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ON LOCALLY H-CLOSED SPACES AND THE FOMIN H-CLOSED EXTENSION

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Locally H-closed spaces, introduced by Obreanu [10], were recently studied by Porter [11] with respect to their one-point H-closed extensions. The aim of this paper is to discuss locally H-closed spaces and their Fomin H-closed extensions. The paper contains a characterization of the Fomin H-closed extension (Fomin [4]; see also Iliadis and Fomin [7]). The other subject of this paper are various kinds of *perfect maps*, i.e., maps inducing on appropriate extensions the maps transforming remainders into remainders. It is proved, e.g., that perfect maps with respect to the Katětov extension coincide with perfect maps with respect to the Fomin extension.

An *ultrafilter* means always a maximal filter in the family of filters consisting of open subsets.

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1. Locally H-closed spaces. A topological space X is said to be *locally H-closed* if it is Hausdorff and for each $x \in X$ there exists an open neighbourhood U of the point x such that $\mathrm{Cl}_X U$ is H-closed.

The property of the space to be locally H-closed is in general not inherited by subspaces, even if they are open or closed, in this case in contrast to the local compactness.

Example. Let $\mathcal{T}$ be the natural topology on the open interval $J = (0, 1)$. Let $\mathcal{T}'$ be the topology on J generated by the family $\mathcal{T} \cup \{W\}$, where W is the set of all rational numbers on J. It is easy to see that $(J, \mathcal{T}')$ is locally H-closed. Obviously, $J \setminus W$ is closed but not locally H-closed: no infinite subset of $J \setminus W$ is H-closed. Analogously, W is open and not locally H-closed.

There exist even regularly open subsets of H-closed spaces which are not locally H-closed.

Example. Let us describe the Urysohn example of a non-compact minimal Hausdorff space. Let

$$X_0 = (\{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \cup \{0\} \cup \{-1, -\tfrac{1}{2}, -\tfrac{1}{3}, \ldots\}) \times \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}$$

and let

$$X = X_0 \cup \{(-1, 0)\} \cup \{(1, 0)\}.$$

Topology in X_0 is the usual product topology. Basic neighbourhoods of point $\{(-1, 0)\}$ are the sets

$$U_k = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\right\} \times \left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} \cup \{(-1, 0)\}$$

and basic neighbourhoods of the point $\{(1, 0)\}$ are the sets

$$V_k = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \times \left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} \cup \{(1, 0)\}.$$

The set

$$A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \cup \{(1, 0)\}$$

is regularly open in X and the subspace A is not locally H-closed. In fact, the point $\{(1, 0)\}$ has no neighbourhood whose closure is H-closed.

In the last section we show that the property of the space to be locally H-closed is inherited by regularly closed subspaces.

2. The Fomin H-closed extension σX. Fomin in [4] constructed for each Hausdorff space X an H-closed extension σX, known in the literature as the Fomin extension. The *Fomin extension* of X is the set

$$\sigma X = X \cup R_X, \tag{1}$$

where R_X is the family of all ultrafilters without adherence points, endowed with the topology generated by the sets $\sigma(U) = U \cup \{\xi \in R_X: U \in \xi\}$, where U is open in X, which form a base.

It is easy to check that X is a dense subspace of σX. The set σX of the Fomin extension is equal to the set of the Katětov H-closed extension τX (Katětov [8]). The topology in τX is stronger than that of σX. Hence σX is H-closed whenever it is Hausdorff. Let us note that

(2) if U and V are open in X and $U \cap V = \emptyset$, then $\sigma(U) \cap \sigma(V) = \emptyset$,

which follows from the calculation:

$$\begin{aligned}\sigma(U) \cap \sigma(V) &= (U \cup \{\xi \in R_X: U \in \xi\}) \cap (V \cup \{\xi \in R_X: V \in \xi\}) \\ &= U \cap V \cup \{\xi \in R_X: U \in \xi\} \cap \{\xi \in R_X: V \in \xi\} = \emptyset\end{aligned}$$

(because an ultrafilter does not contain disjoint sets).

The Hausdorff property for σX is a simple consequence of (2).

LEMMA 1. *If U is open in X, then*

$$\mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_X U \cup \{\xi \in R_X\colon\ U \in \xi\}. \tag{3}$$

Proof. 1. The inclusion $\mathrm{Cl}_X U \subset \mathrm{Cl}_{\sigma X} U$ is obvious.

In order to prove the inclusion $\mathrm{Cl}_X U \cup \{\xi \in R_X\colon\ U \in \xi\} \subset \mathrm{Cl}_{\sigma X} U$ it suffices to show that $\{\xi \in R_X\colon\ U \in \xi\} \subset \mathrm{Cl}_{\sigma X} U$. Let $\xi \in R_X$ be such that $U \in \xi$ and let $\sigma(V)$ be open base neighbourhood of ξ. Hence $V \in \xi$ and $U \cap V \neq \emptyset$. Thus $U \cap \sigma(V) \neq \emptyset$, which means that $\xi \in \mathrm{Cl}_{\sigma X} U$.

