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Commutation relations involving spectrally scattered operators

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Abstract. We are interested in the spectral properties of operators which satisfy some commutation relations together with normal operators having zero dimensional spectrum.

To begin with we take the energetic basis $\{e_n\}_{n\geq 0}$ in the separable complex Hilbert space H. The quantum number operator N is defined by the formula $Nf = \sum_{n=0}^{\infty} n(f, e_n)e_n$ on the manifold

$$D(N) = \{f \in H\colon \sum_{n=0}^{\infty} n^2 |(f, e_n)|^2 < +\infty\}.$$

N is a selfadjoint operator. The question of the quantum phase operator for quantum harmonic oscillator is the question of finding a selfadjoint bounded operator F, which satisfies the commutation relation $[N, F]f = if$ for $f \in M$, where M is a linear manifold dense in H. We refer to [2], [3], [5], [7] for the history and solutions of this problem. See also [4], [6].

There is the nice and deep theorem essentially proved in [1] which reads as follows:

(AD) *Let F be a bounded selfadjoint operator and let $M \subset H$ be a linear dense manifold. Then, if $FM \subset M \subset D(N)$ and $[N, F]f = if$ for $f \in M$, then $\overline{N|M}$ = the closure of the restriction of N to M is not selfadjoint.*

It is shown in [1] that the failure of selfadjointness of $\overline{N|M}$ follows from the property that N has a discrete spectrum, as well as from the Hilbert formula for resolvents $R(\lambda, A)$ (A closed), namely, the equality

$$\text{(H)} \qquad R(\alpha, A) - R(\beta, A) = -(\alpha - \beta)R(\alpha, A)R(\beta, A),$$

where $R(\gamma, A) = (\gamma I - A)^{-1}$, $\gamma \in \varrho(A)$ = the resolvent set of A. (H) implies that

$$\text{(1)} \qquad R'(z, A) = -R^2(z, A), \quad z \in \varrho(A).$$

The essential point in the proof in [1] is the trick with formula (14) below. Our theorem formulated and proved below explains why (AD) is true. Namely, we show that the true reason that (AD) holds true is that $\overline{N|M}$ has a *non-empty* residual spectrum. Next, the proof in [1] uses the property that singletons $\{n\}$ ($n = 0, 1, 2, \ldots$) are Riesz–Dunford spectral sets for N. The spectrum $\delta(N)$ is a *zero-dimensional* closed set. We recall that the following properties of a closed plane set δ are equivalent:

(2) δ is zero-dimensional;

(3) δ is completely disconnected;

(4) δ has a countable basis $\{\delta_n\}$ of clopen bounded sets δ_n.

Let A be a normal operator (bounded or not bounded) and such that the spectrum $\delta(A)$ of A is zero dimensional. Let δ be a clopen bounded subset of $\delta(A)$. Then there is a Cauchy domain $\Omega(\delta)$ such that $\delta \subset \Omega(\delta)$ and $(\delta(A)-\delta) \cap \overline{\Omega(\delta)} = \emptyset$ and $\partial\Omega(\delta)$ is an analytic contour $\Gamma \subset \varrho(A)$, and such that the Riesz–Dunford projection $P(\delta, A)$ corresponding to δ is expressed by the following formula:

$$P(\delta, A) = \frac{1}{2\pi i} \int_{\Gamma} R(z, A)\,dz \tag{5}$$

(with the suitably oriented Γ). We need the following lemma:

LEMMA. *Let A be a normal operator with zero-dimensional spectrum. If $\{\delta_n\}$ is a basis of bounded clopen subsets of $\delta(A)$, $B \in L(H)$ and*

$$P(\delta_n, A)B = BP(\delta_n, A)$$

for all n, then

$$BR(z, A) = R(z, A)B \quad \text{for } z \in \varrho(A).$$

The proof of the lemma follows from the regularity of the spectral measure E of A, the fact that

$$E(\delta_n) = P(\delta_n, A), \qquad R(z, A) = \int_{\delta(A)} \frac{dE(u)}{z-u}$$

for $z \in \varrho(A)$ (see [8], Chapter IX, 135, 136, and Chapter XI, 148) and the property that δ_n are bounded and $\{\delta_n\}$ is a topological basis of $\delta(A)$.

We can say that the normal operator A with zero-dimensional spectrum is *spectrally scattered*, because its spectral measure is living on a very, very thin, extremely disconnected set. Last but not least, we notice that the quantum number operator is spectrally scattered.

