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On functions with vanishing local derivative

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Abstract. There is given the general form of functions f from R^q to a Banach space. The functions are assumed to be locally integrable in the sense of Bochner and to fulfil the differential equation $f^{(m)} = 0$, where m is a multi-index and the derivative is in the sense of Sobolev.

It is known that if a real function f in R^q is of class C^m, where $m = (\mu_1, \ldots, \mu_q)$ (μ_i are non-negative integers), and $f^{(m)}(x) = 0$ for $x = (\xi_1, \ldots, \xi_q) \in R^q$, then

$$f(x) = \sum_{0 \leqslant i \leqslant \mu_1 - 1} \xi_1^i f_{1i}(x) + \ldots + \sum_{0 \leqslant i \leqslant \mu_q - 1} \xi_q^i f_{qi}(x), \tag{1}$$

where the functions f_{ji} are constant with respect to the variables ξ_i, respectively (if $\mu_j = 0$ for some j, then the corresponding sum in (1) should be replaced by 0).

The proof of this fact can be found, for instance, in [5]. Theorem 28.4 in [5] is alternatively formulated also for the case of integrable functions and then $f^{(m)}$ is meant in the distributional sense. However, the proof given there is not adequate for integrable functions. The theorem in this case can be deduced from [4].

In [8], the theorem is generalized to the case of functions from R^q to a Hilbert space, which are locally integrable in the sense of Bochner. But the method applied in the proof fails, if the value of functions are in an arbitrary Banach space.

In this paper, we shall show that the theorems is also true for locally integrable functions from R^q to an arbitrary Banach space.

1. Let $\mathscr{X}$ be a Banach space. The points of the q-dimensional Euclidean space R^q are denoted by $x = (\xi_1, \ldots, \xi_q)$, $y = (\eta_1, \ldots, \eta_q)$, $z = (\zeta_1, \ldots, \zeta_q)$, $\ldots$, and the set of all non-negative integer points of R^q by P^q. We adopt the notation: $x + y = (\xi_1 + \eta_1, \ldots, \xi_q + \eta_q)$, $\lambda x = (\lambda\xi_1, \ldots, \lambda\xi_q)$, $xy = (\xi_1 \eta_1, \ldots, \xi_q \eta_q)$, $x^m = \xi_1^{\mu_1} \ldots \xi_q^{\mu_q}$, where $m = (\mu_1, \ldots, \mu_q) \in P^q$ and λ is a real number. The letter e_i denotes the point whose ith coordinate is 1 and all the remaining ones are 0. The following notation will also be used: $e = (1, \ldots, 1)$, $0 = (0, \ldots, 0)$.

In what follows, we shall use, as far as possible, the notation from [2] and [8].

All integrals considered in this paper are meant as Bochner integrals (see [6]).

We adopt the following definition of the difference operator:

$$\Delta^{(m,h)} f = \Delta_1^{(\mu_1,\chi_1)} \ldots \Delta_q^{(\mu_q,\chi_q)} f, \tag{2}$$

where $m \in P^q$, $h = (\chi_1, \ldots, \chi_q) \in R^q$ and the symbols on the right-hand side mean the iteration of difference operators of one variable:

$$\Delta_i^{(\mu_i,\chi_i)} f(x) = \sum_{0 \leqslant j \leqslant \mu_i} (-1)^{\mu_i - j} \binom{\mu_i}{j} f(x + je_i \chi_i)$$

with the convention:

$$\Delta_i^{(0,\chi_i)} f = f.$$

By the mth *local derivative* of a function $f\colon R^q \to \mathscr{X}$ we mean a function g such that

$$\lim_{h \to 0} \int_I \left| \frac{1}{h^m} \Delta^{(m,h)} f(x) - g(x) \right| dx = 0 \tag{3}$$

for every bounded interval I in R^q (see [8], [9]). In order to have this definition sensible, we assume that the integrand in (3) is a locally integrable function of x. Evidently, if f is locally integrable, and the local derivative $D^m_{\mathrm{loc}} f$ of f exists, then it is also locally integrable.

If a vector valued function f, defined in R^q, is of class C^m in R^q, then its local derivative of order m exists and is equal to the ordinary derivative: $D^m_{\mathrm{loc}} f = D^m f$.