2. Let x be a point of X and $x \in \mathrm{Cl}_{\sigma X} U$. Then for each open set $\sigma(V)$ such that $x \in \sigma(V)$ there is $\sigma(V) \cap U \neq \emptyset$. Hence $U \cap V \neq \emptyset$ and $x \in V$ and, in consequence, $x \in \mathrm{Cl}_X U$.

Let $\xi \in R_X$ and $\xi \in \mathrm{Cl}_{\sigma X} U$. For each $V \in \xi$ the set $\sigma(V)$ is open base neighbourhood of ξ and $\sigma(V) \cap U \neq \emptyset$. Hence, for each $V \in \xi$, $V \cap U \neq \emptyset$ and in virtue of the maximality of the filter ξ, we get $U \in \xi$. Thus $\xi \in \{\eta \in R_X\colon U \in \eta\}$ which ends the proof of the lemma.

COROLLARY. *If X is a Hausdorff space, then the remainder R_X of the Fomin extension of X is* 0-*dimensional.*

Proof. The family $\{\sigma(U) \cap R_X\colon\ U$ is open in $X\}$ forms the base of the topology in R_X. Therefore it suffices to show that $\sigma(U) \cap R_X$ is closed in the remainder. By (3), there is

$$\mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_X U \cup \{\xi \in R_X\colon\ U \in \xi\}.$$

Since X is dense in σX, we have

$$\mathrm{Cl}_{\sigma X}\sigma(U) = \mathrm{Cl}_{\sigma X}\big(\sigma(U) \cap X\big) = \mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_X U \cup \{\xi \in R_X\colon\ U \in \xi\}.$$

Hence

$$\mathrm{Cl}_{\sigma X}\sigma(U) \cap R_X = \{\xi \in R_X\colon\ U \in \xi\} = \sigma(U) \cap R_X$$

is closed in the remainder.

Note. In fact, we have proved that the remainder R_X is 0-*dimensionally embedded* in σX (Flachsmeyer [3]), which means that there exists a base $\mathfrak{B}$ in σX such that for each $U \in \mathfrak{B}$ the boundary of U lies in X.

Extensions μX and $\mu' X$ are said to be *equivalent* if there exists a homeomorphism $h\colon \mu X \to \mu' X$ such that $h \mid X$ is the identity.

The following theorem gives a topological characterization of σX:

THEOREM 1. *An H-closed extension μX is equivalent to the Fomin extension σX iff there exists a base $\mathfrak{B}$ in μX such that*

(4) *for each V open in X there exists $U \in \mathfrak{B}$ such that $V = U \cap X$, and for each $U \in \mathfrak{B}$ there is* $\mathrm{Cl}_{\mu X} U = U \cup \mathrm{Cl}_X(U \cap X)$.

Note. Condition (4) can be reformulated in the form

(4′) *for each V open in X there exists $U \in \mathfrak{B}$ such that $V = U \cap X$, and for each $U \in \mathfrak{B}$ there is* $\mathrm{Bd}_{\mu X} U = \mathrm{Bd}_X(U \cap X)$, which is closely related to the known Katětov characterization of σX from [8]. Characterization given in (4′) is more convenient for our purpose.

Proof. Sufficiency. If $\xi \in R_X$, then let $\mathfrak{F}_\xi = \{U\colon U$ open in μX and $U \cap X \in \xi\}$. Since ξ is an ultrafilter in X and X is dense in μX, $\mathfrak{F}_\xi$ is an ultrafilter of open subsets of μX. Consider the map $\varphi\colon \sigma X \to \mu X$ defined by

$$(5)\qquad \begin{aligned} \varphi(x) &= x \quad \text{for each } x \in X, \\ \varphi(\xi) &= \bigcap \{\mathrm{Cl}_{\mu X} U\colon U \in \mathfrak{F}_\xi\} \quad \text{for each } \xi \in R_X \end{aligned}$$

(the intersection is a one-point set, because $\mathfrak{F}_\xi$ is an ultrafilter).

The map φ maps remainder into remainder. In fact, suppose to the contrary that for some $\xi \in R_X$ there is $\varphi(\xi) \in X$. Then for each $V \in \xi$ there exists $U \in \mathfrak{F}_\xi$ such that

$$\varphi(\xi) \in \mathrm{Cl}_{\mu X} U \cap X = \mathrm{Cl}_X(U \cap X) = \mathrm{Cl}_X V.$$

Since ξ is an ultrafilter without adherence points, we have a contradiction.

Since different ultrafilters have different limits, the map φ is one-to-one.

In order to prove continuity of the map φ we show that for each $U \in \mathfrak{B}$ there is

$$(6)\qquad \{\xi \in R_X\colon \varphi(\xi) \in U\} = \{\xi \in R_X\colon U \cap X \in \xi\}.$$

Let $U \in \mathfrak{B}$. It suffices to show that $\varphi(\xi) \in U$ iff $U \cap X \in \xi$.

