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On ordinary differentiability of Bessel potentials

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Abstract. We study different types of pointwise differentiability of functions and potentials in $\boldsymbol{R}^n$. We first prove general theorems on the connection between the ordinary, the approximative and L^p-differentiability. We prove that the ordinary differentiability of order l is equivalent to the approximative one or L^p-differentiability together with certain conditions defined in terms of Taylor polynomials called A_l and B_l.

Then we apply these results to Bessel potentials $f = G_k * g$ where G_k is the Bessel kernel and g is in some L^p-class. The conditions imposed on g are the existence of the Calderón-Zygmund L^p-derivative and an integral condition involving maximal functions. We consider differentiability of arbitrary order $l > 0$. We also show that condition B_l is of Lipschitz type and give some examples.

0. Introduction.

Several authors have studied smoothness properties of functions using derivatives, differences and maximal functions. It is the purpose of this paper to study the connection between approximative differentiability and ordinary differentiability for general functions with applications to Bessel potentials of L^p-functions. Such potentials are known to have L^q-differentials of certain orders quasi-everywhere in $\boldsymbol{R}^n$. We find necessary conditions under which these potentials are differentiable in the ordinary way at certain points. (See Section 1 for definitions.)

By differentiability of different types we always mean that there exists a Taylor polynomial such that the remainder tends to zero in the appropriate way. It is well known that ordinary differentiability of a function f implies its approximative differentiability and L^p differentiability of the same order. To get a reverse implication we must make some additional assumptions on f. For this purpose we define two different properties of a function f, called A_l and B_l.

The property A_l, $0 < l \leqslant 1$, says roughly that for all $\lambda > 0$ the set

$$\{z;\ |f(z)-f(x)| \leqslant \lambda\cdot|x-a|^l\} \cap B(x, t\cdot|x-a|)$$

has the outer measure $\geqslant c(\lambda, f)\cdot|B(x, t\cdot|x-a|)|$, for all x in a neighbourhood of $a \in \boldsymbol{R}^n$ and certain $t \to 0$. (See Section 2 for the exact definitions.)

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We prove in Section 2 that if f has an approximative differential at $a \in \mathbf{R}^n$ of order l, $0 < l \leqslant 1$, with the constant term $f(a)$ and has property A_l at a, then f is ordinary differentiable at a of order l. This generalizes a lemma of H. Federer [11], Lemma 3.1.5.

The condition B_l, which is stronger than A_l, is of supremum type, but has equivalent forms adopted to L^p-estimates (Sections 2.1 and 2.3). Since both the conditions A_l and B_l, $0 < l \leqslant 1$, are defined by a first difference, they cannot be used in that form for differentiability of order $l > 1$.

In Section 3 we show that there is a natural extension of the properties A_l and B_l to the case $l > 1$ and prove the corresponding differentiability theorems. In Section 4 we begin our study of differentiability properties of Bessel potentials. Several authors have studied different kinds of smoothness properties of functions in Bessel potential spaces, Sobolev spaces and Besov spaces [4]–[10], [13]–[16].

The smoothness is expressed in terms of maximal functions or the Calderón–Zygmund L^q-derivatives [9].

It is well known that Bessel potentials of L^p-functions have L^q-differentials of certain orders (T. Bagby, W. P. Ziemer [5]) and that convolution with the Bessel kernel G_k increases this type of smoothness by the amount of k, A. P. Calderón, A. Zygmund [9].

Let $g \in L^p$ and define

$$f(x) = \int G_k(x-y) \cdot g(y)\, dy, \tag{1.1}$$

whenever the integral converges absolutely. The standard imbedding theorems for Bessel potential spaces imply that f belongs to C^t (f is t times continuously differentiable) if $1 < p < \infty$, $k = n/p + t$, and $t > 0$ ([19], p. 206). See also [3], p. 221.

N. Aronszajn, F. Mulla and P. Szeptycki [4] proved that f has a certain type of pointwise partial derivatives of order l, except for a set of $B_{k-l,p}$-capacity zero if $1 < p < \infty$, $k > l$ and l is a positive integer. Y. Mizuta [14] studies fine differentiability properties of Riesz potentials of the type

$$U_\alpha^\mu(x) = U_\alpha^\mu(a) + L(x-a) + o(|x-a|),$$

as $x \to a$ and $x \notin E$, where $U_\alpha^\mu(x)$ is a Riesz potential, E is an exceptional set and $L \in \mathbf{R}^n$. All those results are of global type.

We are going to find necessary conditions on g in a neighbourhood of a point $a \in \mathbf{R}^n$ such that $f(x)$ defined by (1.1) is differentiable of a certain order $l > 0$ at $x = a$. We express the conditions on g in terms of maximal functions and L^p-differentials. For this purpose we define the maximal functions $N_q^s g(x)$ (A. P. Calderón and R. Scott [8]), see Section 4. We impose the following two types of conditions on g:

(a) g has an L^p-differential of order θ at a,

(b) $N_q^s g(x)$ has certain integrability properties near $x = a$.

Condition (a) is of the Calderón–Zygmund type [9], while (b) seems to be new. The absolute convergence of (1.1) in a neighbourhood of $x = a$ is not explicitly assumed but is a consequence of the properties of g.

A similar problem is studied for $l = 1$ in B.-M. Stocke [18], where it is proved that under suitable assumptions on g, f is differentiable of order one, $B_{k-1,p}$-q.e. in R^n, $k > 1$.

This paper is organized in the following way. The notation and the basic definitions are contained in Section 1. The properties A_l and B_l are defined in Sections 2 and 3. Our results on the differentiability of order l of general functions are found in Sections 2 ($0 < l \leqslant 1$) and 3 ($l > 1$). In Section 4 we apply these results to Bessel potentials in the case $0 < l \leqslant 1$. Section 5 contains some lemmas concerning property B_l and the theorems from Section 4 are proved in Section 6.

We study higher order differentiability of Bessel potentials in Section 7 and we give some examples in Section 8.

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1. Notation and definitions.

We consider the n-dimensional Euclidean space $\boldsymbol{R}^n$, where points are denoted by $x = (x_1, x_2, \ldots, x_n)$. All sets are subsets of $\boldsymbol{R}^n$ and functions f are defined on subsets of $\boldsymbol{R}^n$. Open, closed and compact sets are denoted by V, F and K respectively. The measure and integration are considered with respect to the Lebesgue measure and are denoted by $|E|$ and $\int_E f(x)\,dx$ respectively. We also use the Lebesgue outer measure $|A|^* = \inf\{|E|;\ A \subset E \text{ and } E \text{ measurable}\}$.

For $1 \leqslant p \leqslant \infty$ and E a measurable set, $L^p(E)$ is the usual Lebesgue space of measurable functions defined a.e. on E with norm $\|f\|_{p,E}$. We drop E from the notation when $E = \boldsymbol{R}^n$. L^p_{loc} is the space of measurable functions f which belong to $L^p(K)$ for all compact sets K. The outer density of a set A at a point x is defined by

$$\lim_{r\to 0} |A \cap B(x, r)|^*/|B(x, r)|,$$

if this limit exists.

Unspecified constants depending on some quantities $\alpha, \beta, \ldots$ are denoted by $c(\alpha, \beta, \ldots)$. Constants depending only on n are denoted by c. Both $c(\alpha, \beta, \ldots)$ and c can denote different constants at different occurrences.

A multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where α_i are non-negative integers, has the length $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Differentiation is always denoted by D^α where α is a multiindex.

The Bessel kernel $G_k(x)$ of order $k > 0$ in $\boldsymbol{R}^n$ is the L^1-function whose Fourier transform equals $(1 + |\xi|^2)^{-k/2}$. Basic properties of $G_k(x)$ are found in [4] and [17]. For the reader's convenience we state some of them here. The function $G_k(x)$ is a positive, non-increasing and infinitely differentiable function of radius $r = |x|$ for $r > 0$. It is real analytic for $x \neq 0$ and exponentially decreasing when $|x| \to \infty$. Further we have for $|x| \leqslant 1$

$$|D^\alpha G_k(x)| \leqslant c(n, k, \alpha)\cdot|x|^{k-|\alpha|-n}, \qquad 0 < k < n+|\alpha|,$$

$$|D^\alpha G_k(x)| \leqslant c(n, k, \alpha)\cdot(1+\ln 1/|x|), \qquad k = n+|\alpha|,$$

$$|D^\alpha G_k(x)| \leqslant c(n, k, \alpha), \qquad k > n+|\alpha|.$$

When $|x| \geqslant 1$ we have $|D^\alpha G_k(x)| \leqslant c(n, k, \alpha)\cdot|x|^{(k-n-1)/2}\cdot e^{-|x|}$.

A polynomial P of degree $m = 0, 1, \ldots$ is always of the form

$$P(x) = \sum_{|\alpha|\leqslant m} a_\alpha\cdot x^\alpha, \qquad x\in \boldsymbol{R}^n.$$

Then, for instance, $P(x-a) = \sum_{|\alpha|\leqslant m} a_\alpha\cdot(x-a)^\alpha$. Polynomials of negative degree are by definition identically zero. We denote polynomials by P, Q and R.

A function f defined in some neighbourhood of a point $a\in\boldsymbol{R}^n$ is (ordinarily) differentiable at a of order l, $l > 0$, if there is a polynomial $P(x-a)$ of degree $\leqslant l$ with the constant term $f(a)$ such that

$$f(x)-P(x-a) = o(|x-a|^l),$$

as $x \to a$. Differentiability always means ordinary differentiability. We also consider two types of generalized differentiation, see [9], p. 172, and [11], p. 212.

Definition 1.1. Let $1 \leqslant p \leqslant \infty$, l real, $a\in\boldsymbol{R}^n$ and let f be a function defined in a neighbourhood of a in $\boldsymbol{R}^n$.

(a) f is *L^p-differentiable at a of order l* if f is measurable and there is a polynomial $P(x-a)$ of degree $\leqslant l$ such that

$$\Big(|B(a, r)|^{-1}\cdot \int_{B(a,r)} |f(x)-P(x-a)|^p\,dx\Big)^{1/p} = o(r^l)$$

as $r \to 0$.

