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Lattices with real numbers as additive operators

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CONTENTS

Introduction. In the mathematical analysis we often have to deal with the spaces of real functions defined on some metric space X, satisfying the Lipschitz condition with a constant $C > 0$; if $C = 1$, then we shall call such functions *metric functions*. The above-mentioned spaces, and also many others which are equally important (spaces of continuous real functions with the same continuity modulus, spaces of non-constant increasing real functions on the segment or on the straight line, etc.), are not linear spaces. They are all lattices (see Birkhoff [3]) as regards the normal partial ordering in them, and we can define the addition of any real number to any function in them as the addition of a constant function. (But the constant functions themselves need not belong to those spaces.)

Kaplansky [13] introduced a new axiomatically defined class of algebras — distributive lattices upon which the real numbers act as additive operators. These algebras he called *translation lattices* (in our terminology *distributive metric* d-*lattices*). Kaplansky proved that the translation lattices are isomorphic with the functional d-lattices (see § 1) of real-valued continuous functions defined on any compact space X (see [13], the main part of Theorem 2).

Here we give another proof of this assertion by means of a new notion of "bunch" instead of the classical "ideal" (see § 5). In the proof we do not make use of the representation theorem for distributive lattices.

A more general notion than that of metric d-lattices, namely the notion of a translation semi-lattice was investigated by Pierce [17]. Some of his results and ours are similar.

In this paper we give the representation theorem also for non-metric distributive d-lattices (see § 7). For metric distributive d-lattices we prove (in § 9) theorems which are generalizations of the Banach–Stone's theorem (cf. [2], [5], [7] and [9]) and some others. Then we try to show to what extent the non-distributive d-lattices have the structure of linear spaces (see § 4). The appropriate method appears to be the investigation of the influence of the metric on the algebraic structure of d-lattices (see § 4 and 8), which in itself is interesting. For this purpose we introduce some classes of metric spaces (see § 3). The section devoted to these spaces is independent of the preceding ones.

The following paper is an altered version of the author's dissertation submitted on 4. 10. 1965 at Warsaw University. Parts of it were presented in the lecture during the Symposium on Algebra in Warsaw in September 1964.

§ 1. In this section we introduce d-lattices and fundamental concepts connected with them (such as metric d-lattices, the comparability relation, the d-lattice dual to a given one, homomorphisms and isomorphisms of d-lattices, functional d-lattices). Among other results it is shown that every lattice can be (isomorphically) embedded in a certain metric d-lattice. Next, we show that every metric d-lattice is isometric to a certain d-lattice of real functions. In this lattice the addition of a number to an element, i.e. a function, consists in the addition of a constant function, and the supremum of two functions is given by their maximum. If, in addition, the infimum of two functions is given by their minimum, then we obtain the important class of functional d-lattices. Finally, we give several examples, illustrating certain problems connected with d-lattices.

A set S is called a *lattice* if it is partially ordered by a relation $\subseteq$, such that for arbitrary elements $a, b \in S$ there exists the least upper bound $a \cup b$ and the greatest lower bound $a \cap b$.

Definition 1. A d-*lattice* is a pair composed of a lattice S and a function which maps $S \times R$ (the Cartesian product of S and R, where R is the set of real numbers) into S and assigns an element $a+\alpha \in S$ to every element a of the lattice S and to every real number α, so that the following axioms are satisfied:

1. $a+0 = a$ for $a \in S$.

2. $(a+\alpha)+\beta = a+(\alpha+\beta)$ for $a \in S$ and $\alpha, \beta \in R$.

3. If $a \subseteq b$, then $a+\alpha \subseteq b+\alpha$ for $a, b \in S$ and $\alpha \in R$.

4. $a+\alpha \supset a$ for $\alpha > 0$ and $a \in S$.

5. If $a \supset b$, where $a, b \in S$, then there exists an $\varepsilon > 0$ such that $a \not\subseteq b+\varepsilon$.

6. For every $a, b \in S$ there exists

$$\bigcup_{\alpha \in R} \big(a \cap (b+\alpha)\big) \quad \text{and} \quad \bigcap_{\alpha \in R} \big(a \cup (b+\alpha)\big)$$

and we have

$$a = \bigcup_{\alpha \in R} \big(a \cap (b+\alpha)\big) = \bigcap_{\alpha \in R} \big(a \cup (b+\alpha)\big).$$

We shall prove a simple proposition:

(i) *If* $a \subseteq b+\varepsilon$ *for every* $\varepsilon > 0$, *then* $a \subseteq b$.

Proof. Suppose the contrary. Then $b \subset a \cup b$ and by axiom 5 there exists an $\varepsilon > 0$ such that $a \cup b \not\subseteq b+2\varepsilon$. Then, from axiom 4 it follows that $a \cup b \nsubseteq b+\varepsilon$, i.e. $a \nsubseteq b+\varepsilon$ (as $b \subset b+\varepsilon$), in contradiction to our assumption.

(ii) *Axiom* 1 *follows from axiom* 2, 4 *and* 5.

Proof. It follows from axiom 2 that $(a+0)+\varepsilon = a+\varepsilon$. Applying axiom 4 we obtain $a+0 \subseteq a+\varepsilon$ and $a \subseteq (a+0)+\varepsilon$ for every $\varepsilon > 0$ and, from property (i), $a+0 = a$.

DEFINITION 2. A *metric* d-*lattice* is a d-lattice such that:

6′. For every a, $b \in S$ there exists an $\alpha \in R$ such that $b \subseteq a+\alpha$.

It is evident that axiom 6′ implies axiom 6.

Elements a, b of a d-lattice S are said to be *comparable* if there exists a real number α such that

$$a \subseteq b+\alpha \quad \text{and} \quad b \subseteq a+\alpha.$$

Hence a d-lattice S is metric if and only if every pair of elements of S is comparable. The relation of comparability is an equivalence relation in a d-lattice S and every equivalence class of this relation is a maximal metric d-sublattice of the d-lattice S.

THEOREM 1.1 (duality theorem). *If a set S forms a* d-*lattice with respect to a partial ordering $\subseteq$ and addition $+$ of real numbers to elements of S, then it is also a* d-*lattice with respect to the ordering $\subseteq^\circ$ opposite to $\subseteq$ (i.e. $a \subseteq^\circ b$ iff $b \subseteq a$) and addition $+^\circ$ given by formula $a+^\circ\alpha = a+(-\alpha)$, $a \in S$, $\alpha \in R$.*

The d-*lattice $\langle S, \subseteq^\circ, +^\circ\rangle$ obtained in this way from a* d-*lattice $\langle S, \subseteq, +\rangle$ is called dual to $\langle S, \subseteq, +\rangle$ and will be denoted by S°.*

Proof. It is easy to see that axioms 1, 2 and 3 are satisfied in S°. Axiom 6 is self-dual. We shall prove the remaining two.

It follows from axioms 1 and 2 that $a+(-\alpha)+\alpha = a$. Hence from axiom 4 we obtain $a+(-\alpha) \subset a$ for $\alpha > 0$. Thus axiom 4 holds for S°.

If $a \supset^\circ b$, then $a \subset b$ and there exists (by axiom 5) an $\varepsilon > 0$ such that $a+\varepsilon \nsupseteq b$. This means that $a \nsupseteq b+(-\varepsilon)$. Thus axiom 5 holds for S°. The theorem is thus proved.

It is easy to see that if S is a metric d-lattice, then S° is also a metric one.

A mapping $f\colon S \to S'$ of a d-lattice S into a d-lattice S' is called a *homomorphism* if

$$f(a \cup b) = f(a) \cup f(b),$$
$$f(a \cap b) = f(a) \cap f(b),$$
$$f(a+\alpha) = f(a)+\alpha.$$

A 1-1 homomorphism is called an *isomorphism*. A homomorphism $f\colon S \to R$ is called *functional* (on S).

A subset S_0 of d-lattice S is a *sublattice* of S if $a \cup b \in S_0$ and $a \cap b \in S_0$ for every $a, b \in S_0$.

A sublattice S_0 of d-lattice S is a d-*sublattice* of S if $a + \alpha \in S_0$ for every $a \in S_0$ and $\alpha \in R$.

In the sequel we omit the brackets in the expressions $(a \cup b) + \alpha$, $(a \cup b) - \alpha$, $(a \cap b) + \alpha$ and $(a \cap b) - \alpha$.

THEOREM 1.2. (a) *The following relations hold in every* d-*lattice*:

$$a \cup b + \alpha = (a + \alpha) \cup (b + \alpha),$$
$$a \cap b + \alpha = (a + \alpha) \cap (b + \alpha).$$

(b) *The mapping* $f\colon S \to S$ *given by* $f(a) = a + \alpha$ *is an automorphism* (*isomorphism onto itself*).

Proof. It follows from axiom 3 that $a \cup b + \alpha \supseteq a + \alpha$ and $a \cup b + \alpha \supseteq b + \alpha$. Hence $a \cup b + \alpha \supseteq (a + \alpha) \cup (b + \alpha)$.

On the other hand, for $c = (a + \alpha) \cup (b + \alpha)$ we have $c \supseteq a + \alpha$ and $c \supseteq b + \alpha$. Hence $c + (-\alpha) \supseteq a$ and $c + (-\alpha) \supseteq b$. Hence $c + (-\alpha) \supseteq a \cup b$ and $c \supseteq a \cup b + \alpha$. Thus the first formula is true. The second formula is dual to the first one. Now assertion (b) is obvious and Theorem 1.2 is proved.

COROLLARY 1. *If a mapping* $f\colon S \to S'$ *is a homomorphism of a* d-*lattice* S *into a* d-*lattice* S', *then the mapping* $g\colon S \to S'$ *given by* $g(x) = f(x) + \alpha$, $\alpha \in R$, *is also a homomorphism.*

COROLLARY 2. *Relation of comparability is a lattice congruence.*

Later on we shall use the following important property of d-lattices:

LEMMA 1.3. *If* $b + \varepsilon \subseteq a$ (*respectively* $b - \varepsilon \supseteq a$) *for some* $\varepsilon > 0$ *and if* $a \subseteq b \cup c$ (*resp.* $a \supseteq b \cap c$), *then* $a \subseteq c$ (*resp.* $a \supseteq c$).

Proof. It follows from the assumptions of the lemma that

$$a \cup c \subseteq (a - n\varepsilon) \cup c, \quad n = 0, 1, 2, \ldots \tag{1}$$

In fact, inclusion (1) holds for $n = 0$. If it is true for a certain n, then

$$\begin{aligned} a \cup c &\subseteq (a - n\varepsilon) \cup c \subseteq (b \cup c - n\varepsilon) \cup c \\ &\subseteq (b - n\varepsilon) \cup (c - n\varepsilon) \cup c = (b - n\varepsilon) \cup c \\ &\subseteq \big(a - (n+1)\varepsilon\big) \cup c. \end{aligned}$$

Thus, by induction, inclusion (1) holds for every $n = 0, 1, 2, \ldots$ But

$$\bigcap_{n=0}^{\infty} [(a - n\varepsilon) \cup c] = c$$

(by axiom 6). Hence $a \subseteq c$. The lemma is thus proved.

THEOREM 1.4. *Any lattice* L *can be imbedded as a sublattice in a certain metric* d-*lattice.*

Proof. Let S denote the set of all functions $f\colon L \to R$ such that

$$f(x \cup y) = \min\bigl(f(x), f(y)\bigr)$$

and

$$\sup_{x,y\in L}|f(x)-f(y)| \leqslant 1.$$

Obviously, if $f\in S$ and $g(x) = f(x)+a$ for every $x\in L$, then $g\in S$. Also if $f_t\in S$ for $t\in T$, and

$$\inf_{t\in T, x\in L} f_t(x) > -\infty,$$

then a function $g\colon L \to R$ given by

$$g(x) = \inf_{t\in T} f_t(x), \quad x\in L,$$

belongs to S. Hence the set S partially ordered by relation $\subseteq$, where

$$f \subseteq g \text{ iff } f(x) \leqslant g(x) \text{ for every } x\in L$$

is a metric d-lattice if we put

$$(f+a)(x) = f(x)+a \quad \text{for} \quad f\in S,\, a\in R.$$

Evidently

$$(f \cap g)(x) = \min\bigl(f(x), g(x)\bigr) \quad \text{for} \quad f, g\in S.$$

Let us put

$$f_x(y) = \begin{cases} 1 & \text{if } x \supseteq y \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad x, y\in L.$$

Then $f_x\in S$ and the mapping $x \to f_x$, $x\in L$, is a lattice-isomorphic imbedding of L into S.