If $\varphi(\xi) \in U$, then $\mathfrak{F}_\xi$ is the ultrafilter and, by (5), $U \in \mathfrak{F}_\xi$. Hence $U \cap X \in \xi$. Conversely, if $U \cap X \in \xi$, then $U \in \mathfrak{F}_\xi$ and, by (5), $\varphi(\xi) \in \mathrm{Cl}_{\mu X} U$. In virtue of (4), we get

$$\varphi(\xi) \in \mathrm{Cl}_{\mu X} U = \mathrm{Cl}_X(U \cap X) \cup U.$$

Since $\varphi(\xi) \in \mu X \setminus X$, $\varphi(\xi) \in U$.

Let $U \in \mathfrak{B}$. By (6) and (5) we have

$$\begin{aligned} \varphi^{-1}(U) &= U \cap X \cup \{\xi \in R_X\colon \varphi(\xi) \in U\} = U \cap X \cup \{\xi \in R_X\colon U \cap X \in \xi\} \\ &= \sigma(U \cap X). \end{aligned}$$

Hence $\varphi\colon \sigma X \to \mu X$ is a continuous map.

The map φ maps σX onto μX, because $\varphi(\sigma X)$ is closed in μX as a continuous image of the H-closed space, $X \subset \varphi(\sigma X)$, and X is dense in μX. Now it remains to show that φ is open. Clearly, each open base set in σX is of the form $\sigma(U \cap X)$, where $U \in \mathfrak{B}$. Then, by (6), we have

$$\begin{aligned} \varphi\big(\sigma(U \cap X)\big) &= \varphi(U \cap X) \cup \varphi(\{\xi \in R_X\colon U \cap X \in \xi\}) \\ &= U \cap X \cup \varphi\big(\varphi^{-1}(\{y \in \mu X \setminus X\colon y \in U\})\big) \\ &= U \cap X \cup U \cap (\mu X \setminus X) = U, \end{aligned}$$

which completes the proof.

Necessity follows by Lemma 1.

COROLLARY (Katětov [8], cf. also Flachsmeyer [3]). *The H-closed Fomin extension σX is equivalent to the Čech-Stone compact extension βX iff the boundary of each open set in X is compact.*

Proof. In virtue of (4′), $\mathrm{Bd}_{\sigma X} U$ is compact for each $U \in \mathfrak{B}$. Hence σX is regular and in consequence compact. And if σX is compact, then it is equivalent to βX. In fact, in virtue of (4), every two sets which are completely separated in X have disjoint closures in βX. The converse implication is obvious.

Let X be a Hausdorff space. The set of all H-closed extensions of X can be partially ordered: we say that an extension μX is *not less than an extension* $\mu' X$ iff there exists a map $\varphi\colon \mu X \to \mu' X$ completing the diagram

$$\begin{array}{ccc} X & \subset & \mu X \\ \cap & \swarrow_{\varphi} & \\ \mu' X & & \end{array}$$

It is easy to see that φ maps the remainder into the remainder.

THEOREM 2. *If X is a Hausdorff space, then the Fomin extension σX is the greatest one in the set of all H-closed extensions which have the remainder 0-dimensionally embedded.*

Proof. Let μX be the H-closed extension with the 0-dimensionally embedded remainder. Since the Katětov extension τX is the greatest one in the set of all H-closed extensions, there exists a map

$$\varphi\colon \tau X \xrightarrow{\text{onto}} \mu X$$

being identity on X and carrying the remainder onto the remainder. The set σX is equal to τX. Let ψ be the map equal to φ in the set-theoretical sense, carrying the space σX onto μX. We shall show that the map ψ is continuous.

Let $\mathfrak{B}$ be the base of the topology on μX such that for each $U \in \mathfrak{B}$ the boundary of U lies in X. In order to prove our theorem it suffices to show that for each $U \in \mathfrak{B}$ there is

$$\varphi^{-1}(U) = \sigma(U \cap X). \tag{7}$$

Clearly, $\varphi^{-1}(U) \cap X = U \cap X = \sigma(U \cap X) \cap X$. If $\xi \in \varphi^{-1}(U) \cap R_X$, then $U \cap X \in \xi$ (because for each $\xi \in R_X$ open neighbourhoods of ξ in the topology of τX are the sets $V \cup \{\xi\}$, where $V \in \xi$). Hence

$$\xi \in \{\eta \in R_X\colon\ U \cap X \in \eta\} = \sigma(U \cap X) \cap R_X.$$

Since the boundary of U lies in X, the boundary of $\varphi^{-1}(U)$ lies in X (because φ is continuous). If $\xi \in \sigma(U \cap X) \cap R_X$, then

$$\xi \in \mathrm{Cl}_{\tau X}\big(\sigma(U \cap X)\big) = \mathrm{Cl}_{\tau X}\big(\varphi^{-1}(U) \cap X\big) \subset \mathrm{Cl}_{\tau X}\varphi^{-1}(U).$$

Hence $\xi \in \varphi^{-1}(U)$. Thus condition (7) holds. Since $\psi^{-1}(U) = \varphi^{-1}(U)$, in view of (7) map ψ is continuous.

