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On singular non-linear parabolic differential inequalities in unbounded domains

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Abstract. We consider a system of second order differential inequalities of the type:

$$u_t^i < F^i(t, x, U(t, x), u_x^i(t, x), u_{xx}^i(t, x)) \quad (i = 1, \ldots, m), \tag{0}$$

where $x = (x_1, \ldots, x_n)$, $U = (u^1, \ldots, u^m)$, $u_x^i = (u_{x_1}^i, \ldots, u_{x_n}^i)$ and u_{xx}^i is the matrix of second order derivatives with respect to x. Under suitable assumptions on the functions F^i we prove a theorem on the inequalities of type (0) and a theorem on the existence of the solution of the first boundary problem. The assumptions on the right-hand sides of (0) correspond to those imposed in the case of a singular parabolic or elliptic operator in the linear case (see [4] or [3]).

1. The main purpose of this paper is to extend the maximum principle to the singular differential inequalities of parabolic type. The principal reason for the investigation of such inequalities is the notion of a singular elliptic equation which has been studied extensively by several authors (see [4] and the earlier literature cited there). The existence of the solution of Fourier's problem for singular parabolic equation will be studied by the author in [3]. In connection with the notion of a singular elliptic or parabolic operator, in the natural way the problem of the maximum principle for singular inequalities arises.

Before formulating the results we introduce some definitions and notations (see [5]):

Let $D \subset R^{n+1}$ be a cylindrical domain of the form $D = (0, T) \times \Omega$, where Ω is an unbounded domain in R^n. The boundary of D consists of two unbounded domains S^0 and S^T and a surface Σ lying between the hyperplane $t = 0$ and $t = T$. S^0 is included in $\{(t, x); t = 0\}$ and S^T in $\{(t, x); t = T\}$. We denote by d_pD the parabolic boundary of the cylinder D, $d_pD = S^0 + \Sigma$.

Let M and K be positive constants. The function $f(t, x)$ defined on D will be said to belong to the class $E_2(M, K)$ if:

$$|f(t, x)| \leqslant M\exp(K|x|^2) \quad \text{for all } (t, x) \in D.$$

Let $F^i(t, x, Z, Q, R)$ $(i = 1, \ldots, m)$, where $Z = (z_1, \ldots, z_m)$, $Q = (q_1, \ldots, q_n)$ and $R = \{r_{jk}\}$ is an $n \times n$ real symmetric matrix, be defined for $(t, x) \in D$, Z, Q, R arbitrary.

According to the definition given by J. Szarski in [5], the function $F^i(t, x, Z, Q, R)$ is said to be *elliptic* with respect to $U(t, x) = (u^1, \ldots, u^m)$ of class $C^1(D)$, if for any two real symmetric matrices $R = \{r_{jk}\}, \bar{R} = \{\bar{r}_{jk}\}$ such that:

$$\sum (r_{jk} - \bar{r}_{jk}) \lambda_j \lambda_k \leqslant 0$$

we have

$$F^i\big(t, x, U(t, x), u_x^i(t, x), R\big) \leqslant F^i\big(t, x, U(t, x), u_x^i(t, x), \bar{R}\big).$$

A function $U(t, x) = \big(u^1(t, x), \ldots, u^m(t, x)\big)$ is called *regular in* D, if u^i are continuous in $\bar{D}$ and the derivatives u_x^i, u_{xx}^i, u_t^i are continuous in D.

The system of functions $F^i(t, x, Z, Q, R)$ is said to *satisfy condition* W with respect to Z if for every index i_0 the inequalities $z^j \leqslant \bar{z}^j$ $(j \neq i_0)$, $z^{i_0} = \bar{z}^{i_0}$ imply

$$F^{i_0}(t, x, Z, Q, R) \leqslant F^{i_0}(t, x, \bar{Z}, Q, R)$$

for all $(t, x) \in (0, T) \times R^n$ and Q, R.

Throughout the paper by $\theta(t, x)$ we denote a continuous function on $\bar{D}$, vanishing on a compact set S contained in $d_p D$, having continuous derivatives θ_t, θ_x, θ_{xx} in $\bar{D}$ and such that $\theta(t, x) \equiv b$ for $(t, x) \in \{[0, T] \times \times (|x| \geqslant \bar{R})\} \cap D$, where $\bar{R}$ and b are positive constants.

2. The following theorem is an extension of the maximum principle to the singular differential inequalities of parabolic type.