In [10], it is proved that the local derivative and Sobolev's derivative are equivalent, i.e., if one of the derivatives exists, then the other does and they are equal.

2. By the convolution of two functions f and g we mean the function which assigns to each point $x \in R^q$ the integral

$$\int_{R^q} f(x-t) g(t)\, dt. \tag{4}$$

The convolution exists at a point x, whenever the product $f(x-t)g(t)$ is Bochner integrable with respect to t. We assume that the values of one of the functions f and g is in $\mathscr{X}$ and of the other is in R^1. Convolution (4) will be denoted by $f * g$.

LEMMA 1. *If a locally integrable vector function f has the m-th local derivative and a real function g is of class C^m and of bounded support, then the convolution $f*g$ is of class C^m and*

$$D^m(f*g) = (D^m_{\text{loc}} f)*g.$$

This lemma follows as a simple corollary of Theorem 1 and Lemmas 4 and 5 in [9].

THEOREM 1. *If the function $f\colon R^1 \to \mathscr{X}$ is continuous and*

$$\Delta^{(\mu+1,\chi)} f(\xi) = 0$$

for arbitrary $\xi, \chi \in R^1$, then f is a polynomial of degree $\leqslant \mu$.

Proof. If $\Delta^{(\mu+1,\chi)} f = 0$, then $D^{\mu+1}_{\text{loc}} f = 0$, by the definition of local derivative.

Consider a real delta-sequence $\{\delta_n\}$ (see [2], p. 116). In view of Lemma

$$D^{\mu+1}\varphi_n = (D^{\mu+1}_{\text{loc}} f)*\delta_n,$$

where $\varphi_n = f*\delta_n$. Since $D^{\mu+1}_{\text{loc}} f = 0$, we have $D^{\mu+1}\varphi_n = 0$. This implies, in view of Theorem 8.14.2 (see [3]), that

$$\varphi_n(\xi) = a_{0n} + a_{1n}\xi + a_{2n}\xi^2 + \ldots + a_{\mu n}\xi^\mu.$$

Let $\gamma_i \in R^1$ $(i = 0, 1, \ldots, \mu)$, $\gamma_i \neq \gamma_j$ for $i \neq j$. Consider the system of equations,

$$c_{0n} = a_{0n} + a_{1n}\gamma_0 + \ldots + a_{\mu n}\gamma_0^\mu$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_{\mu n} = a_{0n} + a_{1n}\gamma_\mu + \ldots + a_{\mu n}\gamma_\mu^\mu,$$

where $c_{jn} = \varphi_n(\gamma_j)$ $(j = 0, 1, \ldots, \mu)$. Solving this system of equations we get

$$a_{in} = \sum_{0\leqslant j\leqslant\mu} (-1)^{j+i+2}\frac{W_{j+1,i+1}}{W} c_{jn}, \tag{5}$$

where

$$W - \begin{vmatrix} 1, \gamma_0, \ldots, \gamma_0^\mu \\ \ldots\ldots\ldots \\ 1, \gamma_\mu, \ldots, \gamma_\mu^\mu \end{vmatrix}$$

and $W_{j+1,i+1}$ is the minor of the determinant W obtained by omitting the $(i+1)$th column and the $(j+1)$th line.

Since

$$\varphi_n(\xi) = (f*\delta_n)(\xi) \to f(\xi) \quad \text{almost uniformly}$$

(p. 76, Theorem 3.1.1, see [2]), we have $c_{jn} \to c_j$ as $n \to \infty$. Hence, by (5), the limits $\lim_{n\to\infty} a_{in} = a_i$ exist and

$$a_i = \sum_{0\leqslant j\leqslant\mu} (-1)^{j+i+2} \frac{W_{j+1,i+1}}{W} c_j.$$

Thus we have $f(\xi) = a_0 + a_1\xi + \ldots + a_\mu \xi^\mu$ and the proof is complete.

Remark. Theorem 1 was proved, in [1], by T. Angheluta for the case of real valued functions. The theorem of Angheluta was generalized, in [8], to the case of functions with values in a given Hilbert space. But the method used in the proof cannot be applied in the case of functions with values in an arbitrary Banach space.