THEOREM 3. *For each regularly closed set $A \subset X$ the closure $\mathrm{Cl}_{\sigma X}A$ is an H-closed extension of A equivalent to the Fomin extension σA.*

Proof. First, let us show that $Y = \mathrm{Cl}_{\sigma X}A$, where $A = \mathrm{Cl}_X U$ for some U open in X, is an H-closed extension of A. Clearly, A is dense in Y. Let us note that

$$\mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U) = \mathrm{Cl}_X U \cup \{\xi \in R_X \colon U \in \xi\}, \tag{8}$$

which we shortly write as

$$Y = A \cup \sigma(U). \tag{9}$$

If $\xi \in \mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U) \cap R_X$, then $\sigma(V) \cap \mathrm{Cl}_X U \neq \emptyset$ for each $V \in \xi$. Hence $V \cap U \neq \emptyset$ for each $V \in \xi$, which means that $U \in \xi$.

Thus

$$\mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U) \subset \mathrm{Cl}_X U \cup \{\xi \in R_X \colon U \in \xi\}.$$

Obviously, $\mathrm{Cl}_X U \subset \mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U)$. If $\eta \in \{\xi \in R_X \colon U \in \xi\}$, then, for $U \in \eta$ and for each base neighbourhood $\sigma(V)$ of η, there is $\sigma(V) \cap \mathrm{Cl}_X U \neq \emptyset$. Hence $\eta \in \mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U)$. By (8) and Lemma 1, $Y = \mathrm{Cl}_{\sigma X}\sigma(U)$ is H-closed as a regularly closed subset of the H-closed space σX.

Let $\mathscr{P} = \{\sigma(V) \cap Y \colon V \text{ is open in } X\}$. Clearly, $\mathscr{P}$ is a base in Y. In order to prove the equivalence of Y and σA it is sufficient to show, by Theorem 1, that for each $W \in \mathscr{P}$ there is

$$\mathrm{Cl}_Y W = W \cup \mathrm{Cl}_A(W \cap A).$$

Let $W = \sigma(V) \cap Y$. By (8) and Lemma 1 we get

$$\begin{aligned}
\mathrm{Cl}_Y W &= \mathrm{Cl}_Y[\sigma(V) \cap Y] = Y \cap \mathrm{Cl}_{\sigma X}[\sigma(V) \cap Y] \\
&= Y \cap \mathrm{Cl}_{\sigma X}\big[\sigma(V) \cap \big(\mathrm{Cl}_X U \cup \sigma(U)\big)\big] \\
&= Y \cap \mathrm{Cl}_{\sigma X}[\sigma(U \cap V) \cup \sigma(V) \cap \mathrm{Cl}_X U] \\
&= Y \cap \big[\mathrm{Cl}_X(U \cap V) \cup \sigma(U \cap V) \cup \mathrm{Cl}_{\sigma X}\big(\sigma(V) \cap \mathrm{Cl}_X U\big)\big].
\end{aligned}$$

It is easy to check that

$$\mathrm{Cl}_{\sigma X}\big(\sigma(V) \cap \mathrm{Cl}_X U\big) = \mathrm{Cl}_X(V \cap \mathrm{Cl}_X U) \cup \sigma(U \cap V).$$

One can show in a way analogous to (2) that $\sigma(U \cap V) = \sigma(U) \cap \sigma(V)$. Then by (9) we get

$$\begin{aligned}
\mathrm{Cl}_Y W &= Y \cap [\mathrm{Cl}_X(V \cap \mathrm{Cl}_X U) \cup \sigma(V \cap U)] \\
&= A \cap \mathrm{Cl}_X(V \cap A) \cup Y \cap \sigma(U) \cap \sigma(V) = \mathrm{Cl}_A(W \cap A) \cup W.
\end{aligned}$$

Thus the theorem is proved.

The next theorem is analogous to a known theorem which says that a Tychonoff space X is locally compact iff for each its compactification cX the remainder $cX \setminus X$ is closed (cf. Engelking [2], p. 137); here R_X denotes the remainder in the Fomin extension σX.

THEOREM 4. *Let X be a Hausdorff space. The following conditions are equivalent*:

(I) *X is locally H-closed.*

(II) *The remainder R_X is compact.*

(III) *The remainder R_X is closed.*

Proof. 1. Let X be a locally H-closed space and let $\mathscr{V}$ be a covering of the remainder by open base sets. Since X is locally H-closed space, there exists, for each point $x \in X$, an open set $U \subset X$ such that $\mathrm{Cl}_X U$ is H-closed. Let

$$\mathscr{U} = \{\sigma(U)\colon \mathrm{Cl}_X U \text{ is } H\text{-closed}\} \cup \{\sigma(V)\colon \sigma(V) \cap R_X \in \mathscr{V}\}.$$

The family $\mathscr{U}$ is an open covering of σX. Since σX is H-closed, there exists a finite subfamily $\mathscr{S}$ of $\mathscr{U}$ such that

$$\bigcup \{\mathrm{Cl}_{\sigma X} A\colon A \in \mathscr{S}\} = \sigma X.$$

Hence

$$\bigcup \{(\mathrm{Cl}_{\sigma X} A \cap R_X)\colon A \in \mathscr{S}\} = R_X.$$