(b) f is *approximately differentiable at a of order l* if there is a polynomial $P(x-a)$ of degree $\leqslant l$ such that a is a point of outer density zero for the set

$$E_l(f, a, \varepsilon) = \{x \mid |f(x)-P(x-a)| > \varepsilon|x-a|^l\},$$

for every $\varepsilon > 0$. When $l < 0$, we take $P(x-a) \equiv 0$ in both cases.

We make the usual modifications in case (a) when $p = \infty$. The polynomial $P(x-a)$, which is unique in all the cases, is called the *differential of f at a of order l* of the respective type.

It is obvious that differentiability implies L^p-differentiability if f is measurable, and it is easily proved that L^p-differentiability implies approximative differentiability of the same order. See T. Bagby, W. P. Ziemer [5], Lemma 4.4, for the case $l = 1$. The general case is proved in the same way.

Let L_k^p be the space of Bessel potentials $f = G_k * g$ of L^p-functions g, with norm $\|f\|_{k,p} = \|g\|_p$, $1 \leqslant p \leqslant \infty$, $k > 0$. L_k^p is a Banach space of equivalence classes of functions ([3], p. 219, and [17], p. 130). Closely associated with L_k^p are the Bessel capacities $B_{k,p}$ defined for all subsets of $\boldsymbol{R}^n$. A relation which holds except for a set E with $B_{k,p}(E) = 0$ is said to hold $B_{k,p}$-quasi-everywhere ($B_{k,p}$-q.e.). See N. G. Meyers [12] for details about capacities of this type.

2. Differentiability theorems for general functions.

2.1. In this section we study how approximative differentiability and L^p-differentiability are related to ordinary differentiability at a point in $\boldsymbol{R}^n$. Our main result is a generalization of a lemma due to H. Federer [11], Lemma 3.1.5.

We prove that if a function f has an approximative differential $P(x-a)$ at a and has a suitable additional property, then $P(x-a)$ is also an ordinary differential of f at a.

We start with the following definitions.

Definition 2.1. Let $t > 0$, $\lambda > 0$, $l > 0$, $a \in \boldsymbol{R}^n$ and let f be defined in a neighbourhood of a in $\boldsymbol{R}^n$. Then

$$\Phi_l(f, a; \lambda, t) = \liminf_{x \to a} \frac{|B(x, t \cdot |x-a|) \setminus F_\lambda(x)|^*}{|B(x, t \cdot |x-a|)|}$$

where

$$F_\lambda(x) = \{z \mid |f(z) - f(x)| \geqslant \lambda \cdot |x-a|^l\}.$$

Definition 2.2. Let $0 < l \leqslant 1$. A function f defined in $0 < |x-a| < \delta$, for some $\delta > 0$, has property A_l at a if

$$\limsup_{t \to 0} \Phi_l(f, a; \lambda, t) > 0$$

for every $\lambda > 0$.

2.2. The following theorem is our basic characterization of differentiability of order l, $0 < l \leqslant 1$, for general functions.

Theorem 2.1. *Let $0 < l \leqslant 1$ and let f be defined in a neighbourhood of a in $\boldsymbol{R}^n$. Then f is ordinary differentiable at a of order l if and only if f satisfies the following conditions:*

(i) *f is approximately differentiable at a of order l and the constant term in the approximative differential equals $f(a)$,*

(ii) *f has property A_l at a.*

Remark. Theorem 2.1 contains a result of H. Federer [11], Lemma 3.1.5, as a special case when $l = 1$. In that case $\Phi_l(f, a; \lambda, t) \equiv 1$, for $0 < t < \lambda/M$ and all $\lambda > 0$, where M is a fixed positive number. The idea of the proof of Theorem 2.1 is the same as in [11].

We are going to use some of its technical details later (Lemmas 3.1 and 3.2) and therefore we give a complete proof of Theorem 2.1.

Remark. Let $f \in L^1_{\text{loc}}$; then a.e. $x \in \boldsymbol{R}^n$ is a Lebesgue point for f, i.e.,

$$\lim_{r \to 0} |B(x, r)|^{-1} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0,$$

holds for a.e. $x \in \boldsymbol{R}^n$. If f is approximately differentiable at x of order $l \geqslant 0$, then the constant term in the approximative differential equals $f(x)$ if x is a Lebesgue point of f.

Proof of Theorem 2.1. Assume that f is differentiable at a of order l. Then for every $\lambda > 0$,

$$\Phi_l(f, a; \lambda, t) \equiv 1,$$

for t sufficiently small. Thus (ii) holds. It is clear that (i) holds, too.

Now assume that both (i) and (ii) hold and let ε be an arbitrary positive number. It follows from (ii) that there is t, $0 < t < \min(1, \varepsilon)$ and $k > 0$ such that

$$\Phi_l(f, a; \varepsilon, t) > k,$$

where k is independent of t. Further, we can find $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \text{implies} \quad \frac{|B(x, t \cdot |x-a|) \setminus F_\varepsilon(x)|^*}{|B(x, t \cdot |x-a|)|} > k.$$

Let $P(x-a) = f(a) + L \cdot (x-a)$, $L \in \boldsymbol{R}^n$, be the approximative differential of f at a. Then, if

$$E_\varepsilon = \{z \mid |f(z) - f(a) - L \cdot (z-a)| \geqslant \varepsilon \cdot |z-a|^l\},$$

there is $\delta_2 > 0$ such that $0 < r < \delta_2$ implies

$$\frac{|B(a, r) \cap E_\varepsilon|^*}{|B(a, r)|} < \frac{k}{2} \cdot \left(\frac{t}{t+1}\right)^n, \tag{2.1}$$

by (i). Now let $\delta = \min(\delta_1\ \delta_2, 1)$, $|x-a| < \dfrac{\delta}{1+t}$. Then we have

$$\begin{aligned} &|f(x) - f(a) - L \cdot (x-a)| \\ &\quad \leqslant |f(x) - f(z)| + |f(z) - f(a) - L \cdot (z-a)| + |L| \cdot |z-x|. \end{aligned} \tag{2.2}$$

We complete the proof by showing that z can be chosen to satisfy:

$$(2.3) \qquad |z-x| \leqslant t\cdot|x-a|, \quad |f(z)-f(x)| \leqslant \varepsilon\cdot|x-a|^l,$$

and $z \notin E_\varepsilon$.

If we for a moment assume that this has been done then (2.2) gives

$$\begin{aligned}|f(x)-f(a)-L\cdot(x-a)| &\leqslant \varepsilon\cdot|x-a|^l+\varepsilon\cdot|z-a|^l+|L|\cdot t\cdot|x-a| \\ &\leqslant (1+2^l+|L|)\cdot\varepsilon\cdot|x-a|^l,\end{aligned}$$

provided $|x-a| < \delta/(1+t)$.

Since ε was arbitrary it follows that f is differentiable at a of order l with differential $P(x-a)$.

It remains to prove that there is a z such that (2.3) holds. We define as above

$$F_\varepsilon(x) = \{z \mid |f(z)-f(x)| \geqslant \varepsilon\cdot|x-a|^l\}$$

and we shall prove that

$$(2.4) \qquad G(x) = \big(B(x, t\cdot|x-a|)\setminus E_\varepsilon(x)\big)\setminus E_\varepsilon \neq \emptyset,$$

for $0 < |x-a| < \dfrac{\delta}{1+t}$. In fact we even prove that

$$(2.5) \qquad |G(x)|^* > \tfrac{1}{2}k\cdot\big|B(x, t\cdot|x-a|)\big|.$$

From the inclusion of sets

$$B(x, t\cdot|x-a|)\setminus F_\varepsilon(x) \subset \big(B(x, t\cdot|x-a|)\cap E_\varepsilon\big)\cup G(x)$$

we get

$$\begin{aligned} k &< \frac{|B(x, t\cdot|x-a|)\setminus F_\varepsilon(x)|^*}{\big|B(x, t\cdot|x-a|)\big|} \leqslant \frac{|B(x, t\cdot|x-a|)\cap E_\varepsilon|^*}{\big|B(x, t\cdot|x-a|)\big|}+\frac{|G(x)|^*}{\big|B(x, t\cdot|x-a|)\big|} \\ &\leqslant \left(\frac{1+t}{t}\right)^n\cdot\frac{\big|B\big(a, (1+t)\cdot|x-a|\big)\cap E_\varepsilon\big|^*}{\big|B\big(a, (1+t)|x-a|\big)\big|}+\frac{|G(x)|^*}{\big|B(x, t\cdot|x-a|)\big|} \\ &< \frac{k}{2}+\frac{|G(x)|^*}{\big|B(x, t\cdot|x-a|)\big|}\end{aligned}$$

by (2.1). This proves (2.5) and (2.4). The proof of Theorem 2.1 is now complete.

It is immediate that approximative differentiability can be replaced by L^p-differentiability of the same order if f is measurable.

2.3. In view of our applications in Section 4 we shall replace the condition A_l by a condition which is better adopted to L^p-differentiability. We make the following definitions.

DEFINITION 2.3. Let $1 \leqslant p \leqslant \infty$, $l > 0$, $t > 0$ and let f be a measurable function defined in a neighbourhood of $a \in \mathbf{R}^n$. Then we define

$$(2.6) \qquad \Psi_{l,p}(f, a; t) = \limsup_{x \to a} |x-a|^{-l} \cdot \left(|B(x, t\cdot|x-a|)|^{-1} \times \times \int_{B(x,t\cdot|x-a|)} |f(z)-f(x)|^p dz\right)^{1/p}$$

with the usual modification when $p = \infty$.

A natural replacement for the property A_l would be the condition that

$$\liminf_{t \to 0} \Psi_{l,p}(f, a; t) = 0.$$

However, it turns out that all those conditions, for $1 \leqslant p \leqslant \infty$, are equivalent to the apparently stronger property B_l of the supremum type below. This is proved in Lemma 2.1.

