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On a certain series

by MAREK KUCZMA (Katowice)

In the investigations of the integrable solutions of a functional equation it has turned out [3] that the uniqueness or lack of uniqueness of the solution depends on the convergence or divergence of a series of the form

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n} m(x_i), \tag{1}$$

where m is a positive function and the sequence of points x_i is generated by iteration:

$$x_{i+1} = f(x_i), \quad i = 0, 1, 2, \ldots \tag{2}$$

The purpose of the present paper is to give some criteria for the convergence or divergence of series (1).

In the sequel we make the following general hypothesis concerning the function f:

(H) *The function $f(x)$ is defined and continuous in an interval $I = (0, d)$, and $0 < f(x) < x$ in I. Moreover, $f(x)$ may be written in the form*

$$f(x) = sx + x^{p+1}h(x),$$

where $0 \leqslant s \leqslant 1$ and $p > 0$ are constants, and the function $h(x)$ is bounded in I.

We write

$$h_0 = \liminf_{x\to 0+0} |h(x)|, \qquad H_0 = \limsup_{x\to 0+0} |h(x)|;$$

obviously $0 \leqslant h_0 \leqslant H_0 < \infty$.

Under conditions (H) the sequence x_i tends to zero for every $x_0 \in I$ ([2], [5]). Therefore it is clear that if m_1 and m_2 are two positive functions on I and $m_1(x) \leqslant m_2(x)$ in a neighbourhood of zero, then the convergence of the series $\sum\prod m_2(x_i)$ implies the convergence of series $\sum\prod m_1(x_i)$, whereas the divergence of the latter implies the convergence of the former. For this reason we shall investigate the convergence of series (1) only for certain test functions m. As the test functions we choose:

$$m(x) = 1 - Cx^q, \tag{3}$$

$$m(x) = 1 - C(\log x^{-1})^{-q}, \tag{4}$$

$$m(x) = 1 - C(\log\log x^{-1})^{-q}, \tag{5}$$

where C and q are positive constants. In the case of functions (4) and (5) we assume that d is so small that $m(x)$ is defined and positive in I.

If, besides the sequence x_i, we consider the sequence $y_i = x_{i+j}, j > 0$, then we have

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n} m(x_i) = \sum_{n=1}^{j}\prod_{i=1}^{n} m(x_i) + \prod_{i=1}^{j} m(x_i)\Big\{\sum_{n=1}^{\infty}\prod_{i=1}^{n} m(y_i)\Big\}. \tag{6}$$

Relation (6) shows that we do not spoil the convergence (or divergence) of series (1) if we restrict x_0 to a small neighbourhood of zero.

In the sequel we shall repeatedly use the fact (cf. e.g. [1], § 40) that

$$\sum_{i=i_0}^{n}(ai+b)^{-k} = \begin{cases} \dfrac{(an+b)^{1-k}}{a(1-k)} + E_n & \text{if } k \neq 1,\ k > 0, \\ a^{-1}\log(an+b) + E_n & \text{if } k = 1, \end{cases} \tag{7}$$

where E_n is a convergent sequence. Here $a > 0$ and b are arbitrary constants, and i_0 is such that $ai + b > 0$ for $i \geqslant i_0$. The sequence E_n depends, of course, on a, b, k and i_0.

We write

$$r(x) = 1 - m(x),$$

$$P_n = \prod_{i=1}^{n} m(x_i) = \prod_{i=1}^{n}\big(1 - r(x_i)\big).$$

For an arbitrary number $D > 1$ we have

$$-Dr(x) \leqslant \log\big(1 - r(x)\big) \leqslant -r(x) \tag{8}$$

provided that

$$0 \leqslant r(x) \leqslant (D-1)/D. \tag{9}$$

Since in all cases (3), (4), (5) we have $\lim_{x\to 0+0} r(x) = 0$, $r(x) > 0$, condition (9) may be realized if we restrict ourselves to a suitably small neighbourhood of zero. As was remarked previously (cf. relation (6)) this does not diminish the generality of our considerations.