3. Let T_i^q denote the set of all zero-one systems, in which i elements are equal to 1 and $(q-i)$ elements are equal to 0. For given $k \in T_i^q$, $a = (\alpha_1, \ldots, \alpha_q) \in R^q$ and a function $f\colon R^q \to \mathscr{X}$ we adopt the notation

$$S_k^a f(x) = f(x - kx + ka).$$

In case $k = e_i$ the symbol $S_k^a f(x)$ depends only on the ith coordinate of a and we shall write then $S_i^{\alpha_i} f(x)$ instead of $S_{e_i}^a f(x)$.

We define the operation $A_i = \sum_{1\leqslant j\leqslant i} \eta_j S_i^{\alpha_j}$ for arbitrary $\eta_j, \alpha_j \in R^1$ on functions $f\colon R^q \mapsto \mathscr{X}$ letting

$$[(\sum_{1\leqslant j\leqslant i} \eta_j S_i^{\alpha_j}) f](x) = \sum_{1\leqslant j\leqslant i} \eta_j S_i^{\alpha_j} f(x).$$

Moreover, by 1 we mean the identical operation, i.e., $1f = f$. In particular, the symbol $(1 - A_i) f$ means $f - A_i f$.

It is easy to see that

$$A_i A_j f = A_j A_i f \quad \text{for } i \neq j.$$

Let $m = (\mu_1, \ldots, \mu_q)$ be a fixed element of P^q.

Given a function $f\colon R^q \to \mathscr{X}$ and systems $(\alpha_{i1}, \ldots, \alpha_{i\mu_i})$ and $(\beta_{i1}, \ldots, \beta_{i\mu_i}) = b_i$, where $\alpha_{ij}, \beta_{ij} \in R^1$ for $i = 1, \ldots, q$, we define

$$\nabla_i^{(\mu_i, b_i)} f(x) = \Big(1 - \sum_{0\leqslant j,k\leqslant \mu_i - 1} \alpha_{\mu_i - j, k+1}\, \xi_i^{j+1} S_i^{\beta_{i,k+1}}\Big) f(x) \tag{6}$$

for $i = 1, \ldots, q$, moreover, we adopt

$$\nabla_i^{(0, b_i)} f = f.$$

It is easy to check that

$$\begin{aligned} \nabla_i^{(\mu_i, b_i)} \Delta_j^{(\mu_j, \chi_j)} f &= \Delta_j^{(\mu_j, \chi_j)} \nabla_i^{(\mu_i, b_i)} f \quad \text{for } i \neq j, \\ \nabla_i^{(\mu_i, b_i)} \nabla_j^{(\mu_j, b_j)} f &= \nabla_j^{(\mu_j, b_j)} \nabla_i^{(\mu_i, b_i)} f \quad \text{for } i \neq j. \end{aligned} \tag{7}$$

We have the following

LEMMA 2. *If* $f: R^q \to \mathscr{X}$ *is a continuous function such that*

$$\Delta^{(m+e,h)} f(x) = 0 \quad \text{for } h, x \in R^q$$

and

(8) $$S_i^0 f(x) = 0 \quad \textit{for } i = 1, \ldots, q,$$

then for every $b_i = (\beta_{i1}, \ldots, \beta_{i\mu_i})$, *where* $\beta_{ij} \in R^1$ $(i = 1, \ldots, q;\ j = 1, \ldots, \mu_i)$ *and* $\beta_{ij} \neq \beta_{ik}$ *for* $j \neq k$, *we have*

$$\nabla_1^{(\mu_1, b_1)} \ldots \nabla_q^{(\mu_q, b_q)} f(x) = 0 \quad \textit{for } x \in R^q.$$

Proof. By (2), we have

$$\Delta_1^{(\mu_1+1, \chi_1)} F(x) = 0,$$

where

$$F(x) = \Delta_2^{(\mu_2+1, \chi_2)} \ldots \Delta_q^{(\mu_q+1, \chi_q)} f(x).$$

By Theorem 1 and (8),

$$G(\xi) = A_1 \xi^{\mu_1} + \ldots + A_{\mu_1} \xi,$$

where $G(\xi) = F(\xi, \xi_2, \ldots, \xi_q)$ and A_i $(i = 1, \ldots, \mu_1)$ are functions of the variables $\xi_2, \ldots, \xi_q$.