Indeed, $a \cap b \supseteq x$ iff $a \supseteq x$ and $b \supseteq x$, whence $f_{a\cap b}(x) = 1$ iff $\min\bigl(f_a(x), f_b(x)\bigr) = 1$, i.e.

(2) $$f_{a\cap b} = f_a \cap f_b.$$

Further, from (2) it follows that $f_{a\cup b} \supseteq f_a \cup f_b$. Since

$$(f_a \cup f_b)(a \cup b) = \min\bigl((f_a \cup f_b)(a),\ (f_a \cup f_b)(b)\bigr) \geqslant 1,$$

we have $(f_a \cup f_b)(x) \geqslant 1$ for any $x \subseteq a \cup b$ and evidently $(f_a \cup f_b)(x) \geqslant 0$ for all $x\in L$. Hence $f_{a\cup b} = f_a \cup f_b$.

Thus, it follows from the above theorem that there exist non-distributive and non-modular metric d-lattices (as lattices).

THEOREM 1.5. *Any metric* d-*lattice* S *is isomorphic to a* d-*lattice* S_1 *of real functions defined on the same* S *and such that*

$$(f+a)(x) = f(x)+a, \tag{3}$$

$$(f \cup g)(x) = \max\big(f(x), g(x)\big) \tag{4}$$

for every $f, g \in S_1$, $x \in S$, $a \in R$.

Proof. Let a real function $f_x\colon S \to R$, $x \in S$, be given by

$$f_x(y) = \inf\{a \in R\colon x \subseteq y+a\} \quad \text{for} \quad y \in S.$$

If $f+a$ is given by (3), then $f_{x+a} = f_x + a$. Furthermore,

$$\begin{aligned} f_{x\cup z}(y) &= \inf\{a \in R : x \cup z \subseteq y+a\} \\ &= \max\big(\inf\{a \in R : x \subseteq y+a\}, \inf\{a \in R : z \subseteq y+a\}\big) \\ &= \max\big(f_x(y), f_z(y)\big). \end{aligned}$$

Hence $S_1 = \{f_x\}_{x \in S}$ is closed under (3) and (4), i.e. S_1 is a d-lattice under (3) and (4), and the mapping $x \to f_x$ is an isomorphism of S onto S_1. The theorem is proved.

A set S of real functions defined on a certain set X such that with every pair of functions $f, g \in S$ and a real number a the functions $f+a$, $f \cup g$ and $f \cap g$ belong to S, where $f+a$ and $f \cup g$ are given by (3) and (4), and

$$(f \cap g)(x) = \min\big(f(x), g(x)\big) \tag{5}$$

is a d-lattice. Let us call d-lattices of this type *functional* d-*lattices*.

The following sets are examples of functional d-lattices. X denotes a non-empty set.

EXAMPLE 1. The set of all real functions defined on X.

EXAMPLE 2. The set of all bounded real functions defined on X.

EXAMPLE 3. The set S of all real functions defined on X such that for $f \in S$

$$\sup_{x,y \in X} |f(x)-f(y)| \leqslant 1.$$

EXAMPLE 4. R^n (in particular R) is a d-lattice if we put

$$(a_1, a_2, \ldots, a_n) \subseteq (\beta_1, \beta_2, \ldots, \beta_n) \text{ iff } a_i \leqslant \beta_i \tag{6}$$

for $i = 1, 2, \ldots, n$,

$$(a_1, a_2, \ldots, a_n)+a = (a_1+a, a_2+a, \ldots, a_n+a). \tag{7}$$

Thus we may say that R^n is a functional d-lattice of all real functions defined on the set $\{1, 2, \ldots, n\}$.

EXAMPLE 5. Let X be a topological space. Then the set $C(X)$ of all bounded continuous real functions on X is a functional d-lattice. In particular, if X is a discrete space, then we obtain the d-lattice from Example 2.

Also the set $C^{\cdot}(X)$ of all continuous real functions on X is a functional d-lattice. In particular, if X is a compact space, then $C^{\cdot}(X) = C(X)$. If X is discrete, then we obtain the d-lattice from Example 1.

EXAMPLE 6. Let S be the set of all bounded real functions defined on the set of positive integers N. If $\cup$ and $\cap$ are given in S by (4) and (5), and if, by definition,

$$(f+a)(n) = f(n) + \frac{a}{n} \quad \text{for} \quad f \in S,$$

then S is a non-metric d-lattice. Hence the d-lattice S is not isomorphic to the functional d-lattice of all bounded real functions defined on N (since the last d-lattice is metric), however, they are lattice-isomorphic.

EXAMPLE 7. Let S be given as in Example 3 for $X = \{1, 2\}$ and let S' and S'' be functional d-lattices of real functions defined on $\{1, 2\}$ and such that

$$f \in S' \Leftrightarrow |f(1) - f(2)| \leqslant 2,$$
$$f \in S'' \Leftrightarrow |f(1) - f(2)| \leqslant 1.$$

Then S, S', S'' are metric d-lattices which are isomorphic as lattices. However, they are not isomorphic d-lattices.

Every functional d-lattice is distributive (as a lattice) but there exist non-distributive d-lattices.

EXAMPLE 8. Let S be a subset of R^3 such that

$$(a_1, a_2, a_3) \in S \text{ iff } a_i = a_j \leqslant a_k$$

for a certain substitution (i, j, k) of the numbers $(1, 2, 3)$.

It is easy to see that the set S, partially ordered by relation $\subseteq$, where

$$(a_1, a_2, a_3) \subseteq (\beta_1, \beta_2, \beta_3) \text{ iff } a_1 \leqslant \beta_1, \; a_2 \leqslant \beta_2 \text{ and } a_3 \leqslant \beta_3,$$

is a d-lattice if we put

$$(a_1, a_2, a_3) + a = (a_1 + a, a_2 + a, a_3 + a).$$

Let $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1) \in S$. Then

$$(0, 0, 0) = (a \cap b) \cup (a \cap c) \neq a \cap (b \cup c) = (1, 0, 0). \tag{8}$$

Thus S is a non-distributive d-lattice.

Evidently

$$(\alpha_1, \alpha_2, \alpha_3) \frown (\beta_1, \beta_2, \beta_3) = \big(\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2), \min(\alpha_3, \beta_3)\big)$$

for $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in S$.

The d-lattice S is modular. Indeed, the d-sublattice $S_i = \{x \in S\colon x_i = \min(x_1, x_2, x_3)\}$ is isomorphic to the d-lattice R^2, hence S_i is a distributive d-sublattice, $i = 1, 2, 3$. Hence, if $a_1, a_2, a_3 \in S_i$ for a certain $i \in 1, 2, 3$ and

(9) $$a_1 \subseteq a_3, \quad a_2 \frown a_3 \subseteq a_1, \quad a_3 \subseteq a_1 \smile a_2,$$

then

(10) $$a_1 = a_3.$$

We shall show that (10) follows from (9) for any $a_1, a_2, a_3 \in S$. It is sufficient to verify the case $a_i \notin S_i$ for $i = 1, 2, 3$. We can assume $a_i \nsubseteq a_2$ and $a_2 \nsubseteq a_i$ for $i = 1, 3$, as other cases are trivial.

Thus for $a_i = (a_{i1}, a_{i2}, a_{i3})$, $i = 1, 2, 3$, we have

$$a_2 \frown a_3 = (a_{21}, a_{32}, a_{23}) \quad \text{and} \quad a_1 \smile a_2 = \big(a_{11}, a_{22}, \min(a_{11}, a_{22})\big),$$

as $a_{21} < a_{11} \leqslant a_{31}$ and $a_{12} \leqslant a_{32} < a_{22}$ and $a_{23} < a_{33}$. Since $a_2 \frown a_3 \subseteq a_1$, we have $a_{12} = a_{32}$ and since $a_3 \subseteq a_1 \smile a_2$ we have $a_{11} = a_{31} = a_{33}$. Thus $a_{11} = a_{12} = a_{13} = a_{31} = a_{32} = a_{33}$, whence $a_1 = a_3$. The modularity of S is proved (see [3], V, § 2, Theorem 2).

Example 9. Let S be a subset of R^4 such that

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in S \text{ iff } \alpha_i = \alpha_j \leqslant \alpha_k \leqslant \alpha$$

for a certain substitution (i, j, k, l) of the numbers $(1, 2, 3, 4)$.

It is easy to see that the set S, partially ordered by relation $\subseteq$, where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \subseteq (\beta_1, \beta_2, \beta_3, \beta_4) \text{ iff } \alpha_i \leqslant \beta_i$$

for $i = 1, 2, 3, 4$, is a d-lattice if we put

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + \alpha = (\alpha_1 + \alpha, \alpha_2 + \alpha, \alpha_3 + \alpha, \alpha_4 + \alpha).$$

We shall see that this d-lattice is non-modular.

Definition 3. The d-lattice S is d-*modular* if

$$a \smile \big(b \frown (a + \alpha)\big) = (a \smile b) \frown (a + \alpha)$$

for every $a, b \in S$, $\alpha \geqslant 0$.

Every modular d-lattice is d-modular. The d-lattice S from example 9 is not d-modular (and consequently is non-modular). Indeed, let $a = (0, 0, 2, 4)$, $b = (4, 4, 0, 0)$. Then

$$(1, 1, 2, 4) = a \smile \big(b \frown (a + 1)\big) \neq (a \smile b) \frown (a + 1) = (1, 1, 3, 4).$$

EXAMPLE 10. Let $S^n_{k,l} \subseteq R^n$, where $k, l, n = 1, 2, \ldots$ and $k+l \leqslant n$, be the set of all $(a_1, a_2, \ldots, a_n) \in R^n$ such that

$$a_{i_1} = a_{i_2} = \ldots = a_{i_{k-1}} = a_{i_k} \leqslant a_{i_{k+1}}$$
$$\leqslant a_{i_{k+2}} \leqslant \ldots \leqslant a_{i_{n-l}} \leqslant a_{i_{n-l+1}} = a_{i_{n-l+2}} = \ldots = a_{i_n}.$$

Then under (6) and (7) $S^n_{k,l}$ is a d-lattice.

EXAMPLE 11. Let (X, μ) be a measure space and let $\alpha, \beta \geqslant 0$. Then we define $S^\mu_{\alpha,\beta}$ as a set of measurable real-valued functions f on X such that

$$\mu\big\{x \in X\colon \mu\{y \in X\colon f(y) < f(x)\} = 0\big\} \geqslant \alpha$$

and

$$\mu\big\{x \in X\colon \mu\{y \in X\colon f(y) > f(x)\} = 0\big\} \geqslant \beta$$

which satisfy

$$\lim_{\varepsilon=0} \mu\{f^{-1}[(a-\varepsilon, a) \cup (a, a+\varepsilon)]\} = 0 \tag{11}$$

for every $a \in R$ (if $\mu(X) < \infty$, then this condition holds for every measurable function).

Then we put

$$f \subseteq g \text{ iff } \mu\{x \in X\colon f(x) > g(x)\} = 0$$

and

$$(f+\xi)(x) = f(x) + \xi \quad \text{for} \quad f, g \in S^\mu_{\alpha,\beta},\ \xi \in R.$$

Under this definition $S^\mu_{\alpha,\beta}$ is a d-lattice.

§ 2. In this section we define a metric d in d-lattices. This metric may assume the value ∞ (d does not assume this value if and only if the d-lattice is metric). The operations $\cup$, $\cap$, $+$ turn out to be uniformly continuous relative to the metric d. Then we show that every d-lattice can be isomorphically embedded in a complete metric space. Next, we investigate the interrelations among metric completeness, lattice completeness and hyperconvexity (introduced in [1] by Aronszajn and Panitchpakdi). Thus, for instance, every complete metric d-lattice is a hyperconvex metric space, and thus a complete metric space. At this point we give a partial solution of the problems stated in [1].