DEFINITION 2.4. Let $l > 0$ and let f be defined in $0 < |x-a| < \delta$ for some $\delta > 0$. Then f has property B_l at a if

$$(2.7) \qquad \lim_{t \to 0} \Psi_l(f, a; t) = 0,$$

where

$$\Psi_l(f, a; t) = \limsup_{x \to a} |x-a|^{-l} \cdot \left(\sup_{|z-x| \leqslant t\cdot|x-a|} |f(z)-f(x)|\right).$$

Remark. The possibly infinite number $\Psi_l(f, a; t)$ is well defined for $0 < t < 1$, since then $z = a$ is not allowed. Note that $\Psi_l(f, a; t)$ is a non-decreasing function of t, whenever defined.

LEMMA 2.1. *Let $l > 0$, $1 \leqslant p \leqslant \infty$ and let f be defined and measurable in a neighbourhood of a in $\mathbf{R}^n$. Then f has property B_l at a if and only if*

$$(2.8) \qquad \liminf_{t \to 0} \Psi_{l,p}(f, a; t) = 0.$$

Proof of Lemma 2.1. It suffices to prove that (2.8) implies that f has property B_l at a. Assume that (2.8) holds for some p, $1 \leqslant p \leqslant \infty$. By Hölder's inequality it is no loss of generality to assume $p = 1$. Let $\varepsilon > 0$ be arbitrary, then there is $\delta > 0$ and $0 < s < \min(\varepsilon, 1)$ such that $0 < |x-a| < \delta$ implies

$$|B(x, s\cdot|x-a|)|^{-1} \times \int_{B(x,s|x-a|)} |f(z)-f(x)|\, dz \leqslant \varepsilon \cdot |x-a|^l.$$

Now let $0 < |x-a| < \delta/(1+s)$, $0 < |z-a| < \delta/(1+s)$, $|z-x| = t\cdot|x-a|$, $0 < t \leqslant s/(s+1)$ and

$$B_1 = B(x, s\cdot|x-a|), \qquad B_2 = B(z, s\cdot|z-a|).$$

Consider the inequality

$$(2.9) \qquad |f(x)-f(z)| \leqslant |f(x)-f(u)| + |f(u)-f(z)|.$$

We integrate (2.9) with respect to u over $B_1 \cap B_2$ and notice that $|B_1 \cap B_2| \geq c \cdot \max(|B_1|, |B_2|)$. Then

$$|f(x)-f(z)| \leq c \cdot (|B_1|^{-1} \cdot \int_{B_1} |f(u)-f(x)|\, du + |B_2|^{-1} \cdot \int_{B_2} |f(u)-f(z)|\, du)$$
$$\leq c \cdot \varepsilon \cdot (|x-a|^l + |z-a|^l) \leq c \cdot \varepsilon \cdot |x-a|^l.$$

Hence we have proved that $\Psi_l(f, a; s)$ has lower limit zero as $s \to 0$.

Since $\Psi_l(f, a; s)$ is non-decreasing we can conclude that (2.7) holds. Hence f has property B_l at a and Lemma 2.1 is proved.

Remark. The equivalent form (2.8) of property B_l will be used significantly in the study of differentiability properties of Bessel potentials in Sections 4, 6 and 7.

2.4. The following theorem combines Theorem 2.1 with property B_l and its equivalent form (2.8). It is the most general result of this section.

THEOREM 2.2. *Let $0 < l \leq 1$, $1 \leq p \leq \infty$, and let f be a measurable function defined in a neighbourhood of a in $\boldsymbol{R}^n$. Then f is ordinarily differentiable at a of order l if and only if f has an approximative differential $P(x-a)$ of order l at a with the constant term $f(a)$ and f satisfies one of the following conditions:*

(i) *f has property B_l at a,*

(ii) $\liminf_{t \to 0} \Psi_{l,p}(f, a; t) = 0$ *for some* $1 \leq p \leq \infty$,

(iii) *f has property A_l at a.*

Proof of Theorem 2.2. Lemma 2.1 gives (i)$\Leftrightarrow$(ii) and it is easily seen that (i)$\Rightarrow$(iii). Hence the sufficiency part follows from Theorem 2.1.

Since ordinary differentiability implies both approximate differentiability of the same order and (i), the theorem is proved.

Remark. Theorem 2.2 remains true if we replace approximative differentiability by L^p-differentiability of the same order where p is as in (ii) of Theorem 2.2. If we exclude condition (ii) then Theorem 2.2 is true without assuming f being measurable.

Suppose f satisfies a Lipschitz condition $|f(x)-f(y)| \leq M \cdot |x-y|^l$, $0 < l \leq 1$, in an open set V, then f has property B_l in V. Conversely, if f has property B_l at a point a, it can be proved that f satisfies a certain restricted Lipschitz condition at a, see Section 5.

3. Higher order differentiability for general functions.

3.1. In this section we find the analogues of Theorems 2.1 and 2.2 for differentiability of order $l > 1$. We first define property A_l for all $l > 0$.

DEFINITION 3.1. Let $m < l \leq m+1$, where m is a non-negative integer, and let $f(x)$ be defined for $0 < |x-a| < \delta$, where $\delta > 0$. We say that f has *property* A_l at a if there is a polynomial $Q(x-a)$ of degree $\leqslant m$, without constant term, such that if $f_m(x) = f(x) - Q(x-a)$ then for every $\lambda > 0$ holds

$$\limsup_{t \to 0} \Phi_l(f_m, a; \lambda, t) > 0.$$

When $m = 0$ we take $Q(x-a) \equiv 0$ and our Definitions 2.2 and 3.1 agree.

Examples show that the polynomial $Q(x-a)$ in Definition 3.1 need not be unique when $m \geqslant 1$.

However, if f has an approximative differential $P(x-a)$ of order $l > 0$ at a, then the natural choice for $Q(x-a)$ is to take an appropriate part of $P(x-a)$. We prove that indeed this is the case and thus that Q is then unique.

LEMMA 3.1. *Let $m < l \leqslant m+1$, where m is a positive integer and let f be defined in a neighbourhood of a in $\boldsymbol{R}^n$. Assume that f has property A_l at a with polynomial $Q(x-a)$ and that f has an approximative differential $P(x-a)$ of order l at a. Then if*

$$P(x-a) = \sum_{|\alpha| \leqslant l} c_\alpha (x-a)^\alpha$$

then we have

$$Q(x-a) = \sum_{1 \leqslant |\alpha| \leqslant m} c_\alpha (x-a)^\alpha.$$

Proof of Lemma 3.1. We begin with the case $m < l < m+1$. Let $R(x) = P(x) - P(0) - Q(x)$, then we have the identity

$$\begin{aligned} R(z-a) - R(x-a) &= \big(P(z-a) - f(z)\big) + \big(f_m(z) - f_m(x)\big) + \big(f(x) - P(x-a)\big) \\ &= A + B + C. \end{aligned}$$

We are going to prove that $R \equiv 0$. Assume the contrary and let the minimal degree of the terms in R be r, $1 \leqslant r \leqslant m$, and $R(x-a) = \sum_{r \leqslant |\alpha| \leqslant m} d_\alpha (x-a)^\alpha$. Let $R = R_1 + R_2$, where $R_1(x-a) = \sum_{|\alpha| = r} d_\alpha (x-a)^\alpha$. R_2 is identically zero when $r = m$. Then for $|z-x| \leqslant |x-a| \leqslant 1$ we have

$$|R_2(z-a) - R_2(x-a)| \leqslant N \cdot |x-a|^{r+1},$$

where N depends on R only.

Let $\varepsilon > 0$. Then there are $0 < \delta < 1$ and $0 < t < \min(\varepsilon, 1)$ such that $0 < |x-a| < \delta$ implies that the set

$$G(x) = \{z \mid |A| \leqslant \varepsilon \cdot |x-a|^l \text{ and } |B| \leqslant \varepsilon \cdot |x-a|^l\} \cap B(x, t \cdot |x-a|)$$

satisfies $|G(x)|^* \geqslant k|B(x, t \cdot |x-a|)|$ for some k depending on f only, $0 < k \leqslant 1$. This follows from the proof of Theorem 2.1. Then $0 < |x-a| < \delta$

and $z \in G(x)$ implies that

$$|R_1(z-a)-R_1(x-a)| \leqslant N \cdot |x-a|^{r+1} + 2\varepsilon |x-a|^t + |f(x)-P(x-a)|$$
$$\leqslant \tfrac{1}{2} \cdot \varepsilon \cdot t |x-a|^r + |f(x)-P(x-a)|,$$

if δ is chosen small enough. Fix such a $\delta > 0$. The polynomial R_2 will not be considered any further. We denote R_1 by R.

Next we define the set

$$H(x) = \{z \mid |R(z-a)-R(x-a)| \leqslant \varepsilon \cdot t |x-a|^r\} \cap B(x, t \cdot |x-a|).$$

Let us for a moment assume that

$$(3.1) \qquad |f(x)-P(x-a)| \leqslant \tfrac{1}{2} \cdot \varepsilon \cdot t |x-a|^t.$$

Then $G(x) \subset H(x)$ and

$$(3.2) \qquad |H(x)| \geqslant |G(x)|^* > k |B(x, t \cdot |x-a|)|,$$

provided $0 < |x-a| < \delta$ and δ as above.

We are going to prove that for most points x (in a sense to be described below) $|H(x)| \cdot |B(x, t \cdot |x-a|)|^{-1}$ is close to zero and hence contradicting (3.2).

Let $\omega = (x-a) \cdot |x-a|^{-1}$, $x \neq a$; then

$$|\{z;\ |R(z-a)-R(x-a)| \leqslant \lambda |x-a|^r\} \cap B(x, t \cdot |x-a|)|$$
$$= (t \cdot |x-a|)^n \cdot |\{u;\ |R(\omega+tu)-R(\omega)| \leqslant \lambda\} \cap B(0, 1)|.$$

Let $M_1 = \max\limits_{|z|=1} |\nabla R(z)| > 0$ and $0 < s < M_1$; then the set

$$V = \{z;\ |z| = 1,\ |\nabla R(z)| > s\}$$

is a relatively open subset of $\{|z| = 1\}$. For any $\omega \in V$ we have

$$(3.3) \quad |\{u;\ |R(\omega+tu)-R(\omega)| \leqslant \lambda\} \cap B(0, 1)| \leqslant c \cdot (1/s) \cdot \max(1, M_2) \cdot (\lambda/t + t),$$

where $M_2 = \max\limits_{|\alpha|=2} \max\limits_{|z| \leqslant 2} |D^\alpha R(z)|$.

