^{[(k+1)/2]-[m/2]} = (-1)^{[(1-k)/2]+[-m/2]},$$

$$\psi(\mu, \beta_n(\mu)) = 2 \prod_{\substack{j=1\\ j\neq n}}^{[(k+1)/2]} (\beta_n(\mu)-\beta_j(\mu)) \cdot \prod_{i=1}^{k} (\beta_n(\mu)-a_i(\mu)).$$

Thus, whenever strict inequality holds throughout (18), we have

$$\operatorname{sgn}\psi(\mu, \beta_n(\mu)) = (-1)^{k-2n+1}\cdot(-1)^{[(k+1)/2]-n} = (-1)^{[(1-k)/2]-n+1}.$$

Now consider the signs of ψ evaluated at consecutive terms of the

$$a_{2s-2} \leqslant a_{2s-1} \leqslant \beta_s \leqslant a_{2s} \leqslant a_{2s+1}.$$

Suppressing the constant factor $(-1)^{[(1-k)/2]}$, the signs are, respectively, $(-1)^{1-s}$, $(-1)^{-s}$, $(-1)^{-s+1}$, $(-1)^{-s}$, $(-1)^{-s-1}$. Thus as we progress through the sequence, the signs alternate, giving at least $k+[(k+1)/2]-2$ changes of sign. Since we have noted above that this is the maximum possible number, there are precisely this many sign changes, and by virtue of the factor $(-1)^k$ occurring in $w(\mu, x)$, the conditions of Lemma 1 are fulfilled for each value of x between each adjacent pair $a_i(\mu)$, $\beta_j(\mu)$, and the theorem is proved.

Remarks. We note that for the function f constructed in the proof of the theorem, the zero set of φ, and the type II boundary set of $\mathscr{A}$ coincide, provided that φ is continuous. It is not difficult to see that we may, with only slight modifications in the construction, handle certain functions φ of the form $\varphi(\mu, x) = \prod_{i=1}^{k} (x-a_i(\mu))\cdot\prod_{j=1}^{m} [(x-\beta_j(\mu))^2+\gamma_j(\mu)]$ which will allow us to obtain examples in which the boundary points of type II include isolated points (in contrast to the situation where f is linear in μ, as in Theorem 5) or may include closed curves. We believe that it should be possible to realize the zero set of any function $\varphi(\mu, x) = \sum_{i=1}^{n} c_i(\mu)x^i$ as the type II boundary, provided that the c_i are continuous and, except for an isolated set of values of μ, the roots of φ (as a function of x) are simple, but have been unable to obtain a result of this generality.

7. Converse theorems for type III boundary points. Since a type III boundary point requires of the energy function a certain asymptotic behaviour as x approaches either $+\infty$ or $-\infty$ (or both) there can be at most two type III boundary points for each fixed μ.

In the next two theorems we show how to construct $f(\mu, x)$ generating one and two such boundary curves, respectively.

THEOREM 10. *Let $\beta > 0$ be the solution of $\int_0^\beta xe^x\,dx = \frac{1}{2}$. Then equation* (2) *with*

$$f(\mu, x) = \begin{cases} (x+\beta-a(\mu))e^{x+\beta-a(\mu)}, & x \leqslant a(\mu)-\beta, \\ 2(x+\beta-a(\mu))e^{x+\beta-a(\mu)}, & x > a(\mu)-\beta, \end{cases} \tag{19}$$

has $A-a(\mu) = 0$ as the type III *boundary curve.*

Proof. $A-a(\mu)=0$ is a boundary point of type III if $\int_0^{a(\mu)} f(\mu,x)\,dx = \int_0^{-\infty} f(\mu,x)\,dx$, since by (19) $f(\mu,x)<0$ for $x<a(\mu)-\beta$ and $f(\mu,x)>0$ for $x>a(\mu)-\beta$. But

$$\int_0^{-\infty} f(\mu,x)\,dx = \int_0^{a(\mu)-\beta} f(\mu,x)\,dx + \int_{a(\mu)-\beta}^{-\infty} \big(x-a(\mu)+\beta\big)e^{x-a(\mu)+\beta}\,dx$$

$$= F\big(\mu, a(\mu)-\beta\big) + \int_0^{-\infty} ye^y\,dy = F\big(\mu, a(\mu)-\beta\big)+1,$$

$$\int_0^{a(\mu)} f(\mu,x)\,dx = F\big(\mu, a(\mu)-\beta\big) + \int_{a(\mu)-\beta}^{a(\mu)} 2\big(x-a(\mu)+\beta\big)e^{x-a(\mu)+\beta}\,dx$$

$$= F\big(\mu, a(\mu)-\beta\big) + 2\int_0^{\beta} ye^y\,dy = F\big(\mu, a(\mu)-\beta\big)+1.$$