We shall show that in any d-lattice S we can define a metric d in a canonical way.

First we have to define some auxiliary notions:

$$d^+(a, b) = \begin{cases} \infty & \text{if } a+\alpha \not\supseteq b \text{ for all real } \alpha, \\ \inf\{\alpha\colon \alpha \geqslant 0 \text{ and } a+\alpha \supseteq b\} & \text{otherwise,} \end{cases}$$

and respectively (by duality)

$$d^{-}(a, b) = \begin{cases} \infty & \text{if } a-\alpha \nsubseteq b \text{ for all real } \alpha, \\ \inf\{\alpha\colon \alpha \geqslant 0 \text{ and } a-\alpha \subseteq b\} & \text{otherwise,} \end{cases}$$

where $a, b \in S$.

It is easy to see that:

(i) $d^{-}(a, b) = d^{+}(b, a)$;

(ii) if $a \subset b$, then $d^{+}(a, b) = d^{-}(b, a) > 0$;

(iii) $d^{+}(a, a) = d^{-}(a, a) = 0$;

(iv) $a - d^{-}(a, b) \subseteq b \subseteq a + d^{+}(a, b)$;

(v) $d^{+}(a, b) + d^{+}(b, c) \geqslant d^{+}(a, c)$, $d^{-}(a, b) + d^{-}(b, c) \geqslant d^{-}(a, c)$;

(vi) $d^{+}(a, b) = d^{+}(a, a \cup b) = d^{-}(b, a) = d^{-}(b, a \cap b)$.

Let now

$$d(a, b) = \max\big(d^{+}(a, b), d^{-}(a, b)\big).$$

It follows from (i)-(v) that the function d is a generalized metric, i.e. it differs from an ordinary metric only in that it may assume ∞ as its value.

It is easy to see that if d^{+0}, d^{-0} and d^{0} are defined in S^{0} analogously to d^{+}, d^{-} and d in S (S^{0} is a d-lattice dual to the d-lattice S), then

$$d^{+0} = d^{-}, \quad d^{-0} = d^{+}, \quad \text{and} \quad d^{0} = d.$$

This means that the identity mapping of S onto S^{0} is an isometry.

In the case of functional d-lattices the metric d coincides with the usual metric defined by sup.

We have:

(vii) if $a \subseteq b$, then $d(a, b) = d^{+}(a, b) = d^{-}(b, a)$;

(vi') $d^{-}(a, b) = d(a, a \cap b)$, $d^{+}(a, b) = d(a, a \cup b)$, and $d(a, b) = \max\big(d(a, a \cap b), d(a, a \cup b)\big)$;

(viii) if $a \subseteq a' \subseteq b$ and $a \subseteq b' \subseteq b$, then $d(a', b') \leqslant d(a, b)$;

(ix) $d(a \cap b, a \cup b) = d(a, b)$.

For example we shall prove (ix). It follows from (viii) that $d(a \cap b, a \cup b) \geqslant d(a, b)$. On the other hand,

$$a \cap b + d(a, b) = \big(a + d(a, b)\big) \cap \big(b + d(a, b)\big) \supseteq a \cup b.$$

Thus

$$d(a, b) \geqslant d^{+}(a \cap b, a \cup b) = d(a \cap b, a \cup b).$$

The elements a, b of a d-lattice S are comparable (see § 1) iff $d(a, b) < \infty$ and a d-lattice S is metric iff the function d is an ordinary metric in S.

The d-lattices given by Examples 2-4, 7-10 and also 11 if $a > 0$ and $\beta > 0$, from § 1, are metric d-lattices. If X is infinite, then the d-lattice given by Example 1 from § 1 is not metric.

Let us notice that if $h: S \to S'$ is a homomorphism of a d-lattice S into a d-lattice S', then

$$d(h(x),\ h(y)) \leqslant d(x,\ y), \qquad x,\ y \in S.$$

Moreover,

(x) $d(x \cup y,\ x' \cup y') \leqslant \max\big(d(x,x'),\ d(y,y')\big)$;

(xi) $d(x \cap y,\ x' \cap y') \leqslant \max\big(d(x,x'),\ d(y,y')\big)$;

(xii) $d(x+a,\ y+a) = d(x,\ y)$;

(xiii) $d(x+a,\ x+\beta) = |a-\beta|$.

This shows that any homomorphism of d-lattices is a uniformly continuous mapping, the operators $\cup$ and $\cap$ are uniformly continuous mappings of $S \times S$ into S, and the operation $+$ is a uniformly continuous mapping of $S \times R$ into S. Hence we obtain:

THEOREM 2.1. *If S_0 is a* d*-sublattice of any* d*-lattice S, then the closure of S_0 (in the topology induced by the metric d) is a* d*-sublattice of S as well.*

THEOREM 2.2. *If S_0 is a* d*-sublattice of a* d*-lattice S and $f: S_0 \to S_1$ is a homomorphism of S_0 into a metrically complete* d*-lattice S_1, then there exists (exactly one) continuous extension $g: \bar{S}_0 \to S_1$ of homomorphism f and g is a homomorphism of the* d*-sublattice $\bar{S}_0$ into S_1.*

A metric space X, with a metric ϱ, is said to be a *hyperconvex* (resp. $\mathfrak{m}$-*hyperconvex*) *metric space* if for any subset $X_0 \subseteq X$ (resp. for any subset $X_0 \subseteq X$ with $\mathrm{card}(X_0) < \mathfrak{m}$) and for any real function $f: X_0 \to R$, satisfying the condition $f(x)+f(y) \geqslant \varrho(x, y)$ for any $x,\ y \in X_0$, there exists an element $a \in X$ such that $\varrho(a, x) \leqslant f(x)$ for every $x \in X_0$ (Aronszajn and Panitchpakdi [1]).

THEOREM 2.3. *Every metric* d*-lattice S is an $\aleph_0$-hyperconvex metric space.*

Proof. Let $x_1, x_2, \ldots, x_n \in S$ and let $a_1, a_2, \ldots, a_n$ be real numbers satisfying the condition

$$a_i + a_j \geqslant d(x_i, x_j), \qquad i, j = 1, 2, \ldots, n$$

(n being an arbitrary natural number).

Putting

$$a = \bigcup_{i=1}^{n} (x_i - a_i),$$

we have $x_i - a_i \subseteq a$, i.e. $d^-(x_i, a) \leqslant a_i$.

On the other hand, we have

$$x_i + (a_i + a_j) \supseteq x_j \qquad (\text{for } d(x_i, x_j) \leqslant a_i + a_j).$$

Hence

$$x_i + a_i \supseteq x_j - a_j, \qquad j = 1, 2, \ldots, n,$$

and

$$x_i + a_i \supseteq \bigcup_{j=1}^{n} (x_j - a_j) = a$$

and

$$d^+(x_i, a) \leqslant a_i.$$

Thus $d(x_i, a) \leqslant a_i$ for $i = 1, 2, \ldots, n$ and the theorem is proved.

COROLLARY. *If every bounded subset of a metric* d-*lattice S is totally bounded and S is metrically complete, then S is hyperconvex.*

Indeed, it is true for any $\aleph_0$-hyperconvex complete metric space.

Let us recall that the lattices in which every non-empty bounded (and countable) subset has a least upper bound and a greatest lower bound are called *conditionally complete* (*σ-complete*) lattices.

DEFINITION. A d-lattice S is called a *complete* (*σ-complete*) d-*lattice* if it is conditionally complete (σ-complete) as a lattice.

THEOREM 2.4. *Every complete (σ-complete) metric* d-*lattice S is a hyperconvex ($\aleph_1$-hyperconvex) metric space.*

Proof. The proof of this theorem is analogous to that of Theorem 2.3. Let $A \subseteq S$ be any (countable) subset and let $f\colon A \to R$ be any real function satisfying the condition $f(x)+f(y) \geqslant d(x, y)$ for $x, y \in A$.

The set $\{x-f(x)\}_{x\in A}$ is upper-bounded (indeed, if $x_0 \in A$, then $x-f(x) \subseteq x_0+f(x_0)$ for every $x \in A$). Thus we have an element $a = \bigcup_{x\in A} \big(x-f(x)\big)$ and $d(x, a) \leqslant f(x)$ for arbitrary $x \in A$ and the theorem is proved.

THEOREM 2.5. *Every σ-complete* d-*lattice S is complete as a metric space.*

Proof. Let (a_n), $n = 1, 2, \ldots$, be a Cauchy sequence in S (i.e. $\lim_{n,m=\infty} d(a_n, a_m) = 0$). Then there exists a natural N such that $d(a_N, a_n) \leqslant a < \infty$ for $n > N$. Then $a_N - a \subseteq a_n \subseteq a_N + a$ for $n > N$ and there exists an element

$$a = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} a_k.$$

We shall show that $\lim_{n=\infty} d(a, a_n) = 0$.

In fact, for every $\varepsilon > 0$ there exists a natural number N_ε such that $d(a_{N_\varepsilon}, a_n) \leqslant \varepsilon$ for $n \geqslant N_\varepsilon$, i.e.

$$a_{N_\varepsilon} - \varepsilon \subseteq a_n \subseteq a_N + \varepsilon \qquad \text{for} \qquad n \geqslant N_\varepsilon,$$

whence

$$a_{N_\varepsilon} - \varepsilon \subseteq \bigcap_{n=N_\varepsilon+k}^{\infty} a_n \subseteq a_{N_\varepsilon} + \varepsilon \quad \text{for} \quad k = 0, 1, \ldots$$

and

$$a_{N_\varepsilon} - \varepsilon \subseteq \bigcup_{n=N_\varepsilon}^{\infty} \bigcap_{m=n}^{\infty} a_m = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} a_m \subseteq a_{N_\varepsilon} + \varepsilon .$$

Hence for every $\varepsilon > 0$ there exists a natural number n such that $d(a, a_n) \leqslant \varepsilon$. This shows that $\lim_{n=\infty} a_n = a$ for $(a_n)_{n=1}$ is a Cauchy sequence. The theorem is proved.

Remark. Now it is easy to see that in a σ-complete d-lattice $\lim_{n=\infty} a_n = a$ iff

$$a = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} a_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} a_n .$$

LEMMA 2.6. *If A is a subset of a* d-*lattice S and if $a \in S$ is an upper (lower) bound of A, then $a + \alpha$ is an upper (lower) bound of the set*

$$A + \alpha = \{x + \alpha\}_{x \in A}$$

for arbitrary $\alpha \in R$. In addition, if there exists

$$\bigcup_{x \in A} x \quad (\textit{resp.} \bigcap_{x \in A} x),$$

then there exists

$$\bigcup_{x \in A} (x + \alpha) \quad (\textit{resp.} \bigcap_{x \in A} (x + \alpha))$$

and

$$\bigcup_{x \in A} (x + \alpha) = \bigcup_{x \in A} x + \alpha \quad (\textit{resp.} \bigcap_{x \in A} (x + \alpha) = \bigcap_{x \in A} x + \alpha)$$

for $\alpha \in R$.

Proof. The first part of the above lemma is trivial. For the second part we have $y + \alpha \subseteq \bigcup_{x \in A} x + \alpha$ for every $y \in A$. On the other hand, if b is an upper bound of $A + \alpha$, then $x + \alpha \subseteq b$ and $x \subseteq b - \alpha$ for every $x \in A$. Hence

$$\bigcup_{x \in A} x \subseteq b - \alpha \quad \text{and} \quad \bigcup_{x \in A} x + \alpha \subseteq b .$$

The lemma is proved.