To prove (3.3) we have by Taylor's formula

$$|\nabla R(\omega) \cdot u| \leqslant \lambda/t + c \cdot M_2 \cdot t,$$

and it is easily proved that for $\mu > 0$

$$|\{u;\ |\nabla R(\omega) \cdot u| \leqslant \mu\} \cap B(0, 1)| \leqslant c \cdot \mu |\nabla R(\omega)|^{-1}.$$

This proves (3.3). Now for $0 < |x-a| < \delta'$ and $\omega \in V$ we get by (3.3)

$$(3.4) \qquad |H(x)|/|B(x, t \cdot |x-a|)| \leqslant c \cdot (1/s) \cdot \max(1, M_2) \cdot \varepsilon,$$

provided that (3.1) holds.

It is easy to see that the set

$$\{x \mid 0 < |x-a| < \delta,\ \omega = (x-a)/|x-a| \in V\}$$

contains points satisfying (3.1) for every $\delta > 0$. Thus (3.4) contradicts (3.2) if ε is small enough and thereby Lemma 3.1 is proved in the case $m < l < m+1$. When $l = m+1$ we define

$$R(x) = P(x) - \sum_{|\alpha| = m+1} c_\alpha x^\alpha - P(0) - Q(x),$$

where $P(x) = \sum_{|\alpha| \leqslant m+1} c_\alpha x^\alpha$. Now we get

$$R(z-a) - R(x-a) = \big(P(z-a) - f(z)\big) + \big(f_m(z) - f_m(x)\big) +$$
$$+ \big(f(x) - P(x-a)\big) + \sum_{|\alpha| = m+1} c_\alpha\big((x-a)^\alpha - (z-a)^\alpha\big) = A + B + C + D.$$

The extra term D, compared to the case $m < l < m+1$, satisfies

$$|D| \leqslant c(P) \cdot |z-x| \cdot |x-a|^m \leqslant c(P) \cdot t \cdot |x-a|^{m+1} \leqslant \varepsilon \cdot t \cdot |x-a|^m,$$

if $|z-x| \leqslant t \cdot |x-a|$ and $|x-a|$ is small enough. Now the proof in the case $m < l < m+1$ applies and the proof of Lemma 3.1 is complete.

3.2. We now define property B_l for $l > 1$ analogously to the definition of property A_l in Section 3.1.

DEFINITION 3.2. Let $m < l \leqslant m+1$, where m is a non-negative integer and let f be defined for $0 < |x-a| < \delta$, where $\delta > 0$. Then f has property B_l at a if there is a polynomial $Q(x-a)$ of degree $\leqslant m$, without a constant term, such that if $f_m(x) = f(x) - Q(x-a)$ then

$$\lim_{t \to 0} \Psi_l(f_m, a; t) = 0,$$

where Ψ_l is as in Definition 2.4. When $m = 0$, we take $Q \equiv 0$.

Remark. The polynomial $Q(x-a)$ in Definition 3.2 is unique. This is proved in the same way as Lemma 3.1 was proved. In this case we need not assume that f is approximatively differentiable to get a unique polynomial Q. We omit the details.

Remark. Examples show that property A_l does not imply B_l.

Lemma 2.1 has the following form in the general case $l > 0$.

LEMMA 3.2. *Let m be a non-negative integer, $1 \leqslant p \leqslant \infty$, $m < l \leqslant m+1$ and let f be defined in a neighbourhood of a in $\mathbf{R}^n$. Then f has property B_l at a if and only if there is a polynomial $Q(x-a)$ of degree $\leqslant m$, without a constant term, such that if*

$$f_m(x) = f(x) - Q(x-a), \tag{3.5}$$

then

(3.6) $$\liminf_{t\to 0} \Psi_{l,p}(f_m, a; t) = 0,$$

where $\Psi_{l,p}$ is as in Definition 2.3.

Proof of Lemma 3.2. Assume f has property B_l, then we may take Q as in Definition 3.2 and (3.6) holds. To prove the necessity, suppose that Q exists such that f_m defined by (3.5) satisfies (3.6).

The proof is completed in the same way as the proof of Lemma 2.1. We leave the details to the reader.

3.3. We now have the following characterization of differentiability of order $l > 0$.

THEOREM 3.1. *Let f be a function which is defined in a neighbourhood of a in R^n and let $l > 0$. Then f is ordinary differentiable at a of order l if and only if f has property A_l at a and f has an approximative differential at a of order l with constant term $f(a)$.*

Remark. Property A_l can be replaced by B_l. For measurable functions we can replace approximative differentiability by L^p-differentiability and A_l can be replaced by property (3.6) in Lemma 3.2 with the same p.

Proof of Theorem 3.1. The necessity is obvious, as in the proof of Theorem 2.1. Assume that f has an approximative differential

$$P(x-a) = f(a) + \sum_{1 \leq |\alpha| \leq l} c_\alpha \cdot (x-a)^\alpha,$$

$m < l \leq m+1$, where m is a positive integer. Then by Lemma 3.1 we have

$$f_m(x) = f(x) - \sum_{1 \leq |\alpha| \leq m} c_\alpha \cdot (x-a)^\alpha$$

in the definition of property A_l. Consider the identity

$$f(x) - P(x-a) = \big(f_m(x) - f_m(z)\big) + \big(f(z) - P(z-a)\big) + \sum_{|\alpha| = m+1} c_\alpha \cdot \big((z-a)^\alpha - (x-a)^\alpha\big),$$

where the suming is dropped when $l < m+1$. The rest of the proof follows that proof of Theorem 2.1. We omit the details.

4. Differentiability theorems for Bessel potentials.

4.1. In this section we study ordinary differentiability of Bessel potentials $f = G_k * g$ of functions g defined in R^n. We recall that the function $f = G_k * g$ is well defined at $x \in R^n$ if and only if the integral

(4.1) $$f(x) = \int G_k(x-y) \cdot g(y)\, dy,$$

converges absolutely. It is well known that if $g \in L^p$ then $f = G_k * g$ is well defined a.e. in $\boldsymbol{R}^n$, in fact even $B_{k,p}$-q.e., when $1 < p < \infty$ [12].

Assume that g is a function such that the integral in (4.1) converges absolutely a.e. in $\boldsymbol{R}^n$. Then we have

$$\lim_{r \to 0} |B(a, r)|^{-1} \int_{B(a,r)} |f(x) - f(a)| \, dx = 0 \tag{4.2}$$

for all $a \in \boldsymbol{R}^n$, for which $f(a)$ is well defined by (4.1), [4], p. 293.

Now we give a short description of our theorems. Let $g \in L^p$. Then the differentiability properties of $f = G_k * g$ at $a \in \boldsymbol{R}^n$ depend merely on the local behaviour of $g(y)$ near $y = a$. This follows from the fact that the integral

$$\int_{|y-a| > \delta} G_k(x-y) \cdot g(y) \, dy$$

can be differentiated under the integral sign infinitely many times when $|x-a| < \delta$, since $G_k(x) \to 0$ exponentially as $|x| \to \infty$.

It is easily seen that the same holds for a much larger class of functions g, for instance if g is a polynomial. The following class $M_p(a)$ will be sufficient for our purposes.

Definition 4.1. Let $1 \leqslant p \leqslant \infty$ and $a \in \boldsymbol{R}^n$. We say that $g \in M_p(a)$ if g is locally integrable in $\boldsymbol{R}^n$, $g \in L^p(V)$ for some neighbourhood V of a and

$$\int_{|y-a| \geqslant 1} |g(y)| \cdot e^{-|y|/2} \, dy < \infty .$$

The linear space $M_p(a)$ has the following properties for $1 \leqslant p \leqslant \infty$ and all $a \in \boldsymbol{R}^n$:

(a) $L^p \subset M^p(a)$,

(b) $M^q(a) \subset M^p(a)$, $1 \leqslant p \leqslant q \leqslant \infty$,

(c) all polynomials belong to $M^p(a)$,

(d) if $g \in M_p(a)$, then for all $\delta > 0$ the integral

$$\int_{|y-a| > \delta} G_k(x-y) \cdot g(y) \, dy$$

converges absolutely and is infinitely differentiable under the integral sign for $|x-a| < \delta$,

(e) if $g \in M_p(a)$, then $g \in M_p(b)$ for all points b in a neighbourhood of a.

4.2. Now we state our first result concerning the differentiability of Bessel potentials $f = G_k * g$, $g \in M_p(a)$.

Theorem 4.1. *Let* $1 < p < \infty$, $1 \leqslant q < \infty$, $\theta \geqslant -n/p$, $k > 0$, $k + \theta > 0$, $k + \theta \neq 1$ *and* $l > 0$. *Let* $g \in M_p(a)$ *and* $f = G_k * g$. *Assume that*

(a) *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *and* $0 < t < \min(\varepsilon, 1)$ *such that* $0 < |x-a| < \delta$ *implies that*

$$(4.3) \qquad \int_{|y-a| \leqslant 3\cdot|x-a|} G_k(x-y) \times \times \Big(|B(y, t\cdot|x-a|)|^{-1} \cdot \int_{B(y,t\cdot|x-a|)} |g(u)-g(y)|^q du\Big)^{1/q} dy \leqslant \varepsilon\cdot|x-a|^l.$$

In (4.3) *we replace* $G_k(x-y)$ *by* $|K_0(x-y)|$, *where* $K_0(w) = G_k(w) - G_k(0)$, *when* $n < k \leqslant n+1$.

(b)

$$(4.4) \qquad \Big(|B(a, r)|^{-1} \cdot \int_{B(a,r)} |g(y)-P(y-a)|^p dy\Big)^{1/p} = o(r^\theta), \qquad r \to 0,$$

for some polynomial P *of degree* $\leqslant \theta$.