THEOREM 1. *Suppose that:*

1° *The functions* $F^i(t, x, Z, Q, R)$ $(i = 1, \ldots, m)$, *defined for* $(t, x) \in \in D$, Z, Q, R *arbitrary, satisfy condition* W.

2° *The inequalities*

$$F^i(t, x, Z, Q, R) - F^i(t, x, \bar{Z}, \bar{Q}, \bar{R}) \cdot \operatorname{sgn}(z^i - \bar{z}^i)$$
$$\leqslant \frac{L_0}{\theta(t, x)^{p_1}} \sum |r_{jk} - \bar{r}_{jk}| + \frac{L_1(1 + |x|)}{\theta(t, x)^{p_2}} \sum |q^j - \bar{q}^j| + L_2(1 + |x|^2) \sum |z^i - \bar{z}^i|$$

hold true $(i = 1, \ldots, m, j, k = 1, \ldots, n)$, *where the constants* L_0, L_1, L_2, p_1, p_2 *are positive*, $|x| = \big(\sum x_i^2\big)^{1/2}$.

3° *The functions* $u^i(t, x)$ *and* $v^i(t, x)$ *are regular in* D *and belong to* $E_2(M, K_0)$ $(i = 1, \ldots, m)$.

4° *For every* i_0 *the function* F^{i_0} *is elliptic with respect to* $\{u^i(t, x)\}$.

5° $u_t^i \leqslant F^i(t, x, U, u_x^i, u_{xx}^i)$, $v_t^i \geqslant F^i(t, x, V, v_x^i, v_{xx}^i)$ *for* $(t, x) \in \bar{D} \setminus d_p D$.

6° $u^i(t, x) \leqslant v^i(t, x)$ *for* $(t, x) \in d_p D$.

Under these assumptions we have

$$u^i(t, x) \leqslant v^i(t, x) \quad \textit{for} \ (t, x) \in \bar{D}.$$

The arguments which we give here are based entirely on [1] and [2].

Introduce the growth damping factor:

$$H(t, x) = \exp\big(K\theta^{p+2}|x|^2\exp(at) + \nu t\big), \tag{2.1}$$

where $p = \max\{p_1, p_2 - 1\}$ and K is a constant such that $K \cdot b > K_0$; positive constants a and ν will be chosen later. A direct computation gives:

$$H_{x_j} = (p+2)\theta^{p+1}\theta_{x_j}K|x|^2\exp(at)H + 2Kx_j\theta^{p+2}\cdot\exp(at)\cdot H,$$

$$H_t = (p+2)\theta^{p+1}\theta_t K|x|^2\exp(at)\cdot H + aK\theta^{p+2}|x|^2\exp(at)H + \nu H,$$

$$\begin{aligned} H_{x_jx_k} = {} & (p+2)(p+1)\theta^p\theta_{x_k}\theta_{x_j}K|x|^2\exp(at)H + \\ & + (p+2)\theta^{p+1}\theta_{x_jx_k}|x|^2\exp(at)\cdot KH + (p+2)\theta^{p+1}\theta_{x_j}K2x_k\exp(at)\cdot H + \\ & + (p+2)^2\theta^{2p+2}\theta_{x_j}\theta_{x_k}K^2|x|^4\exp(2at)H + (p+2)\theta^{2p+3}\theta_{x_j}2K^2x_k|x|^2\exp(2at)\cdot H + \\ & + 2K\delta_{jk}\theta^{p+2}\exp(at)H + 2Kx_j(p+2)\theta^{p+1}\theta_{x_k}\exp(at)H + \\ & + 2K^2\theta^{2p+3}\theta_{x_k}x_j(p+2)|x|^2\exp(2at)H + 4K^2x_jx_k\theta^{2p+4}\exp(2at)H. \end{aligned}$$

It follows from the properties of the function θ

$$\sum_j |H_{x_j}| \leqslant C_1\exp(at)\theta^{p+1}|x|H, \tag{2.2}$$

$$\sum_{j,k} |H_{x_jx_k}| \leqslant C_2\exp(2at)\theta^p(1+|x|^2)H. \tag{2.3}$$

Introduce the following operator:

$$\begin{aligned} F(H) = {} & [L_0C_2'\exp(2at)(1+|x|^2) + L_1C_1'(1+|x|)\cdot|x|\cdot\exp(at) + \\ & + L_2(1+|x|^2)\cdot m - (p+2)K\theta^{p+1}\theta_t|x|^2\exp(at) - aK\theta^{p+2}|x|^2\exp(at) - \nu]\cdot H, \end{aligned} \tag{2.4}$$

where

$$C_1' = C_1\max\theta^{p-p_1}, \quad C_2' = C_2\max\theta^{p+1-p_2}.$$

For $|x| \geqslant R$ there exists a $a > 0$ such that $F(H) < 0$ for $(t, x) \in [0, 1/2a] \times (|x| \geqslant R) \cap D$ and every $\nu > 0$.