The following lemma is a variation on the theme of Mac Neille's theorem (see [3]):

LEMMA 2.7. *Any set S partially ordered by a certain relation $\subseteq$ can be imbedded in a conditionally complete lattice $\tilde{S}$, so that inclusion is preserved,* *together with all greatest lower bounds and least upper bounds, and the following two conditions are satisfied:*

(a) *for any subset $A \subseteq S$, there exists a greatest lower bound of $i(A)$ in $\tilde{S}$ iff A is bounded from below in S and there exists a least upper bound of $i(A)$ in $\tilde{S}$ iff it is bounded from above in S.*

(b) $a = \bigcap \{x \in i(S): \ a \subseteq x\} = \bigcup \{x \in i(S): \ x \subseteq a\}$ *for* $a \in \tilde{S}$, *where* $i: S \to \tilde{S}$ *is the imbedding.*

Proof. Let $\tilde{S}$ be a set of all ordered pair (A, B) where A and B are non-empty subsets of S such that

(i) $a \subseteq b$ for any $a \in A$ and $b \in B$,

(ii) if a pair (A', B') satisfies condition (i) and $A \subseteq A'$, $B \subseteq B'$, then $A = A'$ and $B = B'$.

It is easy to see that for $(A, B) \in \tilde{S}$ we have

$$A = \{a \in S: \bigvee_{b \in B} a \subseteq b\}$$

and

$$B = \{b \in S: \bigvee_{a \in A} a \subseteq b\}.$$

We put $(A, B) \subseteq (A', B')$ iff $A \subseteq A'$. Then $(A, B) \subseteq (A', B')$ iff $B \supseteq B'$.

Next, we shall show that $(\tilde{S}, \subseteq)$ is a conditionally complete lattice.

Indeed, let $(A_0, B_0) \in \tilde{S}$ be a lower bound of $U \subset \tilde{S}$. Then $A_0 \subset A$ for every $(A, B) \in U$, whence

$$A_1 = \bigcap_{(A,B) \in U} A \neq \emptyset.$$

Let

$$B_1 = \{b \in S: \bigvee_{a \in A_1} a \subseteq b\}.$$

Then $B_1 \supseteq B = \emptyset$ for any $(A, B) \in U$. Hence the pair (A_1, B_1) satisfies condition (i).

Let $(A', B') \in \tilde{S}$, $A_1 \subseteq A'$ and $B_1 \subseteq B'$. Then $B_1 = B'$. We see also that $B \subseteq B' = B_1$ for any $(A, B) \in U$. Hence $A \supseteq A'$ for any $(A, B) \in U$ and $A_1 \supseteq A'$, i.e. $A_1 = A'$. Thus $(A_1, B_1) \in \tilde{S}$. It is obvious that $(A_1, B_1) = \bigcap U$. Analogously, if $U \subset \tilde{S}$ is bounded from above, then there exists $(A_2, B_2) = \bigcup U \in \tilde{S}$ and

$$B_2 = \bigcap_{(A,B) \in U} B, \qquad A_2 = \{a \in S: \bigvee_{b \in B_2} a \subseteq b\}.$$

Thus $\tilde{S}$ is a conditionally complete lattice.

We put $i(x) = (\{y \in S: y \subseteq x\}, \{y \in S: x \subseteq y\})$ for $x \in S$. It is obvious that $i(S) \subseteq \tilde{S}$ and $(A, B) \in i(S)$ iff $A \cap B \neq \emptyset$, and then $A \cap B$ has exactly one element. If $i(x) = (A, B)$, then $x \in A \cap B$. Thus if $x \neq y$, $x, y \in S$, then $i(x) \neq i(y)$. It is also easy to see that the imbedding i preserves all greatest lower bounds and least upper bounds (hence i preserves inclusion).

We shall verify that conditions (a) and (b) hold. Let $a = (A, B) \in \tilde{S}$. We put $i(u) = (A_u, B_u)$ for $u \in S$. Then $A = \bigcup_{u \in A} A_u$, $B = \bigcup_{v \in B} B_v$. Hence $a = \bigcup i(A) = \bigcap i(B)$. Thus condition (b) holds. (Moreover, $i(A) = \{x \in i(S): x \subseteq a\}$ and $i(B) = \{x \in i(S): a \subseteq x\}$.)

Condition (a) follows from the conditional completeness of $\tilde{S}$ and from condition (b). The lemma is proved.

If S also has the structure of a d-lattice, then $\tilde{S}$ also admits the structure of a d-lattice such that the imbedding of S into $\tilde{S}$ is a d-isomorphism, i.e. the following theorem holds:

THEOREM 2.8. *Any* d-*lattice* S *can be isomorphically imbedded into a complete* d-*lattice* $\tilde{S}$.

Proof. Let $\tilde{S}$ be a lattice satisfying conditions (a) and (b) from Lemma 2.7. We can assume that S is a sublattice of $\tilde{S}$. Let A be a subset of S bounded from above and $a = \bigcup_{x \in A} x \in \tilde{S}$. Then it follows from the first part of the lemma that $\bigcup_{x \in A} (x + a)$ also exists in $\tilde{S}$ and we define $a + a$ as being equal to $\bigcup_{x \in A} (x + a)$ for $a \in R$ (if also $a = \bigcup_{x \in B} x$ for a certain $B \subseteq S$, then of course $\bigcup_{x \in A} (x + a) = \bigcup_{x \in B} (x + a)$).

Then we have

1'. $a + 0 = a$.

2'. $(a + a) + \beta = a + (a + \beta)$.

Indeed,

$$(a + a) + \beta = \bigcup_{x \in A} (x + a) + \beta$$
$$= \bigcup_{x \in A} [(a + a) + \beta] = \bigcup_{x \in A} [x + (a + \beta)] = a + (a + \beta).$$

3'. If $a \subseteq b$, then $a + a \subseteq b + a$ for $a, b \in \tilde{S}$.

In fact,

$$a = \bigcup_{x \in A} x, \quad b = \bigcup_{x \in B} x,$$

where $A = \{x \in S: x \subseteq a\}$, $B = \{x \in S: x \subseteq b\}$. But $a \subseteq b$, whence $A \subseteq B$. Therefore

$$a + a = \bigcup_{x \in A} (x + a) \subseteq \bigcup_{x \in B} (x + a) = b + a.$$

4'. $a+\alpha \supset a$ for $\alpha > 0$, $a \in \tilde{S}$.

Of course, $a+\alpha \supseteq a$ for $\alpha > 0$. Let $a = \bigcup_{x \in A} x$, where $A \subseteq S$, and let $b \supseteq a$, $b \in S$. If $a+\alpha = a$, then $a+n\alpha = a \subseteq b$, $n = 1, 2, \ldots$ Hence we would have $x+n\alpha \subseteq b$ for $x, b \in S$, $x \in A$, $n = 1, 2, \ldots$ and by axiom 6 (§1)

$$b+1 = \bigcup_{\beta \in R} \big((b+1) \cap (x+\beta)\big) = \bigcup_{\beta \in R} (x+\beta) = \bigcup_{n=1}^{\infty} (x+n\alpha) \subseteq b.$$

The contradiction implies $a+\alpha \supset a$.

5'. For any $a, b \in \tilde{S}$ if $a \supset b$, then there exists an $\varepsilon > 0$ such that $a \not\subset b+\varepsilon$.

If $a \subset b+\varepsilon$ for every $\varepsilon > 0$, then for every $x, y \in S$ such that $x \subseteq a$ and $b \subseteq y$ we have $x \subseteq y+\varepsilon$ for every $\varepsilon > 0$, whence $x \subseteq y$. Thus $a \subseteq b$ and this contradicts our assumption.

6'. $a = \bigcup_{\alpha \in R} \big(a \cap (b+\alpha)\big) = \bigcap_{\alpha \in R} \big(a \cup (b+\alpha)\big)$, $a, b \in \tilde{S}$.

Of course, $a \supseteq \bigcup_{\alpha \in R} \big(a \cap (b+\alpha)\big)$. On the other hand, if $a = \bigcup_{x \in A} x$, $b = \bigcup_{y \in B} y$, where $A, B \subseteq S$, then

$$\begin{aligned}
\bigcup_{\alpha \in R} \big(a \cap (b+\alpha)\big) &= \bigcup_{\alpha \in R} \Big(\bigcup_{x \in A} x \cap \big(\bigcup_{y \in B} y+\alpha\big)\Big) \\
&= \bigcup_{\alpha \in R} \Big(\bigcup_{x \in A} x \cap \bigcup_{y \in B} (y+\alpha)\Big) \\
&\supseteq \bigcup_{\alpha \in R} \bigcup_{x \in A} \bigcup_{y \in B} \big(x \cap (y+\alpha)\big) \\
&= \bigcup_{x \in A} \bigcup_{y \in B} \bigcup_{\alpha \in R} \big(x \cap (y+\alpha)\big) \\
&= \bigcup_{x \in A} x = a.
\end{aligned}$$

Theorem 2.8 is proved.

From Theorems 2.1, 2.5, 2.8 we obtain

THEOREM 2.9. *Any* d-*lattice can be isomorphically imbedded into a metrically complete* d-*lattice as a topologically dense* d-*sublattice.*

Remark. Under conditions (a) and (b) from Lemma 2.7, the lattice $\tilde{S}$ from Lemma 2.7 and the d-lattice $\tilde{S}$ from Theorem 2.8 and the isomorphic imbedding $i: S \to \tilde{S}$ are unique in the following sense: if S_1 is a conditionally complete lattice or a complete d-lattice and $i_1: S \to S_1$ is an isomorphic imbedding such that conditions (a) and (b) hold, then there exists an isomorphism $k: \tilde{S} \to S_1$ such that $i_1 = k \circ i$.

THEOREM 2.10. *For any infinite regular* (1) *cardinal number* $\aleph_\mu$, *there exists an* $\aleph_\mu$*-hyperconvex metric* d*-lattice* S, *which is not* $\aleph_{\mu+1}$*-hyperconvex* (2).

Proof. Let X' and X'' be disjoint sets, and card X' = card $X'' = \aleph_\mu$. By S we denote the functional d-lattices of all real functions $f\colon X \to R$, where $X = X' \cup X''$ such that $\operatorname{card}\big(X \setminus f^{-1}(a)\big) < \aleph_\mu$ for a certain $a = a(f) \in R$.

Let us put

$$f_p(x) = \begin{cases} 0 & \text{if} \quad x \neq p,\, x \in X, \\ 1 & \text{if} \quad x = p \in X', \\ -1 & \text{if} \quad x = p \in X'', \end{cases}$$

for $p \in X$. Then $d(f_p, f_q) = 1$ for $p \neq q$ and there exists no $f \in S$ such that $\varrho(f, f_p) \leqslant \frac{1}{2}$ for all $p \in X$. Hence S is not $\aleph_{\mu+1}$-hyperconvex.

Now we shall prove that if card$(A) < \aleph_\mu$, where $A \subseteq S$, and $\varphi\colon A \to R$ satisfies $\varphi(f) + \varphi(g) \geqslant d(f, g)$ for any $f, g \in A$, then there exists an $h \in S$ such that $d(f, h) \leqslant \varphi(f)$ for every $f \in A$.

Indeed,

$$\operatorname{card}\Big(X \setminus \bigcap_{f \in A} f^{-1}\big(a(f)\big)\Big) < \aleph_\mu$$

as $\aleph_\mu$ is regular. We put $h(x) = \sup\limits_{f \in A}\big(f(x) - \varphi(f)\big)$ for $x \in X$. Then $h\colon X \to R$ is constant on $\bigcap\limits_{f \in A} f^{-1}\big(a(f)\big)$, whence $h \in S$, and $d(f, h) \leqslant \varphi(f)$ for $f \in A$. Hence S is $\aleph_\mu$-hyperconvex. The theorem is proved.

Let us remark that under the above notions if $\mu \geqslant 1$, then the d-lattice S is isomorphic (and consequently isometric) to d-lattice $C\big(\beta(X_1)\big)$, where X_1 is a topological space such that $X_1 = \{p\} \cup X$, $p \notin X$, and $G \subseteq X_1$ is an open subset of X_1 if and only if $p \notin G$ or card$(X \setminus G) < \aleph_\mu$. Thus from Theorem 5.2 of paper [1] follows a partial solution of Problem 3 of the same paper.

THEOREM 2.11. *For any regular cardinal number* $\aleph_\mu > \aleph_0$ *there exists an* $\aleph_\mu$*-extremally disconnected compact space which is not* $\aleph_{\mu+1}$*-extremally disconnected.*

§ 3. The concern of the present section is to study metric spaces. We consider a class of metric spaces such that to every pair of points a, b of a space of this class one can assign a point $s(a, b)$, called the *central middle* of a and b, and this assigning is metrically invariant. We can say

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(1) The cardinal number τ = card Z is *regular* if there exists no family P of sets such that card $P < \tau$, card $A < \tau$ for every $A \in P$ and $\bigcup P = Z$.