Then $f(x) = G_k * g(x)$ *is well defined in a neighbourhood of* $x = a$ *in* $\boldsymbol{R}^n$ *and* f *is differentiable at* a *of order* l *in the following cases*:

(i) $0 < k < n$, $0 < l \leqslant \min(k+\theta, 1)$,

(ii) $k = n$, $0 < l < n+\theta$ *and* $0 < l \leqslant 1$,

(iii) $n < k \leqslant n+1$, $0 < l \leqslant \min(k+\theta, 1)$.

Remark. The proof breaks down when $k+\theta = 1$, partly because we cannot conclude that f has an L^p-derivative of order 1 at a (cf. [9], p. 175) and partly because an extra logarithm enters in our calculations.

However, it follows from the proof of Theorem 4.1 that when $k+\theta = 1$, f is differentiable at a of order l in the following case:

(iv) $0 < k \leqslant n+1$ and $0 < l < 1$.

An essential part of the proof of Theorem 4.1 is to show that the assumptions on g imply that the Bessel potential $f(x)$ is well defined in a neighbourhood of a in $\boldsymbol{R}^n$. This differs from the situation considered in Sections 2 and 3 where we just assumed that $f(x)$ was well defined. The rest of the proof is done by means of Theorem 2.2. Assumption (a) in Theorem 4.1 comes from the quantity $\Psi_{l,q}(f, a; t)$ in Section 2 where now $f = G_k * g$. The proof is given in Section 6.

4.3. Following A. P. Calderón and R. Scott [8] we define the maximal function $N_q^\theta f$ (compare also $T_\theta^q(f, x)$ in [9]).

DEFINITION 4.1. Let $1 \leqslant q < \infty$, θ real, $f \in L^q_{loc}$. Then

$$N_q^\theta f(x) = \sup_{r>0} r^{-\theta} \cdot \Big(|B(x, r)|^{-1} \int_{B(x,r)} |f(y)-P(y-x)|^q dy\Big)^{1/q}$$

if there is a polynomial $P(y-x)$ of degree $< \theta$ such that $N_q^\theta f(x) < \infty$. Otherwise we let $N_q^\theta f(x) = \infty$. As usual, we take $P(y-x) = 0$ if $\theta < 0$ The polynomial $P(y-x)$ is unique whenever it exists, and $\theta = 0$, $q = 1$ gives the usual Hardy–Littlewood maximal function [17], p. 4.

The maximal function $N_q^s f$ is lower semicontinuous for $s \leqslant 0$ and a Borel function for $s > 0$.

We now formulate our main result on the ordinary differentiability of order l, $0 < l \leqslant 1$, of Bessel potentials.

THEOREM 4.2. *Let* $1 < p < \infty$, $1 \leqslant q < \infty$, $k > 0$, $l > 0$, $0 \leqslant \alpha \leqslant 1$, $g \in M_p(a)$, $f = G_k * g$. *Assume that*

(a)

$$\int_{|y-a| \leqslant 3\cdot|x-a|} G_k(x-y)\cdot N_q^\alpha g(y)\,dy = o(|x-a|^{l-\alpha}), \quad \text{as } x \to a, \tag{4.5}$$

(b)

$$\left(|B(a,r)|^{-1} \int_{B(a,r)} |g(y)-P(y-a)|^p\,dy\right)^{1/p} = o(t^\theta), \tag{4.6}$$

as $t \to 0$, *for* $\theta = l-k$ *and some polynomial* P *of degree* $\leqslant \theta$.

In (4.5) *we replace* $G_k(x-y)$ *by* $|K_0(x-y)|$ *as in* (4.3) *when* $n < k \leqslant n+1$. *Then* $f(x)$ *is well defined in a neighbourhood of* a *in* $\boldsymbol{R}^n$ *and* f *is differentiable at* a *of order* l *where* l *is defined as in Theorem* 4.1.

Theorem 4.2 follows easily from Theorem 4.1 and the definition of $N_q^s g(x)$. The proof is given in Section 6.

In Section 7 we use the methods of Section 3 to prove higher order differentiability theorems for Bessel potentials.

5. Some lemmas on the property B_l.

5.1. We prove some lemmas which show that functions having property B_l at a point satisfy a certain restricted Lipschitz condition there.

LEMMA 5.1. *Let* $\varepsilon > 0$, $\delta > 0$, $l > 0$ *and* $0 < t < 1$ *be fixed numbers and let* f *be a function such that*

$$\left.\begin{array}{l} |x-a| < \delta \\ |z-x| \leqslant t\cdot|x-a| \end{array}\right\} \quad \text{implies} \quad |f(z)-f(x)| \leqslant \varepsilon\cdot|x-a|^l.$$

Then

$$|f(z)-f(x)| \leqslant k\cdot\varepsilon\cdot\max(|x-a|^l, |z-a|^l)$$

for all $0 < |x-a| < \delta$, $0 < |z-a| < \delta$ *where* k *depends on* t, l *and* n.

Proof of Lemma 5.1. The proof is in three steps.

Step 1. The points x and z lie on the same half ray through a. Assume $|z-a| < |x-a| = r_0 < \delta$. Let L be the line segment between x and a. Define $r_i = r_0\cdot(1-t)^i$, $i = 0, 1, 2, \ldots$ and a sequence of points $\{w_i\}_1^\infty$ such that

$$w_i \in L, \quad |w_i - a| = r_i, \quad w_0 = x, \quad i = 0, 1, 2, \ldots$$

Then $|w_i - w_{i+1}| = r_i - r_{i+1} = t \cdot |w_i - a|$, $i = 0, 1, 2, \ldots$, and $\lim_{i \to \infty} w_i = a$. Suppose that z lies between w_m and w_{m+1} on L, then

$$\begin{aligned} |f(x) - f(z)| &\leqslant \sum_{i=0}^{m-1} |f(w_i) - f(w_{i+1})| + |f(w_m) - f(z)| \\ &\leqslant \varepsilon \cdot \sum_{i=0}^{m-1} |w_i - a|^l + \varepsilon \cdot |w_m - a|^l \leqslant \varepsilon \cdot \big(1 - (1-t)^l\big)^{-1} \cdot |x - a|^l \\ &= \varepsilon \cdot k_1 \cdot |x - a|^l. \end{aligned}$$

Step 2. $|x - a| = |z - a| = r$, $0 < r < \delta$. Let B be the boundary of the unit ball with the centre in a and let $\{u_i\}_1^N$ be points on B such that

(i) $|u_i - u_{i+1}| \leqslant t$, $1 \leqslant i \leqslant N-1$,

(ii) for every $\xi \in B$ there is u_i, $1 \leqslant i \leqslant N$, such that $|u_i - \xi| \leqslant t$.

The integer N depends on n and t only. Define $u_i' = a + r \cdot (u_i - a)$, $1 \leqslant i \leqslant N$. Then we have

$$\begin{aligned} |u_i' - a| &= r, & 1 \leqslant i \leqslant N, \\ |u_{i+1}' - u_i'| &= r \cdot |u_{i+1} - u_i| \leqslant r \cdot t = t \cdot |u_i' - a|, & 1 \leqslant i \leqslant N-1, \end{aligned}$$

and hence

$$|f(u_{i+1}') - f(u_i')| \leqslant \varepsilon \cdot |u_i' - a|^l, \quad 1 \leqslant i \leqslant N-1.$$

Given x, z such that $|x - a| = |z - a| = r$, $0 < r < \delta$, we can find u_m' and u_p', such that $|u_m' - x| \leqslant r \cdot t$ and $|u_p' - z| \leqslant r \cdot t$. Thus we get

$$\begin{aligned} |f(x) - f(z)| &\leqslant |f(x) - f(u_m')| + \sum_{i=m}^{p-1} |f(u_i') - f(u_{i+1}')| + |f(u_p') - f(z)| \\ &\leqslant \varepsilon \cdot r^l + (N-1) \cdot \varepsilon \cdot r^l + \varepsilon \cdot r^l = (N+1) \cdot \varepsilon \cdot |x - a|^l = k_2 \cdot \varepsilon \cdot |x - a|^l. \end{aligned}$$

Step 3. Let $0 < |z - a| < |x - a| < \delta$ be arbitrary. Define y such that $|y - a| = |z - a|$ and y lies on the line segment between a and x. Then the triangle inequality, together with Steps 1 and 2, gives

$$|f(x) - f(z)| \leqslant \varepsilon \cdot (k_1 + k_2) \cdot \max(|x - a|^l, |z - a|^l).$$

This proves Lemma 5.1.

LEMMA 5.2. *Under the assumption of Lemma* 5.1, *the limit*

$$\lim_{\substack{x \to a \\ x \neq a}} f(x) = b \tag{5.1}$$

exists, and

$$|f(x) - b| \leqslant \varepsilon \cdot \big(1 - (1-t)^l\big)^{-1} \cdot |x - a|^l, \tag{5.2}$$

for $0 < |x - a| < \delta$.

Proof of Lemma 5.2. If f is complex valued and satisfies the assumptions of Lemma 5.1, then so do its real and imaginary parts. Assuming f being real valued we see that

$$\limsup_{\substack{x \to a \\ x \neq a}} f(x) \quad \text{and} \quad \liminf_{\substack{x \to a \\ x \neq a}} f(x)$$

exist since f is bounded in $0 < |x-a| < \delta$. But these limits must be equal by Lemma 5.1 which proves (5.1).

Let $0 < |x-a| < \delta$ and define the sequence $\{w_i\}_1^\infty$ as in the proof of Lemma 5.1. Then we get

$$|f(x)-f(w_m)| \leqslant \varepsilon \cdot (1-(1-t)^l)^{-1} \cdot |x-a|^l.$$

Since $f(w_m) \to b$, as $m \to \infty$, this proves (5.2) and completes the proof of Lemma 5.2.

5.2. A function f having property B_l at a, $0 < l \leqslant 1$, satisfies the assumptions of Lemma 5.2. In the general case we have the following lemma.