Let $\beta_i \in R^1$ $(i = 1, \ldots, \mu_1)$ with $\beta_i \neq \beta_j$ for $i \neq j$. From the system of equations

$$G(\beta_1) = A_1 \beta_1^{\mu_1} + \ldots + A_{\mu_1} \beta_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G(\beta_{\mu_1}) = A_1 \beta_{\mu_1}^{\mu_1} + \ldots + A_{\mu_1} \beta_{\mu_1},$$

we get

$$A_i = \sum_{0 \leq j \leq \mu_1 - 1} (-1)^{i+j+1} G(\beta_j) \frac{V_{i,j+1}}{V},$$

where

$$V = \begin{vmatrix} \beta_1^{\mu_1}, & \ldots, & \beta_1 \\ \cdot & \cdot & \cdot \\ \beta_{\mu_1}^{\mu_1}, & \ldots, & \beta_{\mu_1} \end{vmatrix}$$

and $V_{i,j+1}$ is the minor of the determinant V obtained by omitting the ith column and the $(j+1)$th line. Hence

(9) $$G(\xi) = \sum_{0 \leq j,k \leq \mu_1 - 1} \xi^{j+1} \alpha_{\mu_1 - j, k+1} G(\beta_{k+1}),$$

where

(10) $$\alpha_{jk} = (-1)^j \frac{V_{jk}}{V} \quad (j, k = 1, \ldots, \mu_1).$$

Formula (9) can be written in the form

$$F(x) = \sum_{0 \leqslant j,k \leqslant \mu_1 - 1} \xi_1^{j+1} \alpha_{\mu_1 - j, k+1} S_1^{\beta_{1,k+1}} F(x).$$

By the definition of $\nabla_1^{(\mu_1, b_1)}$, we have

$$\nabla_1^{(\mu_1, b_1)} F = \nabla_1^{(\mu_1, b_1)} \Delta_2^{(\mu_2 + 1, \chi_2)} \ldots \Delta_q^{(\mu_q + 1, \chi_q)} f = 0,$$

Now, by induction, we obtain the assertion, in view of (7).

If $b = (\beta_{i,j})$, where $\beta_{ij} \in R$ $(i = 1, \ldots, q;\ j = 1, \ldots, \mu_i)$ and $c = (\gamma_1, \ldots, \gamma_q) \in P^q$ $(1 \leqslant \gamma_i \leqslant \mu_i)$, then we shall write

$$b_c = (\beta_{1\gamma_1}, \ldots, \beta_{q\gamma_q}).$$

It is easy to check that the following equalities are true:

$$\prod_{1 \leqslant j \leqslant i} S_{e_j}^{b_c} f = S_{(e_1 + \ldots + e_i)}^{b_c} f, \qquad \prod_{1 \leqslant j \leqslant i} x^{e_j} = x^{(e_1 + \ldots + e_i)}.$$

In other words, we have the following formulae:

(11) $$\prod_{1 \leqslant j \leqslant i} S_{l_j}^{b_c} f = S_{(l_1 + \ldots + l_i)}^{b_c} f, \qquad \prod_{1 \leqslant j \leqslant i} x^{l_j} = x^{(l_1 + \ldots + l_i)}$$

for $l_j \in T_1^q$.

Let

$$a(k, m, n, c) = \prod_{1 \leqslant i \leqslant q} \alpha_{\varkappa_i(\mu_i - \nu_i), \varkappa_i(\gamma_i + 1)},$$

with the convention: $\alpha_{00} = 1$.

It is easy to prove that

$$\prod_{1 \leqslant j \leqslant i} a(e_j, m, n, c) = a\Big(\sum_{1 \leqslant j \leqslant i} e_j, m, n, c\Big).$$

This can be written in the form

(12) $$\prod_{1 \leqslant j \leqslant i} a(l_j, m, n, c) = a\Big(\sum_{1 \leqslant j \leqslant i} l_j, m, n, c\Big) \quad \text{for } l_j \in T_1^q.$$