In virtue of Lemma 1 we have

$$\begin{aligned} R_X \cap \mathrm{Cl}_{\sigma X} \sigma(U) &= R_X \cap \mathrm{Cl}_{\sigma X} U = R_X \cap (\mathrm{Cl}_X U \cup \{\xi \in R_X\colon U \in \xi\}) \\ &= \{\xi \in R_X\colon U \in \xi\} = R_X \cap \sigma(U). \end{aligned}$$

Hence

$$R_X = \bigcup \{R_X \cap A\colon A \in \mathscr{S}\}. \tag{10}$$

Note that $R_X \cap A = \emptyset$ unless $A \cap R_X \in \mathscr{V}$. In fact, if $A = \sigma(U)$ and $\sigma(U) \cap R_X \notin \mathscr{V}$, then $\mathrm{Cl}_X U$ is H-closed. By Lemma 1 we have

$$\{\xi \in R_X\colon U \in \xi\} \cup \mathrm{Cl}_X U = \mathrm{Cl}_{\sigma X} U \subset \mathrm{Cl}_{\sigma X}(\mathrm{Cl}_X U) = \mathrm{Cl}_X U.$$

Hence

$$\sigma(U) \cap R_X = \{\xi \in R_X\colon U \in \xi\} = \emptyset.$$

Therefore, by (10), the remainder R_X is compact.

2. Let us assume that the remainder R_X is closed. Hence X is open in σX. Then for each $x \in X$ there exists an open base set $\sigma(U)$ such that $x \in \sigma(U) \subset X$. Since $\sigma(U) = U \cup \{\xi \in R_X\colon U \in \xi\}$,

$$\{\xi \in R_X\colon U \in \xi\} = \emptyset. \tag{11}$$

The set $\mathrm{Cl}_{\sigma X} \sigma(U)$ is H-closed, because it is a regularly closed subset of the H-closed space σX. In virtue of Lemma 1 and (11), there is

$$\mathrm{Cl}_{\sigma X} \sigma(U) = \mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_X U \cup \{\xi \in R_X\colon U \in \xi\} = \mathrm{Cl}_X U.$$

Hence each point $x \in X$ has an open neighbourhood the closure of which is H-closed.

To complete the proof of our theorem, it is now sufficient to notice obvious implication (II) $\Rightarrow$ (III).

COROLLARY. *A regularly closed subset of a locally H-closed space is locally H-closed.*

Proof. In fact, if $A \subset X$ is regularly closed, then, by Theorem 3, $\mathrm{Cl}_{\sigma X} A$ is an H-closed extension of A equivalent to σA. By (8) the set $\{\xi \in R_X \colon U \in \xi\}$ is the remainder of this extension. In virtue of Theorem 4, the remainder R_X is closed in σX. Since the set $\{\xi \in R_X \colon U \in \xi\}$ is closed in R_X, it is also closed in $\mathrm{Cl}_{\sigma X} A$. Thus A is locally H-closed.

A continuous map $c\colon X \to Y$ is said to be a *contraction* provided it is one-to-one and onto.

COROLLARY. *Each locally H-closed space X has a contraction onto a minimal Hausdorff space.*

Proof. Let $A = R_X \cup \{x\}$, where $x \in X$. By (II) of Theorem 4, A is compact. Let $\sigma X/A$ be the quotient space and let $\varphi\colon \sigma X \to \sigma X/A$ be the natural map. Since the map $\sigma\colon X \subset \sigma X$ is embedding, the composition $\varphi \circ \sigma\colon X \subset \sigma X \to \sigma X/A$ is a contraction. Clearly, $\sigma X/A$ is a Hausdorff, and therefore it is an H-closed space. If we contract the topology on $\sigma X/A$ to the semi-regular one, we get a contraction of X to a minimal Hausdorff space.

3. Various kinds of perfect maps. A (continuous) map $f\colon X \to Y$ is said to be *τ-proper* ([1]) provided there exists a (unique) map $\tau f\colon \tau X \to \tau Y$ completing the diagram

$$\begin{array}{ccc} X & \subset & \tau X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau f} \\ Y & \subset & \tau Y \end{array} \tag{12}$$

(τZ denoting the Katětov extension of Z).

If, in addition, τf carries the remainder into the remainder, i.e., if $\tau f(\tau X \setminus X) \subset \tau Y \setminus Y$, it is said to be *$\tau$-perfect*.

It was proved in [1] that a (continuous) map $f\colon X \to Y$ is τ-perfect iff the following conditions hold:

(I) f is τ-proper;

(II) for each ultrafilter $\xi \in \tau X \setminus X$ and each $y \in Y$ there exists $U \in \xi$ such that $f^{-1}(y) \cap \mathrm{Cl}_X U = \emptyset$;

(III) $f(A)$ is closed for each regularly closed $A \subset X$.

Let us call a map $f\colon X \to Y$ *σ-perfect* iff there exists a (unique) map σf completing the diagram like (12), with σ instead of τ, and such that $\sigma f(\sigma X \setminus X) \subset \sigma Y \setminus Y$.