LEMMA 5.3. *Let f have property B_l at a, $l > 0$. Then there are numbers $\delta > 0$, $k > 0$ and b such that for $0 < |x-a| < \delta$,*

$$|f(x)-b-Q(x-a)| \leqslant k \cdot |x-a|^l,$$

where $Q(x-a)$ is as in Definition 3.2. In particular, $\lim_{\substack{x \to a \\ x \neq a}} f(x) = b$ exists and f is bounded in $0 < |x-a| < \delta$.

Lemma 5.3 is proved by applying Lemma 5.2 to the function $f_m(x)$ in the definition of the property B_l (Section 3.2). We omit the details.

6. Proof of Theorems 4.1 and 4.2

We start with the proof of Theorem 4.1.

6.1. Recall that $1 < p < \infty$, $\theta \geqslant -n/p$, $k > 0$ and $g \in M^p(a)$. Define $f(x) = G_k * g(x)$, whenever the integral defining the convolution converges absolutely.

We show that it is sufficient to consider the following special case:

(i) $g(y)$ has support in $|y-a| \leqslant 1$ and $g \in L^p$,

(ii) $P = 0$ in (4.4) and (iii) $g \geqslant 0$.

Let $P(x-a)$ be the polynomial in (4.4) and define

$$g_1(x) = \begin{cases} g(x)-P(x-a), & |x-a| \leqslant \delta, \\ 0, & |x-a| > \delta, \end{cases}$$

where $\delta \leqslant 1$ is chosen such that $g_1 \in L^p$. Then g_1, $\operatorname{Re} g_1$, $\operatorname{Im} g_1$ and the positive and negative parts of $\operatorname{Re} g_1$ and $\operatorname{Im} g_1$ satisfy the assumptions of Theorem 4.1 with $P = 0$. It follows from the discussion in Section 4.1 and the fact that Bessel potentials of polynomials are C^∞-functions (in fact polynomials) that it is no loss of generality to assume that (i)–(iii) hold. We are going to prove that f satisfies the assumptions of Theorem 2.2. The first step is to show that f is well defined and that f has an L^p-differential of the right type. In the second step we prove that f has property B_l at a.

6.2. Here we prove that the integral

$$f(x) = \int G_k(x-y)\cdot g(y)\,dy \tag{6.1}$$

converges absolutely in a neighbourhood of $x = a$. First let $x \neq a$ be fixed. Then it suffices to consider the integral in (6.1) over the set $\{y;\ |y-a| \leqslant 3\cdot|x-a|\}$. Let $\varepsilon > 0$ be arbitrary and choose $\delta > 0$ and $0 < \eta < \min(\varepsilon, 1)$ such that (4.3) holds.

Let $0 < |x-a| < \delta$, $z \in \boldsymbol{R}^n$, and consider the inequality

$$g(y) \leqslant |g(y)-g(z-x+y)| + g(z-x+y). \tag{6.2}$$

We define

$$H(z) = \int_{|y-a|\leqslant 3\cdot|x-a|} G_k(x-y)\cdot|g(y)-g(z-x+y)|\,dy.$$

By Minkowski's inequality and (4.3) we get

$$\Big(\int_{|z-x|\leqslant\eta\cdot|x-a|} H(z)^q\,dz\Big)^{1/q} \leqslant \varepsilon\cdot|x-a|^l\cdot\big|B(x, \eta\cdot|x-a|)\big|^{1/q} < \infty,$$

which proves that $H(z) < \infty$ a.e. in $B(x, \eta\cdot|x-a|)$. We also have

$$\int_{|y-a|\leqslant 3\cdot|x-a|} G_k(x-y)\cdot g(z-x+y)\,dy \leqslant \int G_k(y)\cdot g(z-y)\,dy < \infty, \tag{6.3}$$

for a.e. $z \in \boldsymbol{R}^n$.

Multiplying (6.2) by $G_k(x-y)$ and integrating over $|y-a| \leqslant 3\cdot|x-a|$ we get that

$$\int_{|y-a|\leqslant 3\cdot|x-a|} G_k(x-y)\cdot g(y)\,dy$$

converges for $0 < |x-a| < \delta$, since we can choose z such that $H(z) < \infty$ and (6.3) hold. This proves that $f(x)$ is well defined, by (6.1) for $0 < |x-a| < \delta$. Now (4.4) and basic properties of the Bessel kernel (see Section 1) give that the integral (6.1) converges for $x = a$.

It follows from (4.4) and [9], Theorem 4, that f has an L^p-differential of order l at a. Since the integral (6.1) converges absolutely for $x = a$ we can conclude from (4.2) that the constant term in the L^p-differential of f at a equals $f(a)$.

We have proved that $f(x)$ is well defined by (6.1) in a neighbourhood of $x = a$ and that f has an L^p-differential of order l at a with the constant term $f(a)$.

6.3. Our next step is to prove that f has property B_l at a. In view of Lemma 2.1 it suffices to prove that (2.8) holds. We have

$$f(x)-f(z) = \int G_k(x-y)\cdot(g(y)-g(z-x+y))\,dy$$
$$= \int_{|y-a|\leq 3\cdot|x-a|} + \int_{|y-a|>3\cdot|x-a|} = \mathrm{I}+\mathrm{II}.$$

Let $\varepsilon > 0$ be arbitrary and choose δ and t such that (4.3) holds. Then taking L^q-norm we get for $0<|x-a|<\delta$

$$\Big(|B(x,t|x-a|)|^{-1}\int_{B(x,t|x-a|)}|f(z)-f(x)|^q\,dz\Big)^{1/q} \leq \varepsilon|x-a|^l + \sup_{|z-x|\leq t|x-a|}|\mathrm{II}|.$$

We divide II into two terms in the following way:

$$\begin{aligned}\mathrm{II} = &\int_{|y-a|>3\cdot|x-a|}(G_k(x-y)-G_k(z-y))\cdot g(y)\,dy+\\ &+\Big\{\int_{|y-a|>3\cdot|x-a|}G_k(z-y)\cdot g(y)\,dy-\\ &-\int_{|y-z+x-a|>3\cdot|x-a|}G_k(z-y)\cdot g(y)\,dy\Big\}\\ =\ & A+B,\end{aligned}$$

where all the integrals converge absolutely. The term B is majorized by

$$\int_{|x-a|\leq|z-y|\leq 5\cdot|x-a|}G_k(z-y)\cdot g(y)\,dy \qquad \text{for } |z-x|\leq t\cdot|x-a|.$$

Hölder's inequality and (4.4) give

$$\begin{aligned}\sup_{|z-x|\leq t|x-a|}|B| &\leq c(k,n)\cdot|x-a|^k\cdot\Big(|B(a,6|x-a|)|^{-1}\int_{B(a,6|x-a|)}g(y)^p\,dy\Big)^{1/p}\\ &= o(|x-a|^l), \quad \text{as } x\to a,\end{aligned}$$

when $0<k<n$. The case $k=n$ is treated analogously and gives the same result.

We estimate A as in [17], p. 244, using Taylor's formula

$$|A| \leq \int_{|y-a|>3|x-a|} dy\, g(y)\int_0^1 |\nabla G_k(x-y+u(z-x))|\,|z-x|\,du. \tag{6.4}$$

Now from (6.4) we get

$$|A| \leq c(k,n)|z-x|\int_{|y-a|>3|x-a|}|y-a|^{k-n-1}g(y)\,dy, \tag{6.5}$$

when $0<k\leq n+1$. The integral in (6.5) is estimated as in [17], p. 245, and we find that

$$\sup_{|z-x|\leq t\cdot|x-a|}|A| \leq c(k,n)\cdot\varepsilon\cdot(1+\|g\|_p)\cdot|x-a|^l,$$

provided $\delta > 0$ is small enough. When $k > n$ we write

$$f(x)-f(z) = \int K_0(x-y)\cdot(g(y)-g(z-x+y))\,dy$$

$$= \int_{|y-a|\leqslant 3\cdot|x-a|} + \int_{|y-a|>3\cdot|x-a|} = \mathrm{I}+\mathrm{II},$$

and $\mathrm{II} = A+B$ as above. We estimate A and B as in the case $0 < k \leqslant n$. We leave the details to the reader.

Collecting our estimates, we have proved that

$$\Big(|B(x, t|x-a|)|^{-1} \int_{B(x,t|x-a|)} |f(z)-f(x)|^q\,dz\Big)^{1/q} \leqslant \varepsilon|x-a|^l$$

if $0 < |x-a| < \delta$ and δ is small enough. This proves that f has property B_l at a by Lemma 2.1. Now Theorem 2.2 applies and we can conclude that f is differentiable at a of order l. This completes the proof of Theorem 4.1.

6.4. Proof of Theorem 4.2. We only have to prove that (4.5) implies (4.3). Assume that (4.5) holds and $0 < \alpha \leqslant 1$. Then $N_q^\alpha g(y) < \infty$ for a.e. y in some neighbourhood V of a. It follows from the Lebesgue Differentiation Theorem that $P(u-y) = g(y)$ in the definition of $N_q^\alpha g(y)$ for a.e. $y \in V$. The expression in the left-hand member of (4.3) is for $0 < k \leqslant n$ majorized by

$$(t|x-a|)^\alpha \int_{|y-a|\leqslant 3|x-a|} G_k(x-y)\cdot N_q^\alpha g(y)\,dy = o(|x-a|^l),$$

as $x \to a$. This proves (4.3) when $0 < \alpha \leqslant 1$.

Next let $\alpha = 0$, then the left-hand member of (4.3) is majorized by

$$\int_{|y-a|\leqslant 3|x-a|} G_k(x-y)\cdot\big(N_q^0 g(y)+g(y)\big)\,dy. \tag{6.6}$$

It is easily seen that $g(y) \leqslant N_q^0 g(y)$ for a.e. $y \in \mathbf{R}^n$. Inserting this estimate into (6.6) proves (4.3). Thereby Theorem 4.2 is proved for $k \leqslant n$. The case $k > n$ is similar.

