Commutation relations involving spectrally scattered operators

by W. Młak (Kraków)

Zdzisław Opial in memoriam

Abstract. We are interested in the spectral properties of operators which satisfy some commutation relations together with normal operators having zero dimensional spectrum.

To begin with we take the energetic basis \( \{e_n\}_{n \geq 0} \) in the separable complex Hilbert space \( H \). The quantum number operator \( N \) is defined by the formula

\[
Nf = \sum_{n=0}^{\infty} n(f, e_n)e_n
\]

on the manifold

\[
D(N) = \{f \in H : \sum_{n=0}^{\infty} n^2 |(f, e_n)|^2 < +\infty\}.
\]

\( N \) is a selfadjoint operator. The question of the quantum phase operator for quantum harmonic oscillator is the question of finding a selfadjoint bounded operator \( F \), which satisfies the commutation relation \([N, F]f = if \) for \( f \in M \), where \( M \) is a linear manifold dense in \( H \). We refer to [2], [3], [5], [7] for the history and solutions of this problem. See also [4], [6].

There is the nice and deep theorem essentially proved in [1] which reads as follows:

(AD) Let \( F \) be a bounded selfadjoint operator and let \( M \subset H \) be a linear dense manifold. Then, if \( FM \subset M \subset D(N) \) and \([N, F]f = if \) for \( f \in M \), then \( \overline{N|M} \) = the closure of the restriction of \( N \) to \( M \) is not selfadjoint.

It is shown in [1] that the failure of selfadjointness of \( \overline{N|M} \) follows from the property that \( N \) has a discrete spectrum, as well as from the Hilbert formula for resolvents \( R(\lambda, A) \) (\( A \) closed), namely, the equality

(H)

\[
R(\alpha, A) - R(\beta, A) = - (\alpha - \beta) R(\alpha, A) R(\beta, A),
\]

where \( R(\gamma, A) = (\gamma I - A)^{-1}, \gamma \in \varrho(A) \) = the resolvent set of \( A \). (H) implies that

(1)

\[
R'(z, A) = - R^2(z, A), \quad z \in \varrho(A).
\]
The essential point in the proof in [1] is the trick with formula (14) below. Our theorem formulated and proved below explains why (AD) is true. Namely, we show that the true reason that (AD) holds true is that $N|M$ has a non-empty residual spectrum. Next, the proof in [1] uses the property that singletons \{n\} \ (n = 0, 1, 2, \ldots) are Riesz–Dunford spectral sets for $N$. The spectrum $\delta(N)$ is a zero-dimensional closed set. We recall that the following properties of a closed plane set $\delta$ are equivalent:

(2) $\delta$ is zero-dimensional;

(3) $\delta$ is completely disconnected;

(4) $\delta$ has a countable basis \{\delta_n\} of clopen bounded sets $\delta_n$.

Let $A$ be a normal operator (bounded or not bounded) and such that the spectrum $\delta(A)$ of $A$ is zero dimensional. Let $\delta$ be a clopen bounded subset of $\delta(A)$. Then there is a Cauchy domain $\Omega(\delta)$ such that $\delta \subset \Omega(\delta)$ and $(\delta(A) - \delta) \cap \overline{\Omega(\delta)} = \emptyset$ and $\partial \Omega(\delta)$ is an analytic contour $\Gamma \subset \varrho(A)$, and such that the Riesz–Dunford projection $P(\delta, A)$ corresponding to $\delta$ is expressed by the following formula:

$$P(\delta, A) = \frac{1}{2\pi i} \oint_{\Gamma} R(z, A)dz$$

(with the suitably oriented $\Gamma$). We need the following lemma:

**Lemma.** Let $A$ be a normal operator with zero-dimensional spectrum. If \{\delta_n\} is a basis of bounded clopen subsets of $\delta(A)$, $B \in L(H)$ and

$$P(\delta_n, A)B = BP(\delta_n, A)$$

for all $n$, then

$$BR(z, A) = R(z, A)B \quad \text{for} \quad z \in \varrho(A).$$

The proof of the lemma follows from the regularity of the spectral measure $E$ of $A$, the fact that

$$E(\delta_n) = P(\delta_n, A), \quad R(z, A) = \int_{\varrho(A)} \frac{dE(u)}{z-u}$$

for $z \in \varrho(A)$ (see [8], Chapter IX, 135, 136, and Chapter XI, 148) and the property that $\delta_n$ are bounded and \{\delta_n\} is a topological basis of $\delta(A)$.

We can say that the normal operator $A$ with zero-dimensional spectrum is spectrally scattered, because its spectral measure is living on a very, very thin, extremely disconnected set. Last but not least, we notice that the quantum number operator is spectrally scattered.