Relation (8) implies the estimation

$$\exp\Big(-D\sum_{i=1}^{n} r(x_i)\Big) \leqslant P_n \leqslant \exp\Big(-\sum_{i=1}^{n} r(x_i)\Big). \tag{10}$$

THEOREM 1. *Suppose that $s = 1$, $h_0 > 0$, and $m(x)$ is given by* (3). *If $p < q$, or $p = q$ and $C < h_0 p$, then series* (1) *diverges for every $x_0 \in I$.*

Proof. As was shown in [5], for every number $M > (h_0 p)^{-1/p}$ there exists an index N such that

$$x_i \leqslant M i^{-1/p} \quad \text{for } i \geqslant N$$

(x_0 is considered as fixed). Hence

$$r(x_i) \leqslant C M^q i^{-q/p} \quad \text{for } i \geqslant N$$

and

$$\sum_{i=1}^{n} r(x_i) \leqslant C M^q \sum_{i=N}^{n} i^{-q/p} + A \quad \text{for } n \geqslant N,$$

where $A = \sum_{i=1}^{N-1} r(x_i)$ is a constant. If $p < q$, then we have by (7)

$$\sum_{i=1}^{n} r(x_i) \leqslant \frac{CM^q}{1-k} n^{1-k} + CM^q E_n + A \quad \text{for } n \geqslant N, \tag{11}$$

where $k = q/p > 1$. Writing $K = DCM^q(1-k)^{-1}$, $K_n = DCM^q E_n + DA$, we have from (10) and (11)

$$P_n \geqslant \exp(-Kn^{1-k} - K_n) \quad \text{for } n \geqslant N. \tag{12}$$

Since the sequence K_n converges and $k > 1$, the expression on the right-hand side of (12) converges to a positive limit and thus series (1) diverges.

If $p = q$, then by (7)

$$\sum_{i=1}^{n} r(x_i) \leqslant CM^q \log n + CM^q E_n + A \quad \text{for } n \geqslant N,$$

and writing $K = DCM^q$, $K_n = DCM^q E_n + DA$, we have as previously

$$P_n \geqslant \exp(\log n^{-K} + K_n) = n^{-K} \exp(-K_n) \quad \text{for } n \geqslant N. \tag{13}$$

Now, we may choose M very close to $(h_0 p)^{-1/p}$, i.e., since $p = q$, M^q very close to $(h_0 p)^{-1}$. Similarly, D may be chosen very close to 1. Since by hypothesis $C(h_0 p)^{-1} < 1$, we may make $K < 1$ and the divergence of series (1) results from (13).

THEOREM 2. *Suppose that $s = 1$ and $m(x)$ is given by* (3). *If $p > q$, or $p = q$ and $C > H_0 p$, then series* (1) *converges for every $x_0 \in I$.*

Proof. By the results of [5] for every number $M < (H_0 p)^{-1/p}$ there exists an index N such that

$$x_i \geqslant M i^{-1/p} \quad \text{for } i \geqslant N,$$

whence, as previously,

$$\sum_{i=1}^{n} r(x_i) \geqslant CM^q \sum_{i=N}^{n} i^{-q/p} + A \quad \text{for } n \geqslant N.$$

This together with relations (7) and (10) yields in the case $k = q/p < 1$ the estimation

$$P_n \leqslant \exp(-Kn^{1-k} - K_n) \quad \text{for } n \geqslant N, \tag{14}$$

and in the case $p = q$ the estimation

$$P_n \leqslant n^{-K} \exp(-K_n) \quad \text{for } n \geqslant N, \tag{15}$$

where in (14) $K = CM^q(1-k)^{-1} > 0$, and in (15) $K = CM^q > 1$ provided we choose M sufficiently large. K_n is a convergent sequence.

In case (15) the convergence of series (1) is obvious. In case (14) it is enough to note that

$$\lim_{n\to\infty} \frac{-Kn^{1-k} - K_n}{\log n} = -\infty < -1,$$

which implies the convergence of series (1) (cf. [4], p. 43).

THEOREM 3. *Suppose that $0 < s < 1$ and $m(x)$ is given by* (4). *If $q > 1$, or $q = 1$ and $C < \log s^{-1}$, then series* (1) *diverges for every $x_0 \in I$.*

Proof. In the present case the sequence $x_i s^{-i}$ converges to a positive limit ([5], [2], p. 138). Thus there exists a positive constant M such that

$$x_i \leqslant Ms^i, \quad i = 1, 2, \ldots$$

Hence

$$r(x_i) \leqslant C(i \log s^{-1} - \log M)^{-q} \quad \text{for } i \geqslant N,$$

or, with $a = \log s^{-1} > 0$, $b = -\log M$,

$$r(x_i) \leqslant C(ai+b)^{-q} \quad \text{for } i \geqslant N,$$

where N is such that $ai+b > 0$ for $i \geqslant N$. Writing

$$A = \sum_{i=1}^{N-1} r(x_i),$$

we have

$$\sum_{i=1}^{n} r(x_i) \leqslant C \sum_{i=N}^{n} (ai+b)^{-q} + A \quad \text{for } n \geqslant N.$$