7. Higher order differentiability of Bessel potentials.

7.1. In this section we study differentiability of arbitrary order $l > 0$ of Bessel potentials using the results from Section 3. Recall that, for measurable functions, differentiability of order l is equivalent to L^p-differentiability of order l together with property B_l or its equivalent form given in Lemma 3.2. We first sketch what we are going to do. Let $1 < p < \infty$, $k > 0$, $\theta \geqslant -n/p$ and let g be an L^p-function with compact support satisfying

$$\Big(|B(a, r)|^{-1} \int_{B(a,r)} |g(y)|^p\,dy\Big)^{1/p} = o(r^\theta), \qquad r \to 0.$$

Let m be a non-negative integer and assume that $m < k+\theta < m+1$. Then the function $f(x) = G_k * g(x)$ has an L^p-derivative $Q(x-a)$ of order $(k+\theta)$ at x

$= a$ given by

$$(7.1)\qquad Q(x-a) = \sum_{|\alpha| \leqslant m} c_\alpha (x-a)^\alpha, \qquad c_\alpha = \frac{1}{\alpha!} \int_{R^n} D^\alpha G_k(a-y) g(y)\,dy,$$

where the integrals defining c_α, $|\alpha| \leqslant m$, converge absolutely. These facts follow from the proof of [9], Theorem 4.

We prove that, under suitable assumptions on g, the function $f(x) = G_k * g(x)$ is well defined in a neighbourhood of $x = a$. Further, in the case $0 < k \leqslant n$, if we define

$$(7.2)\qquad f_m(x) = f(x) - (Q(x-a) - Q(0))$$

we can prove that f has property B_l, $l = k+\theta$, at a (Definition 3.2). Then by Theorem 3.1 it follows that f is differentiable at a of order $l = k+\theta$. The following theorem treats a slightly more general situation.

THEOREM 7.1. *Let m be a non-negative integer, $k > 0$, $1 < p < \infty$, $1 \leqslant q < \infty$, $-n/p \leqslant \theta \leqslant 1$, $k+\theta > m$, $k+\theta \neq m+1$, $l > 0$. Let $g \in M_p(a)$ and $f = G_k * g$. Assume that*

(a) *For every $\varepsilon > 0$ there is $\delta > 0$ and $0 < t < \min(\varepsilon, 1)$ such that $0 < |x-a| < \delta$ implies*

$$(7.3)\qquad \int_{|y-a| < 3|x-a|} G_k(x-y) \cdot \left(|B(y, t|x-a|)|^{-1} \times \right.$$
$$\left. \times \int_{B(y,t|x-a|)} |g(u) - g(y)|^q\,du\right)^{1/q} dy \leqslant \varepsilon |x-a|^l.$$

In (7.3) we replace $G_k(x-y)$ by $|K_i(x-y)|$ where

$$K_i(w) = G_k(w) - \sum_{|\alpha| \leqslant i} \frac{1}{\alpha!} \cdot D^\alpha G_k(0) \cdot w^\alpha,$$

in the case $n+i < k \leqslant n+i+1$, $0 \leqslant i \leqslant m$.

(b)

$$(7.4)\qquad \left(|B(a, r)|^{-1} \int_{B(a,r)} |g(y) - P(y-a)|^p\,dy\right)^{1/p} = o(r^\theta), \qquad \text{as } r \to 0,$$

for some polynomial P of degree $\leqslant \theta$.

Then $f(x) = G_k * g(x)$ is well defined in a neighbourhood of $x = a$ in R^n and f is differentiable at a of order l in the following cases:

(i) $0 < k < n$, $0 < l \leqslant \min(k+\theta, m+1)$,

(ii) $n \leqslant k \leqslant n+m+1$, $0 < l < k+\theta$, $0 < l \leqslant m+1$.

7.2. Proof of Theorem 7.1. It is no loss of generality to assume that $g(x)$ has support in $|x-a| \leqslant 1$, that $P = 0$ in (7.4) and that $g \geqslant 0$. As in the proof of Theorem 4.1, (7.3) implies that $f(x)$ is well defined by

$$(7.5)\qquad f(x) = \int G_k(x-y) g(y)\,dy$$

in $0 < |x-a| < \delta$, for some $\delta > 0$. It follows from (7.4) and [9], p. 195–198, that $f(x)$ has an L^p-differential at $x = a$ of order l given by (7.1). In particular, $f(x)$ is well defined by (7.5) for $x = a$ and the constant term in the L^p-differential of f at a equals $f(a)$. We first consider the case $0 < k \leqslant n$.

We define $f_m(x)$ by (7.1) and (7.2) in accordance with Lemma 3.1. It is our purpose to prove that f has property B_l at a. Then we can conclude from Theorem 3.1 that f is differentiable at a of order l. According to Lemma 3.2 it suffices to prove that (3.6) holds with L^q-norm.

Let us define the remainder $R_m(u, v)$ of the Bessel kernel G_k by

$$R_m(u, v) = G_k(u) - \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} D^\alpha G_k(v) \cdot (u-v)^\alpha. \tag{7.6}$$

Taylor's formula gives

$$R_m(u, v) = (m+1) \int_0^1 \sum_{|\alpha| = m+1} D^\alpha G_k(v + s(u-v))(u-v)^\alpha \cdot \frac{(1-s)^m}{\alpha!} ds, \tag{7.7}$$

if the line segment between u and v does not contain the origin.

Now let $|z-x| \leqslant \frac{1}{2} \cdot |x-a|$. Then by (7.1), (7.2) and (7.6)

$$\begin{aligned} f_m(x) - f_m(z) &= \int \big(R_m(x-y, a-y) - R_m(z-y, a-y)\big) \cdot g(y)\, dy \\ &= \int \big(R_m(x-y, a-y) \cdot g(y) - R_m(x-y, a-z+x-y) \cdot g(z-x+y)\big) dy \\ &= \int_{|y-a| < 3|x-a|} + \int_{|y-a| \geqslant 3|x-a|} = \mathrm{I} + \mathrm{II}, \end{aligned}$$

where all integrals converge absolutely when $|x-a|$ is small enough, since $f_m(x)$ is well defined in a neighbourhood of $x = a$. We split II into two terms

$$\begin{aligned} \mathrm{II} = & \int_{|y-a| \geqslant 3|x-a|} \big(R_m(x-y, a-y) - R_m(z-y, a-y)\big) \cdot g(y)\, dy + \\ & + \Big(\int_{|y-a| \geqslant 3|x-a|} R_m(z-y, a-y) \cdot g(y)\, dy - \\ & - \int_{|y-a+z-x| \geqslant 3|x-a|} R_m(z-y, a-y) \cdot g(y)\, dy \Big) \\ = & \, A + B. \end{aligned}$$

We estimate II for all $0 < k \leqslant m+n+1$. From (7.7) we get

$$\begin{aligned} |B| &\leqslant \int_{2|x-a| \leqslant |y-a| < 4|x-a|} |R_m(z-y, a-y| \cdot g(y)\, dy \\ &\leqslant c(k, m, n) \cdot |x-a|^{m+1} \cdot \int_{2|x-a| \leqslant |y-a| < 4|x-a|} dy\, g(y) \times \\ &\quad \times \sum_{|\alpha| = m+1} \int_0^1 \big|D^\alpha G_k(a-y+t \cdot (z-a))\big|\, dt. \end{aligned} \tag{7.8}$$

When $0 < k < m+n+1$, (7.8) and properties of the Bessel kernel give

$$|B| \leqslant c(k, m, n) \cdot |x-a|^{k-n} \cdot \int_{|y-a| < 4|x-a|} g(y)\, dy = o(|x-a|^{k+\theta}),$$

as $x \to a$. When $k = m+n+1$ we get analogously $|B| = o(|x-a|^{k+\theta} \cdot \ln 1/|x-a|)$ as $x \to a$.

By using (7.7) we have the identity

$$\begin{aligned}(7.9) \quad & R_m(x-y, a-y) - R_m(z-y, a-y) \\ &= \sum_{|\alpha| = m+1} (m+1) \cdot \int_0^1 dt \big(D^\alpha G_k(a-y+t\cdot(x-a)) - \\ &\qquad - D^\alpha G_k(a-y+t\cdot(z-a))\big)(x-a)^\alpha \cdot \frac{(1-t)^m}{\alpha!} + \\ &+ \sum_{|\alpha| = m+1} (m+1) \cdot \int_0^1 dt\, D^\alpha G_k(a-y+t\cdot(z-a)) \cdot \big((x-a)^\alpha - (z-a)^\alpha\big) \cdot \frac{(1-t)^m}{\alpha!}.\end{aligned}$$

Inserting (7.9) in A and using the Mean Value Theorem on the first sum in (7.9) gives, together with [4], p. 253, when $0 < k \leqslant m+n+1$

$$\begin{aligned}|A| &\leqslant c(k, m, n) \cdot |z-x| \cdot |x-a|^m \int_{|y-a| > 3|x-a|} |y-a|^{k-m-n-1} g(y)\, dy \\ &\leqslant c(k, m, n) \cdot \varepsilon \cdot |x-a|^\alpha, \quad \text{as } x \to a, \text{ where } \alpha = \min(k+\theta, m+1).\end{aligned}$$

Now by (7.6) and a change of variables, I equals to be

$$\begin{aligned}(7.10) \quad & \int_{|y-a| < 3|x-a|} G_k(x-y) \cdot \big(g(y) - g(z-x+y)\big)\, dy - \\ &- \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} \Big(\int_{|y-a| < 3|x-a|} D^\alpha G_k(a-y) \cdot (x-a)^\alpha \cdot g(y)\, dy - \\ &\qquad - \int_{|y-a+x-z| < 3|x-a|} D^\alpha G_k(a-y) \cdot (z-a)^\alpha \cdot g(y)\, dy\Big).\end{aligned}$$

The sum in (7.10) is estimated as in [17], p. 245. It is $o(|x-a|^{k+\theta})$ and $o(|x-a|^{k+\theta} \ln 1/|x-a|)$ when $0 < k < n$ and $k = n$ respectively. In case where $n+i < k \leqslant n+i+1$, $0 \leqslant i \leqslant m$, we first define $h = K_i * g$ so that $(f-h)$ is a polynomial. Next we define

$$\begin{aligned}(7.11) \quad f_m(x) &= \int \Big(K_i(x-y) - \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} \cdot D^\alpha K_i(a-y) \cdot (x-a)^\alpha \Big) \cdot g(y)\, dy \\ &= \int R_{i,m}(x-y, a-y) \cdot g(y)\, dy.\end{aligned}$$

We split $f_m(x) - f_m(z)$ into two terms I and II as in the case $0 < k \leqslant n$. This term II is identical with the term II considered above since $R_{i,m}(u, v) = R_m(u, v)$. The term I is of the form (7.10) with G_k replaced by K_i. As above, it is estimated by using properties of the Bessel kernel. We leave the details to the reader. The first term in (7.10) will be estimated by (7.3).