This proves the theorem.

THEOREM 11. *Given $a_1(\mu)\leqslant a_2(\mu)$, with equality on at most an isolated set, there exists $k>0$, $\beta_1>0$, $\beta_2>0$ such that equation* (2) *with*

$$(20)\quad f(\mu,x) = \begin{cases} \big(x-a_1(\mu)+\beta_1\big)e^{x-a_1(\mu)+\beta_1}, & x\leqslant a_1(\mu)-\beta_1, \\ k\big(x-a_1(\mu)+\beta_1\big)\big(x-\tfrac12 a_1(\mu)-\tfrac12 a_2(\mu)\big)\big(x-a_2(\mu)-\beta_2\big), & \\ \qquad\qquad a_1(\mu)-\beta_1\leqslant x\leqslant a_2(\mu)+\beta_2, & \\ \big(x-a_2(\mu)-\beta_2\big)e^{-x+a_2(\mu)+\beta_2}, & a_2(\mu)+\beta_2\leqslant x, \end{cases}$$

has the zero set of $\big(A-a_1(\mu)\big)\big(A-a_2(\mu)\big)$ as the type III *boundary of $\mathscr{A}$.*

Proof. Define $u(r,s,t)$, $v(r,s,t)$ by

$$u(r,s,t) = \int_0^r ty(y+a_1-a_2-r-s)(y+\tfrac12 a_1-\tfrac12 a_2-r)\,dy,$$

$$v(r,s,t) = \int_0^{-s} ty(y-a_1+a_2+r+s)(y-\tfrac12 a_1+\tfrac12 a_2+s)\,dy.$$

Then u and v are continuous functions from $\boldsymbol{R}^3$ to $\boldsymbol{R}^1$, such that $u(r,s,t) = tu(r,s,1)$, $v(r,s,t) = tv(r,s,1)$.

Now for any fixed positive value β_2 of s, we have $u(0,\beta_2,1)=0$, $u(r,\beta_2,1)$ is of order r^4 for large positive values of r, and $v(0,\beta_2,1)>0$, $v(r,\beta_2,1)$ is of order r for large positive values of r. Therefore there is a value $\beta_1>0$ of r for which $u(\beta_1,\beta_2,1)=v(\beta_1,\beta_2,1)$, and the homogeneity of u, v with respect to t allows us to choose a value k of t for which

$u(\beta_1, \beta_2, k) = v(\beta_1, \beta_2, k) = 1$. Now the remainder of the proof follows along the lines of Theorem 10.

Remarks. We have already observed that for any set which is a type III boundary of an admissible set $\mathscr{A}$, the intersection of that set with any line perpendicular to the μ-axis is necessarily empty, a singleton set or a doubleton set. (The set must also be closed, by definition.) Conversely, given any such set Γ, we may embed it in the image of two arcs $a_1(\mu)$, $a_2(\mu)$ with $a_1(\mu) \leqslant a_2(\mu)$ for all μ and apply Theorem 11 to show that Γ may be realized as a subset of the type III boundary points of an admissible set $\mathscr{A}$.

We have not said much about the smoothness with respect to μ of the functions $f(\mu, x)$ that we have constructed in the converse theorem, but it is clear, at least for these constructions that the smoothness is linked with any regularity conditions with respect to μ that we impose on a set Γ of boundary points of the appropriate type.

In conclusion, we remark that the results of Sections 5–7 make it clear that there is a great generality in those closed sets which may be realized as the set of boundary points of some admissible set of a particular type, somewhat contrasting with the situation when the dependence of the parameter μ is linear.

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