If $q > 1$, we obtain hence in view of (7)

$$\sum_{i=1}^{n} r(x_i) \leqslant \frac{C}{a(1-q)} (an+b)^{1-q} + CE_n + A \quad \text{for } n \geqslant N$$

and by (10)

(16) $$P_n \geqslant \exp[-K(an+b)^{1-q}-K_n] \quad \text{for } n \geqslant N,$$

where $K = \dfrac{CD}{a(1-q)}$, $K_n = CDE_n + DA$. This implies the divergence of series (1), since the sequence on the right-hand side of (16) tends to a positive limit.

If $q = 1$, then by (7)

$$\sum_{i=1}^{n} r(x_i) \leqslant Ca^{-1}\log(an+b)+CE_n+A \quad \text{for } n \geqslant N,$$

i.e., with $K = DCa^{-1}$, $K_n = DCE_n + DA$,

$$P_n \geqslant (an+b)^{-K}\exp(-K_n) \quad \text{for } n \geqslant N.$$

Choosing D sufficiently close to 1 we may make $K < 1$, and the divergence of series (1) follows.

THEOREM 4. *Suppose that* $0 < s < 1$ *and* $m(x)$ *is given by* (4). *If* $q < 1$, *or* $q = 1$ *and* $C > \log s^{-1}$, *then series* (1) *converges for every* $x_0 \in I$.

Proof. Similarly as previously we arrive at the estimations

(17) $$P_n \leqslant \exp[-K(an+b)^{1-q}-K_n] \quad \text{for } n \geqslant N, \text{ if } q < 1,$$

and

(18) $$P_n \leqslant (an+b)^{-K}\exp(-K_n) \quad \text{for } n \geqslant N, \text{ if } q = 1,$$

where in (17) $K = CD/a(1-q) > 0$, and in (18) $K = Ca^{-1} > 1$, whereas in both cases $K_n = CE_n + A$ is a convergent sequence. The convergence of series (1) results now as in the proof of Theorem 2.

THEOREM 5. *Suppose that* $s = 0$ *and* $m(x)$ *is given by* (5). *If* $q > 1$, *or* $q = 1$ *and* $C < \log(p+1)$, *then series* (1) *diverges for every* $x_0 \in I$.

Proof. As was shown in [5], there exists a positive constant $M < 1$ and an index N such that

$$x_i \leqslant M^{(p+1)^i} \quad \text{for } i \geqslant N.$$

Hence

$$r(x_i) \leqslant C\big(i\log(p+1)+\log\log M^{-1}\big)^{-q} \quad \text{for } i \geqslant N,$$

or, with $a = \log(p+1) > 0$, $b = \log\log M^{-1}$,

$$r(x_i) \leqslant C(ai+b)^{-q} \quad \text{for } i \geqslant N,$$

where we assume that N has been chosen so large that $ai+b > 0$ for $i \geqslant N$. Further the proof is identical as that of Theorem 3.

THEOREM 6. *Suppose that* $s = 0$, $h_0 > 0$, *and* $m(x)$ *is given by* (5). *If* $q < 1$, *or* $q = 1$ *and* $C > \log(p+1)$, *then series* (1) *converges for every* $x_0 \in I$.

Proof. The condition $h_0 > 0$ furnishes the estimation

$$x_i \geqslant M^{(p+1)^i} \quad \text{for } i \geqslant N$$

with a positive constant $M < 1$ (cf. [5]). As in the proof of Theorem 5 we derive hence the inequality

$$r(x_i) \geqslant C(ai+b)^{-q} \quad \text{for } i \geqslant N$$

with $a = \log(p+1) > 0$, $b = \log\log M^{-1}$, and further the proof runs as that of Theorem 4.

Remark. Note that our test functions are comparable with each other; i.e., if m_1, m_2, m_3 have form (3), (4), (5), respectively (with different constants C and q), then

$$m_3(x) < m_2(x) < m_1(x)$$

in a neighbourhood of zero.

References

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