THEOREM 5. *If X and Y are Hausdorff and $f\colon X \to Y$ is continuous, then the following conditions are equivalent*:

(I) *f is τ-perfect*;

(II) *f is σ-perfect*;

(III) *for each ultrafilter ξ in X without adherence points, the family $\eta = \{V$ open in $Y\colon f^{-1}(V) \in \xi\}$ is the ultrafilter in Y without adherence points.*

Proof. 1. (II) $\Rightarrow$ (III). Let ξ be an ultrafilter in X without adherence points. Since $\xi \in \sigma X \setminus X = R_X$, $\sigma f(\xi) \in \sigma Y \setminus Y = R_Y$. Clearly, η is a filter. It suffices to show that the ultrafilter $\sigma f(\xi)$ is contained in η. If $V \in \sigma f(\xi)$, then $\sigma f(\xi) \in \sigma(V)$. By the continuity of σf, there exists an open base set $\sigma(U)$ such that $\xi \in \sigma(U)$ and $\sigma f(\sigma(U)) \subset \sigma(V)$. Then $U \in \xi$ and

$$f(U) = f(\sigma(U) \cap X) = \sigma f(\sigma(U) \cap X) \subset \sigma f(\sigma(U)) \cap Y \subset \sigma(V) \cap Y = V.$$

Thus $U \subset f^{-1}(V)$ which means that $f^{-1}(V) \in \xi$ and in consequence $V \in \eta$. Hence η is an ultrafilter without adherence points.

2. (III) $\Rightarrow$ (II). The map σf is defined by the formula

$$\begin{aligned} &\sigma f(x) = f(x) \quad \text{for each } x \in X, \\ &\sigma f(\xi) = \{V \text{ open in } Y\colon f^{-1}(V) \in \xi\} \quad \text{for each } \xi \in R_X. \end{aligned} \tag{13}$$

By the assumption, σf maps the remainder into the remainder, i.e., $\sigma f(R_X) \subset R_Y$. To show the continuity of σf, let us take a base open set $\sigma(V)$ in σY. We get

$$\begin{aligned} (\sigma f)^{-1}(\sigma(V)) &= (\sigma f)^{-1}(V \cup \{\eta \in R_Y\colon V \in \eta\}) \\ &= (\sigma f)^{-1}(V) \cup (\sigma f)^{-1}(\{\eta \in R_Y\colon V \in \eta\}) \\ &= f^{-1}(V) \cup \{\xi \in R_X\colon f^{-1}(V) \in \xi\} = \sigma(f^{-1}(V)), \end{aligned}$$

which means that σf is continuous.

3. (I) $\Rightarrow$ (III). Let ξ be an ultrafilter in X without adherence points. By the assumption, there exists a map τf filling up the diagram (12) and such that $\tau f(\tau X \setminus X) \subset \tau Y \setminus Y$. Hence $\tau f(\xi)$ is an ultrafilter without adherence points. It is now sufficient to show that $\tau f(\xi) \subset \eta$. To do this let $V \in \tau f(\xi)$. Then $V \cup \{\tau f(\xi)\}$ is an open neighbourhood of the point $\tau f(\xi) \in \tau Y \setminus Y$. By the continuity of τf, there exists an open neighbourhood $U \cup \{\xi\}$ of the point ξ such that $f(U \cup \{\xi\}) \subset V \cup \{\tau f(\xi)\}$. Hence $f(U) \subset V$ and, in consequence, $U \subset f^{-1}(V)$. Since $U \in \xi$, there is $f^{-1}(V) \in \xi$ which means that $V \in \eta$.

4. (III) $\Rightarrow$ (I). The map $\tau f\colon \tau X \to \tau Y$ is defined as follows:

$$\begin{aligned} &\tau f(x) = f(x) \quad \text{for each } x \in X, \\ &\tau f(\xi) = \{V \text{ open in } Y\colon f^{-1}(V) \in \xi\} \quad \text{for each } \xi \in \tau X \setminus X. \end{aligned} \tag{14}$$

Clearly, $\tau f(\tau X\setminus X) \subset \tau Y\setminus Y$. It remains to show that τf is continuous map. Continuity at points of X is obvious, and if $\xi \in \tau X\setminus X$, then each open neighbourhood of the point $\tau f(\xi)$ is the form $V\cup\{\tau f(\xi)\}$, where $V \in \tau f(\xi)$. From the definition of τf, we have $f^{-1}(V) \in \xi$. Hence $f^{-1}(V)\cup\{\xi\}$ is an open neighbourhood of ξ and $\tau f(f^{-1}(V)\cup\{\xi\}) = V\cup\{\tau f(\xi)\}$ which ends our proof.

It is known from Henriksen and Isbell [5] that in the case of compact extensions, the family of perfect maps with respect to the Čech-Stone extension is equal to the family of perfect maps in the usual sense (i.e., closed maps such that $f^{-1}(y)$ are compact for $y \in Y$). It was proved in [1] that there exist maps which are perfect but not τ-perfect.

From Theorems 4 and 5 it follows immediately that if $f\colon X \xrightarrow{\text{onto}} Y$ is τ-perfect and Y is locally H-closed, then X is locally H-closed. In fact, the remainder of σX is the counter-image by σf of the remainder of σY which is closed.