Our theorem reads as follows:

**Theorem.** Let $A$ be a normal, spectrally scattered operator in $H$. Suppose that $B$ and $C$ are bounded linear operators in $H$. We assume that

$$R(z, A)C = CR(z, A) \quad \text{for} \quad z \in \varrho(A),$$

(6)
The linear manifold \( M \subset H \) is dense in \( H \) and \( M \subset D(A) \) and \( BM \subset M \).

\[
ABf - BAf = [A, B]f = Cf \quad \text{for } f \in M.
\]

Then either

\[
C = 0
\]

or

\[
The \text{ closure } A_M = \overline{A|M} \text{ of the restriction of } A \text{ to } M \text{ has a non-empty residual spectrum.}
\]

Proof. If \( f \in M \) and \( z \in \mathfrak{g}(A) \), then there exists a unique vector \( s(f, z) \) such that \( f = R(z, A)s(f, z) \). The commutation relation (8) yields that \( B(zI - A)f - (zI - A)Bf = Cf \) and consequently, by (6) and the equality \( s(f, z) = (zI - A)f \), we get that

\[
B(s(f, z) - (zI - A)BR(z, A)s(f, z) = CR(z, A)s(f, z) = R(z, A)Cs(f, z).
\]

It follows that

\[
R(z, A)B(s(f, z) - BR(z, A)s(f, z) = R^2(z, A)Cs(f, z).
\]

We have two possibilities:

(12) for each \( z \in \mathfrak{g}(A) \), the linear manifold \( M(z) = (zI - A)M \) of vectors \( s(f, z) (f \in M) \) is dense in \( H \),

or

(13) there exists \( z_0 \in \mathfrak{g}(A) \) such that \( M_{z_0} = (z_0I - A)M \) is not dense in \( H \).

Suppose that (12) holds true. Then by (11) we get that for \( z \in \mathfrak{g}(A) \)

\[
R(z, A)B - BR(z, A) = R^2(z, A)C.
\]

Let \( \{\delta_n\} \) be the topological basis of \( \delta(A) \) of clopen bounded sets, and take the projections

\[
P(\delta_n, A) = \frac{1}{2\pi i} \int_{I_n} R(z, A)dz
\]

for suitable \( I_n \). We derive now from (14), that for all \( n \)

\[
\frac{1}{2\pi i} \int_{I_n} R(z, A)Bhdz - \frac{1}{2\pi i} \int_{I_n} BR(z, A)hdz = \frac{1}{2\pi i} \int_{I_n} R^2(z, A)Chdz
\]

for each \( h \in H \). It follows from (1) that the last integral is equal to zero. Consequently,

\[
P(\delta_n, A)B - BP(\delta_n, A) = 0
\]

for each \( n \). By our lemma, \( R(z, A)B = BR(z, A) \) for \( z \in \mathfrak{g}(A) \), which by (14) implies that \( C = 0 \), i.e., (9) holds true.

Suppose that (13) holds true. Then there is a non-zero vector \( q \) orthogonal to \( (z_0I - A)M \). Since \( A_M = \overline{A|M} \subset A \) and \( (z_0I - A)^{-1} \) exists, \( (z_0I - A_M)^{-1} \) exists. It follows that \( z_0 \) is in the residual spectrum of \( A_M \), which completes the proof.
Let us come back to Theorem (AD). Since there appears the assumption that, in our notation, $C = iI \neq 0$, we see by our theorem that, for $A = N$, the closure of the restriction of $N$ to $M$ has a non-empty residual spectrum and consequently is not a selfadjoint operator. So, the part of $N$ in $M$ after closing is not an orthodox quantum observable. Hence, since the candidates for quantum phase operator $F$ have been constructed ([3], [7]), the part of quantum number operator, that one, which intervenes in the commutation relation $[N, F] f = if (f \in M)$, after closing will be never an observable. A similar effect appears for angle quantum variable $\varphi$. To be more precise, we take in the space $L^2(0, 2\pi)$ of functions $f(\varphi)$ the operator

$$A = \frac{1}{i} \frac{d}{d\varphi},$$

with domain

$D(A) = \{ f : f$ absolutely continuous on $[0, 2\pi]$, $f' \in L^2(0, 2\pi)$, $f(0) = f(2\pi) \}$. 

The spectrum of $A = A^*$ is the whole totality of integers $-\delta(A) = \{ 0, \pm 1, \pm 2, \pm 3, \ldots \}$; $A$ has a pure point simple spectrum. Let $\Phi$ be the selfadjoint operator of multiplication by an independent variable, that is, $(\Phi f)(\varphi) = \varphi f(\varphi)$ for $f \in L^2(0, 2\pi)$. $\Phi$ is bounded and selfadjoint. Next, for $f \in \{ g \in L^2(0, 2\pi) : g$ absolutely continuous and $g' \in L^2(0, 2), g(0) = 0 = g(2\pi) \} = M$ we have

$$[A, \Phi] f = -if.$$

By the theorem, the closure $\overline{A} \mid M = A_M$ is not selfadjoint. So again, the selfadjointness is lost at the cost of the commutation relation.

References