(2) Recently the author [10] proved that for any finite cardinal $m > 3$ there exists an m-hyperconvex metric space, which is not an $(m+1)$-hyperconvex metric space (this is the solution of Problem 1 from [1]).

of a central space, i.e. a metric space X in which an operation $s: X \times X \to X$ has been defined, that it has been given some (though rather weak) algebraic structure. We also define some smaller classes of metric spaces — but all these classes contain the class of hyperconvex spaces. For hyperconvex spaces we prove that the operation s is continuous (this result and a stronger one are contained in Theorem 3.5). Besides the operation s we introduce another operation ${}_as$. ${}_as$ is a partial operation of a type similar to s, but it is more regular than s (see Theorem 3.9). The above-mentioned results can be applied to metric d-lattices, for every complete metric d-lattice is a central space (see § 4).

Let (X, ϱ) be a metric space. For a bounded non-empty set $A \subseteq X$ we denote by $S_1^X(A)$, or simply $S_1(A)$, the set of all $x \in X$ such that $\varrho(x, y) \leqslant \frac{1}{2}$ diam A for every $y \in A$, and we put

$$S_{n+1}(A) = S_n(A) \cap S_1(S_n(A)) \quad \text{for} \quad n = 1, 2, \ldots$$

Since

$$\operatorname{diam} S_1(A) \leqslant \operatorname{diam} A \quad \text{and} \quad \operatorname{diam} S_{n+1}(A) \leqslant \tfrac{1}{2} \operatorname{diam} S_n(A)$$

for $n = 1, 2, \ldots$, we have

$$\lim_{n=\infty} \operatorname{diam} S_n(A) = 0.$$

Thus the set $\bigcap_{n=1}^{\infty} S_n(A)$ contains a single-point or is empty. It is obvious that $S_1(A)$ and consequently the sets $S_n(A)$ are closed. If X is complete and $S_n(A)$ is a non-empty set for every $n = 1, 2, \ldots$, then, from the Cantor theorem, there exists a point $s(A) \in \bigcap_{n=1}^{\infty} S_n(A)$ (exactly one).

For $A = \{a, b\} \subseteq X$ we shall write $S_n(a, b)$ instead of $S_n(\{a, b\})$ and $s(a, b)$ instead of $s(\{a, b\})$.

The set $S_1(A)$ is the set of all metric middles of A (of a and b, if $A = \{a, b\}$). The point $s(A)$ (respectively $s(a, b)$) will be called the *central middle* of A (of a and b) (³).

Definition 1. A metric space (X, ϱ) is said to be:

an *absolutly central space* if a point $s(A)$ exists for any non-empty bounded subset $A \subseteq X$;

a *strongly central space* if a point $s(A)$ exists for any non-empty totally bounded set $A \subseteq X$;

a *central space* if a point $s(a, b)$ exists for any $a, b \in X$.

(³) The idea of a central middle $s(a, b)$ (for normed linear spaces) is due to Mazur and Ulam [16].

For example if a metric space is strongly convex (see [4]), then $S_1(a, b) = \{s(a, b)\}$ for any points a, b. Hence a strongly convex space is a central space. On the other hand, an Euclidean plane is a strongly convex space, but it is not a strongly central space.

It is easy to see that:

(1) if $A \subseteq B$ and $\operatorname{diam} A = \operatorname{diam} B$, then $S_1(A) \supseteq S_1(B)$;

(2) if $x \in \bar{A}$ and $y \in S_1(A)$, then $d(x, y) \leqslant \frac{1}{2} \operatorname{diam} A$.

Hence

(3) $S_1(A) = S_1(\bar{A})$.

Thus if a point $s(A)$ exists for every compact subset A of a complete metric space, then this space is strongly central.

THEOREM 3.1. *Every hyperconvex metric space (X, ϱ) is absolutly central.*

Proof. Let A be a bounded non-empty subset of X. Then $S_1(A) \neq \emptyset$ and $S_1(A)$ is a hyperconvex subspace of X. Hence $S_2(A) \neq \emptyset$ as $S_2(A) = S_1^{S_1(A)}(S_1(A))$ and also $S_2(A)$ is a hyperconvex subspace. Similarly, by induction $S_n(A) \neq \emptyset$ and $S_n(A)$ is a hyperconvex subspace of X for any $n = 1, 2, \ldots$ Since any hyperconvex space is complete, the theorem is proved.

If X is a compact strongly convex space, then the mapping $s\colon X \times X \to X$ is continuous. On the other hand, there exist examples of complete strongly convex spaces X such that the mapping $s\colon X \times X \to X$ is not continuous (see [8], [15]). We shall prove that the mapping $s\colon X \times X \to X$ is continuous for any hyperconvex space X.

LEMMA 3.2. *Let (X, ϱ) be a hyperconvex space. Then*

$$\varrho_H\big(S_n(A), S_n(B)\big) \leqslant 2^n \varrho_H(A, B)$$

for any $A, B \in 2^X$ and $n = 0, 1, 2, \ldots$, where by definition $S_0(Y) = Y$ for any $Y \subseteq X$ [4].

Proof. The case of $n = 0$ is trivial. Let us assume that for $k \leqslant n$ the lemma holds and let $x \in S_{n+1}(A)$. We define a real function $f_n\colon B \cup S_1(B) \to R$ as follows:

$$f_n(p) = \begin{cases} \frac{1}{2} \operatorname{diam} B & \text{if} \quad p \in B \setminus S_1(B), \\ \frac{1}{2} \operatorname{diam} S_k(B) & \text{if} \quad p \in S_k(B) \setminus S_{k+1}(B),\ k = 1, 2, \ldots, n-1, \\ \frac{1}{2} \operatorname{diam} S_n(B) & \text{if} \quad p \in S_n(B). \end{cases}$$

---

[4] 2^X is the space of all closed bounded non-empty subsets of X and $\varrho_H(A, B) \stackrel{\text{df}}{=} \max\big(\sup_{x \in A} \inf_{y \in B} \varrho(x, y), \sup_{x \in B} \inf_{y \in A} \varrho(x, y)\big)$ for any $A, B \in 2^X$.

Then

$$\{q\in X:\ \bigvee_{p\in B\cup S_1(B)} \varrho(p,q)\leqslant f_n(p)\} = S_{n+1}(B).$$

Since $x\in S_{n+1}(A)$, by the induction assumption it follows that for $p\in S_k(B)$, where $0\leqslant k\leqslant n$, we have

$$\begin{aligned}\varrho(x,p) &\leqslant \tfrac{1}{2}\operatorname{diam} S_k(A)+\varrho_H\big(S_k(A),S_k(B)\big)\\ &\leqslant \tfrac{1}{2}\Big(\operatorname{diam} S_k(B)+2\varrho_H\big(S_k(A),S_k(B)\big)\Big)+\varrho_H\big(S_k(A),S_k(B)\big)\\ &\leqslant \tfrac{1}{2}\big(\operatorname{diam} S_k(B)+2\cdot 2^k\varrho_H(A,B)\big)+2^k\varrho_H(A,B)\\ &\leqslant \tfrac{1}{2}\operatorname{diam} S_k(B)+2^{k+1}\varrho_H(A,B)\\ &\leqslant \tfrac{1}{2}\operatorname{diam} S_k(B)+2^{n+1}\varrho_H(A,B)\\ &\leqslant 2^{n+1}\varrho_H(A,B)+f_n(p).\end{aligned}$$

Thus $\varrho(x,p)\leqslant 2^{n+1}\varrho_H(A,B)+f_n(p)$. Since X is a hyperconvex space, there exists a point $y\in S_{n+1}(B)$ such that

$$\varrho(x,y)\leqslant 2^{n+1}\varrho_H(A,B).$$

Analogously, for any $x\in S_{n+1}(B)$ there exists a $y\in S_{n+1}(A)$ such that $\varrho(x,y)\leqslant 2^{n+1}\varrho_H(A,B)$. Hence

$$\varrho_H\big(S_{n+1}(A),S_{n+1}(B)\big)\leqslant 2^{n+1}\varrho_H(A,B).$$

The lemma is proved.

LEMMA 3.3. *Let (X,ϱ) be a hyperconvex space. Then*

$$\varrho_H\big(S_{n+1}(A),S_{n+1}(B)\big)-\varrho_H\big(S_n(A),S_n(B)\big)\leqslant 2^{-n+1}\big(\operatorname{diam} A+\varrho_H(A,B)\big)$$

for any $A,B\in 2^X$ and $n=1,2,\ldots$

Proof. We have

$$\begin{aligned}&\varrho_H\big(S_{n+1}(A),S_{n+1}(B)\big)\\ &\quad\leqslant \varrho_H\big(S_{n+1}(A),S_n(A)\big)+\varrho_H\big(S_n(A),S_n(B)\big)+\varrho_H\big(S_n(B),S_{n+1}(B)\big)\\ &\quad\leqslant \tfrac{1}{2}\operatorname{diam} S_n(A)+\tfrac{1}{2}\operatorname{diam} S_n(B)+\varrho_H\big(S_n(A),S_n(B)\big)\\ &\quad\leqslant 2^{-n}\operatorname{diam} A+2^{-n}\operatorname{diam} B+\varrho_H\big(S_n(A),S_n(B)\big)\\ &\quad\leqslant 2^{-n+1}\big(\operatorname{diam} A+\varrho_H(A,B)\big)+\varrho_H\big(S_n(A),S_n(B)\big).\end{aligned}$$

LEMMA 3.4. *Let (X,ϱ) be a hyperconvex space. Then*

$$\varrho_H\big(S_n(A),S_n(B)\big)\leqslant 4\sqrt{2\varrho_H(A,B)\big(\operatorname{diam} A+\varrho_H(A,B)\big)} \tag{4}$$

for any $A,B\in 2^X$ and $n=1,2,\ldots$ Consequently

$$\varrho\big(s(A),s(B)\big)\leqslant 4\sqrt{2\varrho_H(A,B)\big(\operatorname{diam} A+\varrho_H(A,B)\big)}. \tag{5}$$

Proof. Evidently if $A = B$, then inequalities (4) and (5) are satisfied. Thus let $A \neq B$. From Lemmas 3.2 and 3.3 it follows that

$$\varrho_H(S_n(A), S_n(B)) < 2^{a+1} \cdot \varrho_H(A, B) + \sum_{k=0}^{\infty} 2^{(-a+1)-k}[\operatorname{diam} A + \varrho_H(A, B)]$$
$$= 2.2^a \cdot \varrho_H(A, B) + 4.2^{-a}[\operatorname{diam} A + \varrho_H(A, B)]$$

for any $a > 0$.

Let

$$a = \tfrac{1}{2}\left(1 + \lg_2(\operatorname{diam} A + \varrho_H(A, B)) - \lg_2 \varrho_H(A, B)\right).$$

Then

$$2^a = \sqrt{\frac{2(\operatorname{diam} A + \varrho_H(A, B))}{\varrho_H(A, B)}}$$

and we obtain (4). The lemma is proved.

THEOREM 3.5. *Let (X, ϱ) be a hyperconvex space. The mapping $A \to s(A)$ is uniformly continuous on any subspace $P_a \subseteq 2^X$ of all sets $A \in 2^X$ such that* diam $A < a$, $a > 0$. *The mapping $(a, b) \to s(a, b)$ is uniformly continuous on any subspace $D_a \subseteq X \times X$ of all pairs $(a, b) \in X \times X$ such that $\varrho(a, b) < a$, $a > 0$ (in $X \times X$ we consider the metric given by $\varrho((a, b), (a', b')) = \max(\varrho(a, a'), \varrho(b, b'))$).*

Proof. The first part immediately follows from (5). The second part follows from (5) and from the inequality

$$\varrho_H(\{a, b\}, \{a', b'\}) \leqslant \varrho((a, b), (a', b')).$$

Remark 1. P_a is an open subset of 2^X and $\bigcup_{a>0} P_a = 2^X$. D_a is an open subset of $X \times X$ and $\bigcup_{a>0} D_a = X \times X$. Any bounded subset of 2^X (resp. of $X \times X$) is contained in some P_a (resp. D_a).