A (continuous) map $f\colon X \to Y$ is said to be *skeletal* ([9], p. 13) if, for each open subset V of Y, we have

$$\mathrm{Int}_X f^{-1}(\mathrm{Cl}_Y V) \subset \mathrm{Cl}_X f^{-1}(V). \tag{15}$$

It is easy to see that a (continuous) map is skeletal iff the counter-image of each dense and open set is dense and open. Herrlich and Strecker introduced (cf. [6]) *pseudo-open* maps (i.e., maps such that, for each $A \subset X$, $\mathrm{Int}_X A \neq \emptyset$ implies $\mathrm{Int}_Y \mathrm{Cl}_Y f(A) \neq \emptyset$). It is worth to mention that both notions coincide.

THEOREM 6. *A (continuous) map $f\colon X \to Y$ is pseudo-open iff it is skeletal.*

Proof. 1. Let f be a skeletal map and let $A \subset X$ be such that $\mathrm{Int}_X A \neq \emptyset$. Suppose that $\mathrm{Int}_Y \mathrm{Cl}_Y f(A) = \emptyset$. Since f is skeletal, $f^{-1}(\mathrm{Cl}_Y f(A))$ is the boundary set in X. Then we get

$$\emptyset = \mathrm{Int}_X f^{-1}(\mathrm{Cl}_Y f(A)) \supset \mathrm{Int}_X f^{-1}(f(A)) \supset \mathrm{Int}_X A,$$

a contradiction.

2. Let f be a pseudo-open map. It is sufficient to show that the counter-image of each boundary and closed set in Y is boundary in X. Suppose that $B \subset Y$ is boundary and closed and $\mathrm{Int}_X f^{-1}(B) \neq \emptyset$. Since the map f is pseudo-open, $\mathrm{Int}_Y \mathrm{Cl}_Y f(f^{-1}(B)) \neq \emptyset$. Then $\mathrm{Int}_Y B \neq \emptyset$, a contradiction.

We shall prove the following

LEMMA 2. *A (continuous) map $f\colon X \to Y$ is skeletal iff, for each ultrafilter ξ of open subsets of X, the family $\eta = \{V \text{ open in } Y\colon f^{-1}(V) \in \xi\}$ is an ultrafilter.*

Proof. 1. Let us assume that, for each ultrafilter ξ, the family $\eta = \{V \text{ open in } Y\colon f^{-1}(V)\in\xi\}$ is an ultrafilter and let G be an arbitrary dense and open set in Y. Clearly, $G\in\eta$. Hence $f^{-1}(G)\in\xi$ for each ultrafilter ξ. Thus $f^{-1}(G)$ is dense in X.

2. Let us assume that $f\colon X\to Y$ is a skeletal map and let ξ be an arbitrary ultrafilter in X. We show that $\eta = \{V \text{ open in } Y\colon f^{-1}(V)\in\xi\}$ is an ultrafilter. Clearly, η is a filter in Y. Suppose that an open set $U\subset Y$ does not belong to η. Let $V = Y\setminus \mathrm{Cl}_Y U$. Clearly, $U\cup V$ is dense and open in Y, therefore $f^{-1}(U\cup V)$ is dense and open in X. Then $f^{-1}(U)\cup \cup f^{-1}(V)\in\xi$. Since $f^{-1}(U)\notin\xi$, $f^{-1}(V)\in\xi$. Thus $V\in\eta$ and $U\cap V=\emptyset$ which means that η is an ultrafilter.

It is easy to see that all Hausdorff spaces and all skeletal maps form a category. It seems natural to require that all maps in consideration be elements of this category.

THEOREM 7. *For a map $f\colon X\to Y$ there exists a skeletal map $\sigma f\colon \sigma X\to\sigma Y$ completing the diagram*

$$\begin{array}{ccc} X & \subset & \sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma f} \\ Y & \subset & \sigma Y \end{array} \tag{16}$$

and carrying the remainder into the remainder iff f is skeletal and

(I) *for each $y\in Y$ and each $\xi\in R_X$ there exists an open set U such that $f^{-1}(y)\subset U$ and $U\notin\xi$,*

(II) *$f(A)$ is closed for each A regularly closed in X.*

Proof. 1. Let us assume that a map $\sigma f\colon \sigma X\to\sigma Y$ completes diagram (16) and $\sigma f(R_X)\subset R_Y$. First, we show that f is a skeletal map. If $V\subset Y$ is dense and open, then $\sigma(V) = V\cup R_Y$ is dense and open in σY. Since σf is skeletal, $(\sigma f)^{-1}(\sigma(V)) = f^{-1}(V)\cup(\sigma X\setminus X)$ is dense and open in σX. Thus $f^{-1}(V)$ is dense and open in X. Now we verify condition (I). If $y\in Y$ and $\xi\in R_X$, then $y\neq\sigma f(\xi)$. Then there exist sets U and V such that $y\in U$; $V\in\sigma f(\xi)$ and $U\cap V=\emptyset$. Therefore $f^{-1}(U)\cap f^{-1}(V)=\emptyset$, $f^{-1}(y)\subset f^{-1}(U)$ and $\xi\in(\sigma f)^{-1}(\sigma V)$. Since σf maps remainder into remainder, $f^{-1}(V)\in\xi$. Hence $f^{-1}(U)\notin\xi$.