THEOREM 2 (cf. [8]). *If $F\colon R^q \to \mathscr{X}$ is a continuous function such that*

$$\Delta^{(m+e,h)} F(x) = 0 \quad \textit{for } h, x \in R^q$$

and

$$S_i^0 F(x) = 0 \quad \textit{for } i = 1, \ldots, q,$$

then for arbitrary $b_c = (\beta_{1\gamma_1}, \ldots, \beta_{q\gamma_q})$ *with* $\beta_{ij} \neq \beta_{ik}$ *for* $j \neq k$ *we have*

$$(13)\qquad F(x) = \sum_{i=1}^{q} \sum_{k \in T_i^q} \sum_{0 \leqslant c, n \leqslant m-e} (-1)^{i-1} a(k, m, n, c)\, x^{k(n+e)} S_k^{b_c} F(x)$$

for $x \in R^q$, *where* $c = (\gamma_1, \ldots, \gamma_q)$, $n = (\nu_1, \ldots, \nu_q)$, $m = (\mu_1, \ldots, \mu_q)$ *and* $a(k, m, n, c)$ *are some constants. The constants* $a(k, m, n, c)$ *are given by the formula*

$$a(k, m, n, c) = \prod_{i=1}^{q} \alpha_{\varkappa_i(\mu_i - \nu_i), \varkappa_i(\gamma_i + 1)}, \qquad \alpha_{00} = 1,$$

where the numbers α_{jk} $(j, k = 1, \ldots, \mu_i)$ *are defined by* (10).

Proof. By Lemma 2 and formula (6) for arbitrary $b_c = (\beta_{1\gamma_1}, \ldots, \beta_{q\gamma_q})$ $(1 \leqslant \gamma_i \leqslant \mu_i;\ 1 \leqslant i \leqslant q)$, with $\beta_{ij} \neq \beta_{ik}$ for $i \neq k$, we get

$$(14)\qquad \prod_{i=1}^{q} \Big(1 - \sum_{0 \leqslant \gamma_i, \nu_i \leqslant \mu_i - 1} \alpha_{\mu_i - \nu_i, \gamma_i + 1}\, \xi_i^{\nu_i + 1} S_i^{\beta_{i,\gamma_i + 1}}\Big) F(x) = 0;$$

if $\mu_j = 0$ for some j, then the corresponding sum in (14) should be replaced by 0.

Let

$$a(k, m, n, c) = \prod_{i=1}^{q} \alpha_{\varkappa_i(\mu_i - \nu_i), \varkappa_i(\gamma_i + 1)}$$

with the convention: $\alpha_{00} = 1$.

Equality (14) can be replaced by the equality

$$(15)\qquad \prod_{l \in T_1^q} \Big(1 - \sum_{0 \leqslant c, n \leqslant m-e} a(l, m, n, c)\, x^{l(n+e)} S_l^{b_c}\Big) F(x) = 0.$$

By multiplying factors on the left-hand side of (15) and applying identities (11) and (12) we obtain the equation

$$\Big(1 - \sum_{i=1}^{q} \sum_{k \in T_i^q} \sum_{0 \leqslant c, n \leqslant m-e} (-1)^{i-1} a(k, m, n, c)\, x^{k(n+e)} S_k^{b_c}\Big) F(x) = 0$$

which is equivalent to (13).

4. The symbol

$$\int_{x_0}^{x} f(t)\, dt^k$$

will denote the iterated integral of order $k \in P^q$ of a locally integrable function f (see [2], p. 69).

We say that a function f is *constant* with respect to x^k $(k \in T_i^q)$, if f is constant with respect to these coordinates ξ_i for which $\varkappa_i = 1$. A hyperplane

π is said to be *perpendicular to the axis* x^k ($k \in T_i^q$) if π is perpendicular to the axis ξ_i for which $\varkappa_i = 1$.

We shall use the following lemmas, proved in [9] and [10].

LEMMA 3 (cf. [9], Theorem 3). *Let* $f\colon R^q \to \mathscr{X}$ *be a locally integrable function. We have* $D^m_{\text{loc}} f = 0$ *iff* $\Delta^{(m,h)} f(x) = 0$ *holds for each fixed* $h \in R^q$ *and almost all* $x \in R^q$.

LEMMA 4 (cf. [9], Theorem 4). *Let* $f\colon R^q \to \mathscr{X}$ *be a locally integrable function. Then*

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

is a local primitive for f.