Remark 2. If $A \neq B$, $(A, B \in 2^X)$, then in (4) and also in (5) we can write $<$ instead of $\leqslant$.

The following theorem is evident:

THEOREM 3.6. *If X is an absolutely central space, in particular if X is a hyperconvex space and $f\colon X \to X$ is an isometry of X onto itself such that $f(A) = A$ for a bounded subset A of X, then $f(x) = x$ for some $x \in X$. Evidently $f(s(A)) = s(A)$.*

Analogous theorems hold for strongly central spaces and for central spaces.

In the sequel we shall use the notions ${}_aS_n(A)$ and ${}_as(A)$, which are analogous to the notions $S_n(A)$ and $s(A)$. We now begin to define the notions ${}_aS_n(A)$ and ${}_as(A)$.

Let (X, ϱ) be a metric space. For a bounded non-empty set $A \subseteq X$ and a real number $a > \operatorname{diam} A$ we denote by ${}_aS_1^X(A)$, or simply ${}_aS_1(A)$, the set of all $x \in X$ such that $\varrho(x, y) \leqslant a/2$ for every $y \in A$, and we put

$${}_aS_{n+1}^X(A) = {}_aS_{n+1}(A) = {}_aS_n(A) \cap {}_{\frac{a}{2^{n-1}}}S_1\big({}_aS_n(A)\big) \quad \text{for} \quad n = 1, 2, \ldots$$

Let us remark that $\operatorname{diam} {}_aS_1(A) \leqslant a$ and if

$$\operatorname{diam} {}_aS_n(A) \leqslant \frac{a}{2^{n-1}}, \tag{6}$$

then, by definition, $\operatorname{diam} {}_aS_{n+1}(A) \leqslant a/2^n$. Thus inequality (6) holds for every $n = 1, 2, \ldots$ and the definition of the sets ${}_aS_n(A)$ is correct and there exists at most one point ${}_as(A) \in \bigcap_{n=1}^{\infty} {}_aS_n(A)$.

For $A = \{a, b\} \subseteq X$ we shall write ${}_aS_n(a, b)$ instead of ${}_aS_n(\{a, b\})$ and ${}_as(a, b)$ instead of ${}_as(\{a, b\})$.

The proof of the following theorem is quite analogous to the proof of Theorem 3.1.

THEOREM 3.7. *For any bounded subset A of a hyperconvex metric space (X, ϱ) and any real $a \geqslant \operatorname{diam} A$ there exists an ${}_as(A)$.*

LEMMA 3.8. *Let (X, ϱ) be a hyperconvex space, $A, B \in 2^X$, $A \neq \emptyset \neq B$ and* $\max(\operatorname{diam} A, \operatorname{diam} B) \leqslant a$. *Then*

$$\varrho_H\big({}_aS_n(A), {}_aS_n(B)\big) \leqslant \varrho_H(A, B) \quad \textit{for any } n = 1, 2, \ldots, \tag{7}$$

where ${}_aS_0(Y) = Y$ for any $Y \in 2^X$.

Proof. If $n = 0$, then inequality (7) holds. Let us assume that it holds for every $k \leqslant n$. We define a real function $f_n\colon B \cup {}_aS_1(B) \to R$ as follows:

$$f_n(p) = \begin{cases} a/2 & \text{if} \quad p \in {}_a\big(B \cup {}_aS_1(B)\big) \setminus {}_aS_2(B), \\ a/2^k & \text{if} \quad p \in {}_aS_k(B) \setminus {}_aS_{k+1}(B), \qquad k = 1, 2, \ldots, n-1, \\ a/2^n & \text{if} \quad p \in {}_aS_n(B). \end{cases}$$

Then

$$\Big\{q \in X\colon \bigvee_{p \in B \cup {}_aS_1(B)} \varrho(p, q) \leqslant f(p)\Big\} = {}_aS_{n+1}(B).$$

Let $x \in {}_aS_{n+1}(A)$ and $p \in {}_aS_k(B)$, where $0 \leqslant k \leqslant n$. Since $x \in {}_aS_{k+1}(A)$ and

$$\varrho_H\big({}_aS_k(A), {}_aS_k(B)\big) \leqslant \varrho_H(A, B),$$

we have

$$\varrho(x, p) \leqslant \varrho_H(A, B) + \frac{a}{2^k}.$$

Thus $\varrho(x,p) \leqslant \varrho_H(A,B)+f_n(p)$ for any $x \in {}_aS_{n+1}(A)$ and $p \in B \cup {}_aS_1(B)$, whence $\varrho(x,y) \leqslant \varrho_H(A,B)$ for a certain $y \in {}_aS_{n+1}(B)$ (since X is a hyperconvex space). Analogously, if $x \in {}_aS_{n+1}(B)$, then there exists a $y \in {}_aS_{n+1}(A)$ such that $\varrho(x,y) \leqslant \varrho_H(A,B)$. Hence $\varrho_H({}_aS_{n+1}(A), {}_aS_{n+1}(B)) \leqslant \varrho_H(A,B)$. The lemma is proved.

The following theorem is a direct consequence of Lemma 3.8:

THEOREM 3.9. *Let (X, ϱ) be a hyperconvex space. Then the mapping ${}_as\colon P_a \to X$ and ${}_as\colon D_a \to X$* (see Theorem 3.5) *are metric mappings.*

Each of the functions s and ${}_as$ has its own advantages. The function s is defined on a larger set of elements. Moreover, the sets $S_n(A)$ converge to $s(A)$ faster than the sets ${}_aS_n(A)$ converge to ${}_as(A)$; sometimes we can obtain the point $s(A)$ after a finite number of steps. But, on the other hand, for the function ${}_as$ Theorem 3.9 holds. Therefore, an especially interesting class of spaces is formed by those spaces for which the functions s and ${}_as$ coincide on the set D_a. Unfortunately, this class does not contain all the metric d-lattices. We shall show, however, that every metric d-lattice whose d-sublattices generated by two of its elements are all distributive belongs to this class.

§ 4. In this section we use the concepts introduced in the preceding section and the results of that section. We prove that every metrically complete metric d-lattice is a strong central space. Moreover, it does not matter whether the central middle of a totally bounded subset of such a d-lattice is considered in a given d-lattice or in its arbitrary extension. In this sense the central middle is absolute (see Theorems 4.4 and 4.6). The operation $A \to {}_as(A)$, where A runs over all compact subsets, is continuous, just as the operation ${}_as$ (a stronger result is given in Theorem 4.7). In the case of functional d-lattices it turns out that $s(f, g) = (f+g)/2$ (see Theorem 4.8). Hence it is immediate that a metrically complete functional metric d-lattice is a linearly convex set of functions (cf. [13] and [17]). The operation $A \to {}_as(A)$, though less universal, has a more regular algebraic character (see Theorem 4.13) than the operation $A \to s(A)$, for the first operation is preserved under homomorphisms, i.e. $h({}_as(A)) = {}_as(h(A))$ for any homomorphism h of two d-lattices (see Theorem 4.14). If every d-sublattice of a given d-lattice generated by two elements is distributive, then the two operations coincide (on the set on which ${}_as$ is defined, see Theorem 4.16). Thus every homomorphism of metric functional d-lattices is affine (Theorem 4.17).

Let S be a d-lattice and $\emptyset \neq A \subseteq S$. Then $\operatorname{diam} A < \infty$ iff there exist a lower bound p and an upper bound q of A such that $d(p,q) < \infty$. Hence if $\operatorname{diam} A < \infty$, $p = \bigcap A$ and $q = \bigcup A$, then $d(p,q) < \infty$.

LEMMA 4.1. *If $p = \bigcap A$ and $q = \bigcup A$ for a subset A of* d-*lattice S, then* $\operatorname{diam} A = d(p, q)$.

Proof. From (viii) of § 2 follows diam $A \leqslant d(p, q)$. We shall prove the converse inequality under the assumption of diam $A < \infty$. Then $d(p, q) < \infty$. For any $\varepsilon > 0$ there exists an element $x \in A$ such that

$$x \nsubseteq (p + d(p, q) - \varepsilon) \cap q.$$

But $x \subseteq q$, whence $x \nsubseteq p + d(p, q) - \varepsilon$ and $x - d(p, q) + \varepsilon \nsubseteq p$. Thus there exists a $y \in A$ such that

$$y \nsupseteq (x - d(p, q) + \varepsilon) \cup p$$

and then $y \nsupseteq x - d(p, q) + \varepsilon$. Hence $d(x, y) > d(p, q) - \varepsilon$. The lemma is proved.

LEMMA 4.2. *If A is a non-empty subset of a* d-*lattice S such that* diam $A < \infty$ *and* $p = \bigcap A$ *and* $q = \bigcup A$, *then*

(1) $S_1(A) = S_1(p, q) = \{x \in S\colon q - \frac{1}{2} d(p, q) \subseteq x \subseteq p + \frac{1}{2} d(p, q)\} \neq \emptyset$.

In particular

(2) $S_1(a, b) = \{x \in S\colon a \cup b - \frac{1}{2} d(a, b) \subseteq x \subseteq a \cap b + \frac{1}{2} d(a, b)\}$

for any comparable $a, b \in S$.

Proof. If

(3) $$q - \tfrac{1}{2} d(p, q) \subseteq x \subseteq p + \tfrac{1}{2} d(p, q)$$

and $y \in A$, then

$$x - \tfrac{1}{2} d(p, q) \subseteq p \subseteq y \subseteq q \subseteq x + \tfrac{1}{2} d(p, q)$$

whence

$$\max\big(d(x, p), d(x, q), d(x, y)\big) \leqslant \tfrac{1}{2} d(p, q),$$

i.e. $x \in S_1(A)$ and $x \in S_1(p, q)$. On the other hand, if $x \in S_1(A)$, then

$$x - \tfrac{1}{2} d(p, q) \subseteq y \subseteq x + \tfrac{1}{2} d(p, q)$$

for any $y \in A$, whence

$$x - \tfrac{1}{2} d(p, q) \subseteq \bigcap A = p \subseteq q = \bigcup A \subseteq x + \tfrac{1}{2} d(p, q),$$

i.e. $x \in S_1(p, q)$. But if $x \in S_1(p, q)$, then evidently condition (3) holds. Thus equalities (1) are proved.

Since $q \subseteq p + d(p, q)$ for any comparable $p, q \in S$, we have $q - \frac{1}{2} d(p, q) \subseteq p + \frac{1}{2} d(p, q)$. Thus for $x = q - \frac{1}{2} d(p, q)$ condition (3) is satisfied and consequently $S_1(A) \neq \emptyset$.

Equality (2) follows from the above equality (1) and from equality (ix) of § 2. The lemma is proved.

DEFINITION 1. We put $x \| y$ iff $x + a = y$ for a certain $a \in R$ and we put xIy iff $x \cap y \| x \cup y$ but the relation $x \| y$, where x, y are the elements of a d-lattice S, does not hold.