Now we verify condition (II). Let A be a regularly closed set in X, i.e. $A=\mathrm{Cl}_X U$, where U is open in X. By Lemma 1 we get

$$\mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_X U\cup\{\xi\in R_X\colon U\in\xi\}.$$

Since $\mathrm{Cl}_{\sigma X} U = \mathrm{Cl}_{\sigma X}\sigma(U)$ is regularly closed in σX, it is H-closed. Then

$$\sigma f(\mathrm{Cl}_{\sigma X} U) = \sigma f(\mathrm{Cl}_X U)\cup\sigma f(\{\xi\in R_X\colon U\in\xi\})$$

is closed in σY. By the assumption that $\sigma f(R_X)\subset R_Y$, $f(\mathrm{Cl}_X U) = f(A)$ is closed in X.

2. Let us assume that $f\colon X \to Y$ is skeletal and satisfies conditions (I) and (II). The map $\sigma f\colon \sigma X \to \sigma Y$ is defined as follows:

$$(17)\qquad \begin{aligned} \sigma f(x) &= f(x) \quad \text{for each } x \in X, \\ \sigma f(\xi) &= \{V \text{ open in } Y\colon f^{-1}(V) \in \xi\} \quad \text{for each } \xi \in R_X. \end{aligned}$$

From Lemma 2 it follows immediately that $\sigma f(\xi)$ is an ultrafilter, provided ξ is an ultrafilter belonging to R_X. We show that $\sigma f(\xi)$ is an ultrafilter without adherence points. Suppose, a contrario, that

$$\bigcap \{\mathrm{Cl}_Y V\colon V \in \sigma f(\xi)\} = y \in Y.$$

Hence, by (17),

$$\bigcap \{\mathrm{Cl}_Y V\colon f^{-1}(V) \in \xi\} = y.$$

By (I), there exists a set U such that $f^{-1}(y) \subset U$ and $U \notin \xi$. Hence $f^{-1}(y) \subset \mathrm{Int}_X\mathrm{Cl}_X U$ and $\mathrm{Int}_X\mathrm{Cl}_X U \notin \xi$. By (II), $V = Y \setminus f(X \setminus \mathrm{Int}_X\mathrm{Cl}_X U)$ is open and $y \in V$. Since y is an adherence point of the ultrafilter $\sigma f(\xi)$, there is $V \in \sigma f(\xi)$. Thus $f^{-1}(V) \in \xi$. On the other hand,

$$f^{-1}(V) = X \setminus f^{-1}\big(f(X \setminus \mathrm{Int}_X\mathrm{Cl}_X U)\big) \subset \mathrm{Int}_X\mathrm{Cl}_X U.$$

Hence $\mathrm{Int}_X\mathrm{Cl}_X U \in \xi$, a contradiction. Therefore, $\sigma f\colon \sigma X \to \sigma Y$ is the map from σX to σY and $\sigma f(R_X) \subset R_Y$.

In order to finish our proof it is sufficient to show continuity and skeletality of σf. To do this let us note that, for each V open in Y, there is

$$(18)\qquad (\sigma f)^{-1}(\{\eta \in R_Y\colon V \in \eta\}) = \{\xi \in R_X\colon f^{-1}(V) \in \xi\}.$$

If $\xi \in (\sigma f)^{-1}(\{\eta \in R_Y\colon V \in \eta\})$, then $V \in \sigma f(\xi)$. Hence, by (17), $f^{-1}(V) \in \xi$. Conversely, let $f^{-1}(V) \in \xi$. Then we have $V \in \sigma f(\xi)$ and, in consequence, $\xi \in (\sigma f)^{-1}(\{\eta \in R_Y\colon V \in \eta\})$. By (18) we get

$$\begin{aligned} (\sigma f)^{-1}\big(\sigma(V)\big) &= (\sigma f)^{-1}(V \cup \{\eta \in R_Y\colon V \in \eta\}) \\ &= (\sigma f)^{-1}(V) \cup (\sigma f)^{-1}(\{\eta \in R_Y\colon V \in \eta\}) \\ &= f^{-1}(V) \cup \{\xi \in R_X\colon f^{-1}(V) \in \xi\} = \sigma\big(f^{-1}(V)\big). \end{aligned}$$

Therefore, the counterimage of each base open set $\sigma(V)$ is open. The map σf is skeletal. In fact, if $W \subset \sigma Y$ is dense and open, then $W \cap Y$ is dense and open in Y. Since f is skeletal, $f^{-1}(W \cap Y)$ is dense in X. Hence $(\sigma f)^{-1}(W)$ is dense and open in σX.

From Theorem 7 it follows immediately assertion of the corollary to Theorem 4, which says that each regularly closed subset A of a locally H-closed space X is locally H-closed. In fact, embedding $A \subset X$ is a skeletal map and conditions (I) and (II) of Theorem 7 are evidently satisfied. Hence, in virtue of Theorem 7 and Theorem 4, the set A is locally H-closed.

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