Let $0 < \varepsilon < \frac{1}{2}$ be arbitrary and choose $0 < t < \min(\varepsilon, 1)$ and $0 < \delta < \frac{1}{3}$ such that (7.3) holds. Then for $0 < |x-a| < \delta$ we have in the case $0 < k \leqslant n$

$$\Big(|B(x, t|x-a|)|^{-1} \int_{B(x,t|x-a|)} |f_m(z)-f_m(x)|^q dz\Big)^{1/q}$$
$$\leqslant \int_{|y-a| \leqslant 3|x-a|} G_k(x-y)\cdot\Big(|B(x, t|x-a|)|^{-1} \times$$
$$\times \int_{B(x,t|x-a|)} |g(y)-g(z-x+y)|^q dz\Big)^{1/q} + c(k, m, n)\cdot\varepsilon\cdot|x-a|^l,$$

and analogously for $k > n$.

Combining this estimate with our assumption (7.3) we can conclude that (3.6) holds (with p replaced by q) and hence by Lemma 3.2; $f(x)$ has property B_l at $x = a$. This completes the proof of Theorem 7.1.

7.3. Recall the maximal function $N_q^\alpha g(x)$ defined in Section 4.3. The following theorem is in complete analogy with Theorem 4.2.

THEOREM 7.2. *Let m, k, p, q, θ and l be as in Theorem 7.1. Let $g \in M_p(a)$ and $f = G_k * g$. Assume that*

(a)

$$(7.12)\qquad \int_{|y-a| < 3|x-a|} G_k(x-y)\cdot N_q^\alpha g(y)\,dy = o(|x-a|^{l-\alpha}) \quad \text{as } x \to a, \text{ where } 0 \leqslant \alpha \leqslant 1.$$

Here we replace $G_k(x-y)$ by $|K_i(x-y)|$ when $n+i < k \leqslant n+i+1$, $0 \leqslant i \leqslant m$, as in (7.3).

(b) $$\Big(|B(a, r)|^{-1}\cdot \int_{B(a,r)} |g(y)-P(y-a)|^p dy\Big)^{1/p} = o(r^\theta)$$

as $r \to 0$, for some polynomial P of degree $\leqslant \theta$. Then $f(x)$ is well defined in a neighbourhood of $x = a$ in R^n and f is differentiable at a of order l.

Proof of Theorem 7.2. The proof is only a repetition of the argument in the proof of Theorem 4.2 and is omitted.

8. Some examples.

8.1. The famous Rademacher–Stepanoff Theorem [17], p. 250, states that if $f\colon V \to R$ satisfies the Lipschitz condition

$$(8.1)\qquad f(x+h)-f(x) = O(|h|), \quad |h| \to 0$$

for every $x \in E$ where V is an open set in R^n and $E \subset V$ is a measurable set, then f is ordinary differentiable of order 1 a.e. on E. We prove that a certain generalization is false.

We are going to construct a Cantor set K with positive Bessel capacity and a Bessel potential f such that f has an L^p-differential and satisfies a Lipschitz condition on K but f is nowhere differentiable on K.

To be more specific, we consider the following situation.

EXAMPLE. Let $1 < p < \infty$, $1 < q < \infty$, $1 \leqslant r < \infty$, $0 < \alpha \cdot p \leqslant n$, $0 < \beta \cdot q < n$ and $m > l > 0$ be given. We shall define a Cantor set K and $f \in L^p_\alpha$ such that

(8.2) (a) $B_{\beta,q}(K) > 0$,

(8.3) (b) f is infinitely differentiable outside K,

(8.4) (c) $|f(x+h)-f(x)| \leqslant |h|^l$ for $x \in K$ and $h \in R^n$,

(8.5) (d) f has an L^r-differential of order m everywhere on K,

(8.6) (e) f is differentiable of order l at no point $x \in K$.

We define K and f and prove (8.2)–(8.6) in Section 8.2.

This example shows that we cannot conclude ordinary differentiability $B_{\beta,q}$-q.e in the Rademacher–Stepanoff Theorem when we know that f is a Bessel potential of an L^p-function and (8.1) holds.

8.2. Let $I = \{x;\ 0 \leqslant x_i \leqslant 1,\ 1 \leqslant i \leqslant n\}$ be the unit cube in R^n. All cubes have sides parallel to the axes. Let $J = J(a, r)$ denote a cube with centre a and side length r and define $t \cdot J = J(a, t \cdot r)$, when $t > 0$.

Let $K = \bigcap_1^\infty K_j$ be a generalized Cantor set contained in I as defined in [1], p. 899.

We define $l_j = t^j$, $t = 2^{(\varepsilon - n)/(n - \beta q)}$, $j = 1, 2, \ldots$, for some $0 < \varepsilon < \min(\beta q, 1)$. Then $H_h(K) > 0$ for $h(u) = u^{n-\beta q}(\ln 1/u)^{1-s}$, $0 \leqslant u \leqslant u_0 < 1$ and $s > q$, [1], p. 899. It now follows that $B_{\beta,q}(K) > 0$, [1], p. 895, which proves (8.2).

As usual $K_0 = I$ and K_j, $j \geqslant 1$, is the disjoint union of 2^{jn} cubes, each with side length l_j.

We define f to be a sum $f = \sum_1^\infty f_j$ in the following way. Let $j \geqslant 1$ be fixed and let J be one of the $2^{(j-1)n}$ cubes in K_{j-1}. In the process of constructing K_j we remove a cube J' from J, having the same centre as J and side length $d_j = l_{j-1} - 2l_j = (1-2t) \cdot t^{j-1}$. There are $2^{(j-1)n}$ such cubes J'. For each cube J' we choose $g \in C_0^\infty$ such that

(8.7) (a) $0 \leqslant g(x) \leqslant (d_j/4)^l$, (b) $\|g\|_\infty = (d_j/4)^l$ and (c) $\operatorname{supp} g \subset \frac{1}{2} \cdot J'$.

Now we define f_j as the sum of these $2^{(j-1)n}$ functions g, one for each cube J'.

It is possible to choose the functions g such that

(8.8) $\|f_j\|_{\alpha,p}$ and $\|f_j\|_r$ are arbitrary small, $j \geqslant 1$.

It is well known that $B_{\alpha,p}(\{x\}) = 0$ for all $x \in \boldsymbol{R}^n$, [12]. Then (8.8) is a consequence of the alternative definition of the Bessel capacity given in [2].

We now choose f_j, $j \geq 1$, such that $\sum_1^\infty \|f_j\|_{\alpha,p}$ converges and hence $f \in L_\alpha^p$.

It is clear that the sum $f(x) = \sum_1^\infty f_j(x)$ has at most one non-zero term for each $x \in \boldsymbol{R}^n$, that (8.3) holds and that $f(x) = 0$, when $x \in K$. Let $x \in K$ and $h \in \boldsymbol{R}^n$; then for some j we have

$$0 \leq f(x+h) = f_j(x+h) \leq (d_j/4)^l \leq |h|^l,$$

by (8.7) (a) and (c). This proves (8.4).

Next we show that (8.8) implies (8.5). Let $x \in K$ and $d_{s+1}/4 \leq t < d_s/4$; then

$$\int_{B(x,r)} |f(y)|^r \, dy = \sum_{j=s+1}^{\infty} \|f_j\|_r^r.$$

It is now clear that (8.5) holds, with the L^r-differential of order m at a identically zero, provided $\|f_j\|_r$, $j \geq 1$, are chosen small enough.

It remains to prove (8.6). Assume that $f(y)$ has an ordinary differential $P(y-x)$ of order l at x. Then $P(y-x)$ is also an L^r-differential of f at x of the same order. The uniqueness of the L^r-differential then gives that $P(y-x) \equiv 0$. Hence, it suffices to prove that

$$\limsup_{h \to 0} f(x+h) \cdot |h|^{-l} > 0, \qquad x \in K. \tag{8.9}$$

Let $x \in K$; then for every $j \geq 1$ there is $h_j \in \boldsymbol{R}^n$ such that $f_j(x+h_j) = (d_j/4)^l$ and $|h_j| \leq l_{j-1} \cdot \sqrt{n}$. This follows from (8.7) (b) and the fact that x belongs to one of the cubes J in the construction of f_j above. Hence we get for $j \geq 1$,

$$f(x+h_j) \geq f_j(x+h_j) = (d_j/4)^l \geq \big((1-2t)/4\sqrt{n}\big)^l \cdot |h_j|^l,$$

which proves (8.9), since $h_j \to 0$ as $j \to \infty$. This completes the discussion of the example.

8.3. Let $1 < p < \infty$, $0 < k \leq n/p$. Then there is a function $f \in L_k^p$ which is essentially unbounded in the neighbourhood of every point in $\boldsymbol{R}^n$ and hence f is non-differentiable of any order $l > 0$ everywhere in $\boldsymbol{R}^n$. We just choose $h \in L^p$, $h \geq 0$, such that $G_k * h(0) = \infty$, [12], p. 260, and put $g(y) = \sum_1^\infty 2^{-i} \cdot h(y-a_i)$, $f = G_k * g$, where $\{a_i\}$ is a dense set in $\boldsymbol{R}^n$. See also [17], p. 159. It is, however, proved in [4], Theorem 13.5, that functions in L_k^p have pointwise partial derivatives outside a certain exceptional set, when $1 < p < \infty$ and $k > 1$.

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