LEMMA 5 (cf. [10]). *Let* $f\colon R^q \to \mathscr{X}$ *be a locally integrable function. Then*

$$\Delta^{(e,h)} \int_{x_0}^{x} f(t)\,dt = \int_{x}^{x+h} f(t)\,dt$$

and

$$\Delta^{(m,h)} \int_{x}^{x+h} f(t)\,dt = \int_{x}^{x+h} \Delta^{(m,h)} f(t)\,dt$$

for $m \in P^q$ *and* $x, h \in R^q$.

THEOREM 3. *Let* $f\colon R^q \to \mathscr{X}$ *be a locally integrable function. If* $D^m_{\text{loc}} f = 0$ *then*

$$f(x) = \sum_{1 \leqslant i \leqslant q} \sum_{k \in T_i^q} \sum_{0 \leqslant n \leqslant m-e} (-1)^{i-1} x^{kn} f_{kn} \quad \text{a.e.,} \tag{16}$$

where the functions f_{kn} *are locally integrable in the* $(q-i)$*-dimensional hyperplane perpendicular to the axis* x^k *and constant with respect to* x^k *(if* $\mu_j = 0$ *for some* j*, then the corresponding sum in* (16) *is adopted to be* 0).

Proof. It follows from Lemma 3 that

$$\Delta^{(m,h)} f(x) = 0 \tag{17}$$

for each fixed $h \in R^q$ and almost all $x \in R^q$.

Let

$$F(x) = \int_{0}^{x} f(t)\,dt. \tag{18}$$

By Lemmas 4, 5 and equality (17), we obtain

$$\Delta^{(m+e,h)} F = \Delta^{(m,h)} \Delta^{(e,h)} \int_{0}^{x} f(t)\,dt = \Delta^{(m,h)} \int_{x}^{x+h} f(t)\,dt = \int_{x}^{x+h} \Delta^{(m,h)} f(t)\,dt = 0.$$

Since the function F satisfies assumptions of Theorem 2, we have

$$\int_0^x f(t)\,dt = \sum_{i=1}^{q} \sum_{k\in T_i^q} \sum_{0\le c,n\le m-e} (-1)^{i-1} a(k, m, n, c)\, x^{k(n+e)} S_k^{b_c} F(x)$$

for $x \in R^q$.

Using the definition of the iterated integral and the Fubini theorem, we have

$$\begin{aligned} x^{k(n+e)} S_k^{b_c} F(x) &= x^{k(n+e)} \int_0^x \Big(\int_0^{b_c} f(t)\,dt^k\Big) dt^{e-k} \\ &= x^{kn} \int_0^x \Big(\int_0^{b_c} f(t)\,dt^k \int_0^x dt^k\Big) dt^{e-k} = x^{kn} \int_0^x \Big(\int_0^{b_c} f(t)\,dt^k\Big) dt. \end{aligned}$$

Thus

$$\int_0^x f(t)\,dt = \sum_{i=1}^{q} \sum_{k\in T_i^q} \sum_{0\le c,n\le m-e} (-1)^{i-1} a(k, m, n, c)\, x^{kn} \int_0^x \Big(\int_0^{b_c} f(t)\,dt^k\Big) dt.$$

Since

$$knx^{k(n-e)} \int_0^x \Big(\int_0^{b_c} f(t)\,dt^k\Big) dt = knx^{kn} S_k^{b_c} F,$$

we have, in view of Lemma 4 and the Fubini theorem,

$$\sum_{i=1}^{q} \sum_{k\in T_i^q} \sum_{0\le c,n\le m-e} (-1)^{i-1} x^{kn} a(k, m, n, c) \int_0^{b_c} f(t)\,dt^k + knS_k^{b_c} F(x) = f(x),$$

where the above equality holds almost everywhere.

Denoting

$$\sum_{0\le c\le m-e} a(k, m, n, c)\Big[\int_0^{b_c} f(t)\,dt^k + knS_k^{b_c} F\Big] = f_{kn},$$

we have for almost all $x \in R^q$ the equation

$$f(t) = \sum_{1\le i\le q} \sum_{k\in T_i^q} \sum_{0\le n\le m-e} (-1)^{i-1} x^{kn} f_{kn},$$

where f_{kn} are locally integrable functions in the $(q-i)$-dimensional hyperplane perpendicular to the axis x^k and constant with respect to x^k.

Remark. Note that in view of the result of paper [10] the local derivative can be replaced by Sobolev's derivative.

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