Now choose $\nu > 0$ sufficiently large to have $F(H) < 0$ for $|x| < R$ and $0 < t < 1/\nu$.

Hence it is clear that $F(H) < 0$ for $(t, x) \in [0, A] \times R^n$, where $A = \min\{1/\nu, 1/2a\}$.

Now, as in [2], putting

$$u^i = \bar{u}^iH, \quad v^i = \bar{v}^iH \quad (i = 1, \ldots, m),$$

from 3° we get

$$u^i - v^i \leqslant 2M\exp(K_0|x|^2).$$

Observe that θ is constant for $|x| > \bar{R}$ and $t \in [0, T]$; hence for every $\varepsilon > 0$ there is a $R_0 > \bar{R}$ such that $\bar{u}^i - \bar{v}^i < \varepsilon$ for all $t \in [0, A]$, $|x| \geqslant R_0$.

Finally, using the classical argument, we prove that the last inequalities hold true for $|x| < R_0$ and $0 < t < A$, $(t, x) \in D$ (for details see [1]). If $A \geqslant T$, this completes the proof; otherwise the proof can be completed in finitely many steps by applying the same argument to $[A, 2A] \times \Omega$, $[2A, 3A] \times \Omega$ etc.

3. In Theorem 2 we prove the existence of a solution of Fourier's first problem in unbounded domains.

We shall need the following additional assumptions:

(I) *$\Phi^i(t, x)$ is a continuous function belonging to E_2 in $\bar{D}$,*

(II) *For every $R > 0$ there exists in D_R the regular solution $U(t, x)$ of the following Fourier's first problem:*

$$u_t^i = F^i(t, x, U, u_x^i, u_{xx}^i) \tag{3.1}$$

with the condition

$$u^i(t, x) = \Phi^i(t, x) \quad \text{for } (t, x) \in d_p D_R, \tag{3.2}$$

where F^i are elliptic with respect to $U(t, x) = \big(u^1(t, x), \ldots, u^m(t, x)\big)$ and $D_R = D \cap \{(t, x);\ 0 < t < T,\ |x| < R\}$, $d_p D_R$ denotes the parabolic boundary of D_R.

We write $f^i(t, x) = F^i(t, x, 0, \ldots, 0)$.

THEOREM 2. *If hypotheses* (I) *and* (II) *and* 2°, 3° *and* 4° *from Theorem* 1 *are satisfied, and if*

(III) *f^i belong to E_2 $(i = 1, \ldots, m)$,*

(IV) *$\varphi^i(t, x)$ are arbitrary given continuous functions belonging to E_2 in $d_p D$, then for $0 < h_0 < A$ problem* (3.1), (3.2) *has in $D^{h_0} = \{(t, x) \in D;\ 0 < t < h_0\}$ the unique solution which belongs to E_2.*

The proof is almost identical to that Theorem 3 of [1], so we give only an outline.

Introducing

$$u_a^i = w_a^{i*} H(t, x, K + k_0),$$

where k_0 is a positive constant, and using the fact that $H(t, x, K)/ /H(t, x, K + k_0)$ converge to 0 if $x \to \infty$, we prove that u_a^i converge uniformly, and the limit function is the solution of problem (3.1), (3.2) (for details see [1]).

4. Note that Theorem 1 would fail to hold, were the assumption of the compactness of S omitted.

To see this consider two examples:

EXAMPLE 1. The equation

$$u_{xx}+\frac{u_x}{t}-u_t-u=0 \quad \text{for } (t,x)\in(0,T)\times(-\infty,\infty),$$

possesses the solution $u(t,x) = t\exp(x)$ vanishing for $t = 0$, $S = \{(t,x);\ t = 0\}$.

EXAMPLE 2. The equation

$$u_{xx}+\frac{xu_x}{t}-u_t=0 \quad \text{for } (t,x)\in(0,T)\times(0<x<\infty)$$

has the solution $u(t,x) = tx$ vanishing for $(t,x)\in S$. In this case S is unbounded, but different from d_pD, $S = \{(t,x);\ t = 0,\ 0\leqslant x<\infty\}$.

References

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