Our theorem reads as follows:

THEOREM. *Let A be a normal, spectrally scattered operator in H. Suppose that B and C are bounded linear operators in H. We assume that*

$$R(z, A)C = CR(z, A) \quad \text{for } z \in \varrho(A), \tag{6}$$

(7) *The linear manifold $M \subset H$ is dense in H and $M \subset D(A)$ and $BM \subset M$,*

(8) $$ABf - BAf = [A, B]f = Cf \quad \textit{for } f \in M.$$

Then either

(9) $$C = 0$$

or

(10) *The closure $A_M = \overline{A|M}$ of the restriction of A to M has a non-empty residual spectrum.*

Proof. If $f \in M$ and $z \in \varrho(A)$, then there exists a unique vector $s(f, z)$ such that $f = R(z, A)s(f, z)$. The commutation relation (8) yields that $B(zI - A)f - (zI - A)Bf = Cf$ and consequently, by (6) and the equality $s(f, z) = (zI - A)f$, we get that

$$Bs(f, z) - (zI - A)BR(z, A)s(f, z) = CR(z, A)s(f, z) = R(z, A)Cs(f, z).$$

It follows that

(11) $$R(z, A)Bs(f, z) - BR(z, A)s(f, z) = R^2(z, A)Cs(f, z).$$

We have two possibilities:

(12) for each $z \in \varrho(A)$, the linear manifold $M(z) = (zI - A)M$ of vectors $s(f, z)$ $(f \in M)$ is dense in H,

or

(13) there exists $z_0 \in \varrho(A)$ such that $M_{z_0} = (z_0 I - A)M$ is not dense in H.

Suppose that (12) holds true. Then by (11) we get that for $z \in \varrho(A)$

(14) $$R(z, A)B - BR(z, A) = R^2(z, A)C.$$

Let $\{\delta_n\}$ be the topological basis of $\delta(A)$ of clopen bounded sets, and take the projections

$$P(\delta_n, A) = \frac{1}{2\pi i} \int_{\Gamma_n} R(z, A)dz$$

for suitable Γ_n. We derive now from (14), that for all n

$$\frac{1}{2\pi i} \int_{\Gamma_n} R(z, A)Bhdz - \frac{1}{2\pi i} \int_{\Gamma_n} BR(z, A)hdz = \frac{1}{2\pi i} \int_{\Gamma_n} R^2(z, A)Chdz$$

for each $h \in H$. It follows from (1) that the last integral is equal to zero. Consequently,

$$P(\delta_n, A)B - BP(\delta_n, A) = 0$$

for each n. By our lemma, $R(z, A)B = BR(z, A)$ for $z \in \varrho(A)$, which by (14) implies that $C = 0$, i.e., (9) holds true.

Suppose that (13) holds true. Then there is a non-zero vector g orthogonal to $(z_0 I - A)M$. Since $A_M = \overline{A|M} \subset A$ and $(z_0 I - A)^{-1}$ exists, $(z_0 I - A_M)^{-1}$ exists. It follows that z_0 is in the residual spectrum of A_M, which completes the proof.

Let us come back to Theorem (AD). Since there appears the assumption that, in our notation, $C = iI \neq 0$, we see by our theorem that, for $A = N$, the closure of the restriction of N to M has a non-empty residual spectrum and consequently is not a selfadjoint operator. So, the part of N in M after closing is not an orthodox quantum observable. Hence, since the candidates for quantum phase operator F have been constructed ([3], [7]), the part of quantum number operator, that one, which intervenes in the commutation relation $[N, F]f = if$ $(f \in M)$, after closing will be never an observable. A similar effect appears for angle quantum variable φ. To be more precise, we take in the space $L^2(0, 2\pi)$ of functions $f(\varphi)$ the operator

$$A = \frac{1}{i}\frac{d}{d\varphi},$$

with domain

$D(A) = \{f\colon f$ absolutely continuous on $[0, 2\pi]$, $f' \in L^2(0, 2\pi)$, $f(0) = f(2\pi)\}$.

The spectrum of $A = A^*$ is the whole totality of integers $-\delta(A) = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$; A has a pure point simple spectrum. Let Φ be the selfadjoint operator of multiplication by an independent variable, that is, $(\Phi f)(\varphi) = \varphi f(\varphi)$ for $f \in L^2(0, 2\pi)$. Φ is bounded and selfadjoint. Next, for $f \in \{g \in L^2(0, 2\pi)$: g absolutely continuous and $g' \in L^2(0, 2)$, $g(0) = 0 = g(2\pi)\} = M$ we have

$$[A, \Phi]f = -if.$$

By the theorem, the closure $\overline{A|M} = A_M$ is *not selfadjoint*. So again, the selfadjointness is lost at the cost of the commutation relation.

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