Thus $S_1(x, y)$ is a one-element set iff $x \frown y \| x \smile y$, i.e. $x \| y$ or xIy.
We put for a comparable pair a, b of elements of a d-lattice S

$$s_1(a, b) = a \smile b - \tfrac{1}{2}\, d(a, b),$$

$$s^1(a, b) = a \frown b + \tfrac{1}{2}\, d(a, b),$$

$$s_{n+1}(a, b) = s_n(a, b) \smile \big(s^n(a, b) - \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big)\big),$$

$$s^{n+1}(a, b) = s^n(a, b) \frown \big(s_n(a, b) + \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big)\big)$$

for $n = 1, 2, \ldots$

It is easy to prove by induction that

$$(4) \qquad s_n(a, b) \subseteq s^n(a, b).$$

We shall prove that

$$(5) \qquad S_n(a, b) = \{x \in S\colon\ s_n(a, b) \subseteq x \subseteq s^n(a, b)\}, \qquad n = 1, 2, \ldots$$

For $n = 1$ formula (5) holds. Let it holds for a certain n. Then it is obvious that

$$\begin{aligned} S_{n+1}(a, b) &= S_n(a, b) \frown S_1\big(S_n(a, b)\big) \\ &= S_n(a, b) \frown S_1\big(s_n(a, b), s^n(a, b)\big) \\ &= \{x \in S\colon\ s_n(a, b) \subseteq x \subseteq s^n(a, b)\} \frown \big\{x \in S\colon\ s^n(a, b) - \\ &\qquad - \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big) \subseteq x \subseteq s_n(a, b) + \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big)\big\} \\ &= \big\{x \in S\colon\ s_n(a, b) \smile \big(s^n(a, b) - \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big)\big) \\ &\qquad \subseteq x \subseteq s^n(a, b) \frown \big(s_n(a, b) + \tfrac{1}{2}\, d\big(s_n(a, b), s^n(a, b)\big)\big)\big\} \\ &= \{x \in S\colon\ s_{n+1}(a, b) \subseteq x \subseteq s^{n+1}(a, b)\}, \end{aligned}$$

thus (5) holds and by (4) $S_n(a, b)$ is a non-empty and closed set for every $n = 1, 2, \ldots$

Hence if $A \subseteq S$, diam $A < \infty$, $p = \bigcap A$ and $q = \bigcup A$, then

$$(6) \qquad S_n(A) = S_n(p, q) \neq \emptyset \quad \text{for} \quad n = 1, 2, \ldots$$

LEMMA 4.3. *If A is a totally bounded (but not necessarily bounded) subset of a metrically complete* d-*lattice S, then there exist $\bigcup A$ and $\bigcap A$.*

Proof. Let A_n be a finite $(1/n)$-net of A (i.e. $A_n \subseteq A$ and $\min_{a \in A_n} d(a, x) \leqslant 1/n$ for every $x \in A$) for $n = 1, 2, \ldots$ Then

$$\bigcap A_n - \frac{1}{n} \subseteq x \subseteq \bigcup A_n + \frac{1}{n}$$

for any $x \epsilon A$, whence

$$d(\bigcap A_n, \bigcap A_m) \leqslant \max\left(\frac{1}{n}, \frac{1}{m}\right)$$

and

$$d(\bigcup A_n, \bigcup A_m) \leqslant \max\left(\frac{1}{n}, \frac{1}{m}\right),$$

since, for example,

$$\bigcap A_n - \frac{1}{n} \subseteq \bigcap A_m \quad \text{and} \quad \bigcap A_m - \frac{1}{m} \subseteq \bigcap A_n,$$

where m, $n = 1, 2, \ldots$ Thus from (i) of § 1 it follows that

$$\bigcup A = \lim_{n=\infty} \bigcup A_n \quad \text{and} \quad \bigcap A = \lim_{n=\infty} \bigcap A_n.$$

The lemma is proved.

From the above we obtain

THEOREM 4.4. *Any metrically complete metric* d*-lattice S is a strongly central metric space.*

Remark 1. The assumption that the d-lattice S is complete as a metric space is essential. For example, let S be the set of all real-valued functions defined on $I = [0, 1]$ such that for every $f \epsilon S$ there exist $0 < a_1 < a_2 < \ldots < a_n < 1$ such that

$$f(t) = f(a_i) + \varepsilon_i (t - a_i),$$

where $\varepsilon_i = 1$ or -1, $a_i \leqslant t \leqslant a_{i+1}$, $i = 0, 1, \ldots, n$, $a_0 = 0$, $a_{n+1} = 1$.

Then S is a functional metric d-lattice and for $f, g \epsilon S$ such that $f(t) = t$, $g(t) = -t$ for $t \epsilon I$, there exists no $s(f, g)$.

LEMMA 4.5. *Let A be a totally bounded subset of a* d*-sublattice S of a* d*-lattice $S' = (S', \bigcup', \bigcap', +)$ and let $a = \bigcup' A \epsilon S'$ (resp. $a = \bigcap' A \epsilon S'$). Then $a = \bigcap' \{x \epsilon S\colon a \subseteq x\}$ (resp. $a = \bigcup' \{x \epsilon S\colon x \subseteq a\}$). Hence if $a \epsilon S$, then $a = \bigcup A$ (resp. $a = \bigcap A$).*

Proof. Let $a = \bigcup A \epsilon S'$ and $a^n = \bigcup A_n$, where A_n is a finite $(1/n)$-net of A, $n = 1, 2, \ldots$ Then $a^n \epsilon S$ as well as $a^n + 1/n \epsilon S$ and $a^n \subseteq a \subseteq a^n + 1/n$, whence

$$a = \bigcup_{n=1}^{\infty} a^n = \bigcap \{x \epsilon S\colon a \subseteq x\}.$$

The lemma is proved.

THEOREM 4.6. *Let A be a non-empty totally bounded subset of a* d*-sublattice S of a metric* d*-lattice $S' = (S', \bigcup', \bigcap', +)$ and let $a \epsilon S$. Then $a = s(A)$ in S iff $a = s(A)$ in S'.*

Remark 2. If $A = \{x, y\}$, then Theorem 4.6 immediately follows from the fact that the point $s_n(x, y)$ just as $s^n(x, y)$, is the same in S and in S', $n = 1, 2, \dots$

Proof of Theorem 4.6. The d-lattice S is a d-sublattice of a certain metrically complete metric d-lattice S''. Thus it is sufficient to prove the theorem under the assumption that S' is metrically complete. Then there exist a $p = \bigcap' A$, a $q = \bigcup' A \in S'$ and

$$S_1(A) = \{x \in S\colon\ s_1(p, q) \subseteq x \subseteq s^1(p, q)\},$$

where we write $S_n(A)$ for $S_n^S(A)$. From Lemma 4.5 it follows that

$$\bigcap{}' S_1(A) = s_1(p, q) \quad \text{and} \quad \bigcup{}' S_1(A) = s^1(p, q).$$

Thus if $a = s(A)$ in S or in S', we obtain

$$\bigcap{}' S_1(A) = s_1(p, q) \subseteq a \subseteq s^1(p, q) = \bigcup{}' S_1(A).$$

Let us assume that

$$S_n(A) = \{x \in S\colon\ s_n(p, q) \subseteq x \subseteq s^n(p, q)\} \tag{7}$$

and

$$\bigcap{}' S_n(A) = s_n(p, q) \subseteq a \subseteq s^n(p, q) = \bigcup{}' S_n(A). \tag{8}$$

Then in both cases ($a = s(A)$ in S or S') we shall prove

$$S_{n+1}(A) = \{x \in S\colon\ s_{n+1}(p, q) \subseteq x \subseteq s^{n+1}(p, q)\} \tag{9}$$

and

$$\bigcap{}' S_{n+1}(A) = s_{n+1}(p, q) \subseteq a \subseteq s^{n+1}(p, q) = \bigcup{}' S_{n+1}(A). \tag{10}$$

Equality (9) immediately follows from (7) and (8).

Let $a_k = \bigcap A_k$ and $a^k = \bigcup A_k$, where A_k is a $(1/k)$-net of A, $k = 1, 2, \dots$ Then $s_{n+1}(a_k, a^k)$, $s^{n+1}(a_k, a^k) \in S$ and

$$\lim_{k=\infty} s_{n+1}(a_k, a^k) = s_{n+1}(p, q), \qquad \lim_{n=\infty} s^{n+1}(a_k, a^k) = s^{n+1}(p, q).$$

From (8), in the both cases, it follows that $a \in S_{n+1}(A)$ and $s_{n+1}(p, q) \subseteq a \subseteq s^{n+1}(p, q)$ (if one of these assertions is an assumption, then the second one is a corollary).

Moreover,

$$s_{n+1}(p, q) = \bigcap_{k=1}^{\infty} \Big[\big(s_{n+1}(a_k, a^k) + d\big(s_{n+1}(a_k, a^k), s_{n+1}(p, q)\big)\big) \frown a\Big],$$

whence $s_{n+1}(p, q) = \bigcap S_{n+1}(A)$ and similarly $s^{n+1}(p, q) = \bigcup S_{n+1}(A)$. Thus formula (10) is proved. Hence (7) and (8) hold for any positive integer n. It means that if $a = s(A)$ in S, then $a = s(A)$ in S' and conversely. The theorem is proved.

From Theorems 2.8, 3.5, 4.5 and 4.6 we immediately obtain

THEOREM 4.7. *Let S be a metrically complete metric* d*-lattice. The mapping $A \to s(A)$ is uniformly continuous on any subspace $C_a \subseteq 2^S$ of all compact sets $A \in 2^S$ such that* diam $A < a$, $a > 0$. *The mapping $(a, b) \to s(a, b)$ is uniformly continuous on any subspace $D_a \subseteq S \times S$ of all pairs $(a, b) \in S \times S$ such that $d(a, b) < a$, $a > 0$.*

THEOREM 4.8. *If S is a functional* d*-lattice of functions defined on X, then $s(f, g) = (f+g)/2$ (if $s(f, g)$ exists) for every comparable $f, g \in S$. Hence if, in addition, S is a metrically complete metric* d*-lattice, then S is a convex subset of the linear space S' of all functions on X.*

Proof. The notions of $s_n(a, b)$ and $s^n(a, b)$ are dual. The map $\varphi\colon S' \to S'$, where $\varphi(u) = f+g-u$, $u \in S'$, is a dual automorphism of the d-lattice S' and $\varphi(f) = g$, $\varphi(g) = f$. Thus

$$\varphi\big(s_n(f, g)\big) = s^n(f, g) \quad \text{and} \quad \varphi\big(s^n(f, g)\big) = s_n(f, g) \tag{11}$$

whence

$$\varphi\big(S_n(f, g)\big) = S_n(f, g), \quad n = 1, 2, \ldots$$

and

$$\varphi\big(\{s(f, g)\}\big) = \varphi\Big(\bigcap_{n=1}^{\infty} S_n(f, g)\Big) = \bigcap_{n=1}^{\infty} \varphi\big(S_n(f, g)\big)$$
$$= \bigcap_{n=1}^{\infty} S_n(f, g) = \{s(f, g)\},$$

whence $f+g-s(f,g) = s(f,g)$ and $s(f,g) = (f+g)/2$. The theorem is proved.

Remark 3. The assumption that the d-lattice S is metric is essential, i.e. there exists a metrically complete functional non-metric d-lattice S of the real functions defined on an infinite set X which is not a convex subset of the linear space of all functions on X.

Remark 4. We have also

$$\frac{1}{2}\big(s_n(f, g) + s^n(f, g)\big) = s(f, g) = \frac{f+g}{2} \tag{12}$$

since by (11)

$$\tfrac{1}{2}\big(s_n(f, g) + s^n(f, g)\big) = \tfrac{1}{2}\Big(\varphi\big(s_n(f, g)\big) + \varphi\big(s^n(f, g)\big)\Big)$$
$$= \varphi\left(\frac{s_n(f, g) + s^n(f, g)}{2}\right).$$

The following example shows that Theorem 4.6 does not hold for any bounded subset A of a d-lattice S.

EXAMPLE. Let $S' = C(N)$ (see § 1, Example 5) and let S be a d-sublattice of all real functions $f\colon N \to R$ such that $\lim_{n=\infty} f(n) = f(1)-2$. We put

$$f_n(k) = \begin{cases} 2 & \text{for} \quad k \leqslant n \\ 0 & \text{for} \quad k > n \end{cases} \qquad n, k = 1, 2, \ldots$$

Then for $A = \{f_1, f_2, \ldots\}$ we find that a real function $g \in S$ given by

$$g(k) = \begin{cases} 3 & \text{for} \quad k = 1 \\ 1 & \text{for} \quad k > 1 \end{cases} \qquad k = 1, 2, \ldots$$

is the unique middle of the set A in S, whence $g = s(A)$ in S, but in S we have $h = s(A)$, where

$$h(k) = \begin{cases} 2 & \text{for} \quad k = 1 \\ 1 & \text{for} \quad k > 1 \end{cases} \qquad k = 1, 2, \ldots$$

as $h = \frac{1}{2}(\bigcap' A + \bigcup' A)$.

Now we shall show that if $h\colon S \to S'$ is a homomorphism of a metrically complete distributive metric d-lattice S into a d-lattice S', then $h(s(A)) = s(h(A))$ for any non-empty totally bounded subset A of S. For this purpose we shall consider the sets ${}_aS_n(A)$ and we shall show that for such a d-lattice S we have $s(A) = {}_as(A)$. More precisely, it is sufficient to consider the notions of ${}_as_n(a, b)$ and ${}_as^n(a, b)$, which are defined below. But first let us remark that

LEMMA 4.9. *If A is a non-empty totally bounded subset of a metrically complete* d-*lattice S and $h\colon S \to S'$ is a homomorphism of S into a* d-*lattice S', then*

$$h(\bigcup A) = \bigcup h(A) \qquad \textit{and} \qquad h(\bigcap A) = \bigcap h(A).$$

Proof. $\bigcup A = \lim_{n=\infty} \bigcup A_n$, where A_n is a finite $(1/n)$-net of A. Thus

$$\bigcup h(A) \subseteq h(\bigcup A) = \lim_{n=\infty} h(\bigcup A_n) = \lim_{n=\infty} \bigcup h(A_n) \subseteq \bigcup h(A),$$

since h is a continuous mapping.

The proof of the equality $h(\bigcap A) = \bigcap h(A)$ is dual. The lemma is proved.

Now let S be an arbitrary d-lattice and let a, b be a comparable pair of the elements of S. We put

$${}_as_1(a, b) = a \cup b - \tfrac{1}{2}a,$$

$${}_as^1(a, b) = a \cap b + \tfrac{1}{2}a$$

and

$$ {}_as_{n+1}(a,b) = {}_as_n(a,b) \cup \left({}_as^n(a,b) - \frac{1}{2^n}a\right), $$

$$ {}_as^{n+1}(a,b) = {}_as^n(a,b) \cap \left({}_as_n(a,b) + \frac{1}{2^n}a\right) $$

for $n = 1, 2, \ldots$

It is easy to prove by induction that

$$ {}_as_n(a,b) \subseteq {}_as^n(a,b), $$

and

$$ d\big({}_as_n(a,b), {}_as^n(a,b)\big) \leqslant \frac{1}{2^{n-1}}a $$

LEMMA 4.10. *If A is a non-empty subset of the* d*-lattice S such that* $\operatorname{diam} A < \infty$, $p = \bigcap A$ *and* $q = \bigcup A$, *then*

$$ {}_aS_n(A) = \{x \in S\colon {}_as_n(p,q) \subseteq x \subseteq {}_as^n(p,q)\}, \qquad n = 1, 2, \ldots $$

for $a \geqslant \operatorname{diam} A$. *In particular, if $a, b \in S$ and $a \geqslant d(a,b)$, then*

$$ {}_aS_n(a,b) = \{x \in S\colon {}_as_n(a,b) \subseteq x \subseteq {}_as^n(a,b)\}. $$

The following theorem is a direct consequence of this lemma:

THEOREM 4.11. *For every non-empty totally bounded subset A of a metrically complete metric* d*-lattice S there exists a point ${}_as(A)$.*

We have also

THEOREM 4.12. *Let A be a non-empty totally bounded subset of a* d*-sublattice S of a metric* d*-lattice S' and $a \in S$ and* $a \geqslant \operatorname{diam} A$. *Then $a = {}_as(A)$ in S iff $a = {}_as(A)$ in S'.*

THEOREM 4.13. *Let S be a metrically complete metric* d*-lattice. The mapping $A \to {}_as(A)$ is metric on the space C_a* (see Theorem 4.7). *The mapping $(a, b) \to {}_as(a, b)$ is metric on the space D_a.*

The proofs of Theorems 4.12 and 4.13 are analogous to the proofs of Theorems 4.6 and 4.7.

Evidently for any homomorphism h of d-lattices we have

$$ {}_as_n\big(h(a), h(b)\big) = h\big({}_as_n(a), {}_as_n(b)\big), $$

$$ {}_as^n\big(h(a), h(b)\big) = h\big({}_as^n(a), {}_as^n(b)\big), $$

where $a \geqslant d(a, b)$. Thus from Lemma 4.9 it follows that

$$ {}_as_n\big(h(A)\big) = h\big({}_as_n(A)\big), \qquad {}_as^n\big(h(A)\big) = h\big({}_as^n(A)\big) $$

for any totally bounded A. Thus we obtain

THEOREM 4.14. *Let $h\colon S \to S'$ be a homomorphism of a metrically complete metric* d*-lattice S into a metric* d*-lattice S'. Then*

$$ s\big(h(A)\big) = h\big({}_as(A)\big) $$

for any non-empty totally bounded subset A of S and real $a \geqslant \operatorname{diam} A$.

Now, let S be a distributive metric d-lattice. Any such d-lattice is isomorphic to a metric functional d-lattice (see Kaplansky [13] and also § 5 of this paper). Thus we can assume that S is a functional d-lattice defined on a certain set X. Then for $a, b \in S$, $x \in X$, $\alpha \geqslant d(a, b)$, as is easy to see, we have

$$(13) \quad \tfrac{1}{2}\big({}_{\alpha}s_n(a, b) + {}_{\alpha}s^n(a, b)\big)(x) = \tfrac{1}{2}\big(a(x) + b(x)\big) \quad \text{for} \quad n = 1, 2, \ldots$$

Hence, if in addition S is a metrically complete space, then

$$_{\alpha}s(a, b) = s(a, b).$$

Let us remark that all the elements ${}_{\alpha}s_n(a, b)$ and ${}_{\alpha}s^n(a, b)$ are contained in a d-sublattice generated by a and b. Thus we obtain

THEOREM 4.15. *Let S be a metrically complete metric* d-*lattice such that every* d-*sublattice of S generated by a pair of its elements is a distributive* d-*lattice. Then ${}_{\alpha}s(A) = s(A)$ for any totally bounded $A \subseteq S$ and real* $\alpha \geqslant \operatorname{diam} A$.

The following theorem is a direct consequence of Theorem 4.14 and Theorem 4.15:

THEOREM 4.16. *If, under the assumption of Theorem* 4.15 *about S, a mapping $h\colon S \to S'$ is a homomorphism of a* d-*lattice S into* d-*lattice S', then $h\big(s(A)\big) = s\big(h(A)\big)$ for any totally bounded subset A of S.*

EXAMPLE. Let S be a d-lattice given by Example 9 of § 1, $a = (0, -1, -3, -3)$, $b = (0, 0, 2, 3)$. Then

$$s_1(a, b) = {}_6s_1(a, b) = (-3, -3, -1, 0),$$
$$s^1(a, b) = {}_6s^1(a, b) = (3, 2, 0, 0),$$

and

$$s_2(a, b) = {}_6s_2(a, b) = (0, -1, -1, 0),$$
$$s^2(a, b) = {}_6s^2(a, b) = (0, 0, 0, 0).$$

Thus from (12) and (13) it follows that

$$s(a, b) = {}_6s(a, b) = (0, \tfrac{1}{2}, \tfrac{1}{2}, 0),$$

as each of the elements $s_2(a, b) = {}_6s_2(a, b)$ and $s^2(a, b) = {}_6s^2(a, b)$ belongs to a functional d-sublattice $S' \subseteq S$ of all the elements $(a_1, a_2, a_3, a_4) \in R^4$ such that $a_1 \geqslant a_2 = a_3 \leqslant a_4$.

On the other hand,

$$_8s_1(a, b) = (-4, -4, -2, -1), \qquad {}_8s^1(a, b) = (4, 3, 1, 1),$$
$$_8s_2(a, b) = (0, -1, -1, -1), \qquad {}_8s^2(a, b) = (0, 0, 1, 1),$$
$$_8s_3(a, b) = (0, -1, -1, -1), \qquad {}_8s^3(a, b) = (0, 0, 1, 1),$$
$$_8s_4(a, b) = {}_8s^4(a, b) = {}_8s(a, b) = (0, 0, 0, 0).$$

Hence $s(a, b) \neq {}_8 s(a, b)$.

Let us remark that the d-lattice S, as a complete d-lattice, is a hyperconvex space (see Theorem 2.4).

THEOREM 4.17. *If S and S_1 are functional metric* d-*lattices defined on the sets X and X_1 respectively and $f\colon S \to S_1$ is a homomorphism, then there exists an affine mapping $F\colon L \to L_1$ (i.e. such that $F - F(0)$ is a linear mapping), where L and L_1 are the linear spaces of all real functions on X and X_1 respectively, such that $F|S = f$.*

Proof. Let $\bar{S}$ and $\bar{S}_1$ be the closures of S and S_1 in L and L_1 (under uniform metric). Then $\bar{S}$ and $\bar{S}_1$ are the metrically complete d-sublattices of L and L_1 (see Theorem 2.1) and they are convex sets (Theorem 4.8). By Theorem 2.2 there exists an extension $\bar{f}\colon \bar{S} \to \bar{S}_1$, which is a lattice homomorphism. It follows from Theorems 4.8 and 4.9 that

$$\bar{f}\left(\frac{a+b}{2}\right) = \bar{f}\big(s(a, b)\big) = s\big(\bar{f}(a), \bar{f}(b)\big) = \frac{\bar{f}(a)+\bar{f}(b)}{2}$$

for any $a, b \in \bar{S}$. Hence

$$\bar{f}\left(\frac{k}{2^n}a + \frac{2^n-k}{2^n}b\right) = \frac{k}{2^n}\bar{f}(a) + \frac{2^n-k}{2^n}\bar{f}(b)$$

for any $a, b \in \bar{S}$ and positive integers n, k such that $0 < k < n$.

Since $\bar{f}$ is a metric mapping, we have

$$\bar{f}\big(\alpha a + (1-\alpha)b\big) = \alpha\bar{f}(a) + (1-\alpha)\bar{f}(b)$$

for any $a, b \in \bar{S}$ and real α such that $0 \leqslant \alpha \leqslant 1$. Such a mapping $\bar{f}$ has an affine extension $F\colon L \to L_1$. The theorem is proved.

§ 5. In this section we give a new proof of the representation theorem for distributive metric d-lattices. The proof is based on the notion of bunch which plays a role similar to that of a coset relative to an ideal in rings.

The following is the main theorem of this paragraph:

THEOREM 5.1 (Kaplansky's representation theorem [13]). *Every distributive metric* d-*lattice S is isomorphic to a functional* d-*lattice.*

This theorem implies

THEOREM 5.2. *Every distributive metric* d-*lattice is isomorphic to a certain functional* d-*lattice of bounded real-valued functions.*

Indeed, it follows from Theorem 5.1 that S may be considered as a certain d-lattice of functions defined on a set X. Let $f_0 \in S$. For every $g \in S$ let $g' = g - f_0$. The map $g \to g'$ is an isomorphism of S onto a certain d-lattice of bounded real functions.

For the proof of Theorem 5.1 we have to construct a set X and a canonical isomorphism of S into the functional lattice of real functions defined on X. We begin with the following remark.

Let $f\colon S \to S'$ be a homomorphism of a d-lattice S into a d-lattice S' and let $p \in S'$. Then the set $P = f^{-1}(p)$ has the following properties:

(B1) P is a sublattice of d-lattice S.

(B2) If $a, b \in P$, then the elements a, b are linked, briefly aQb, where by definition the elements a, b are *linked* iff

$$a+\varepsilon \nsubseteq b \quad \text{and} \quad b+\varepsilon \nsubseteq a \quad \text{for any } \varepsilon > 0.$$

(B3) If $a \subseteq b \subseteq c$ and $a, c \in P$, then $b \in P$.

DEFINITION 1. A non-empty subset P of a d-lattice S is called a *bunch* if conditions (B1) and (B2) are satisfied for P and it is called a *pseudo-bunch* if the condition (B2) is satisfied.

Obviously no pseudo-bunch is a d-sublattice.

Now X is defined as the set $\mathscr{P}(S)$ of all maximal bunches of S. In order to define the canonical isomorphism Φ we shall prove first that:

If S is a metric distributive d-lattice, then a bunch P is maximal in S iff for every $x \in S$ there exists a ξ (exactly one) such that $x-\xi \in P$.

We define $\Phi(x)$ as the function which at the maximal bunch $P \in \mathscr{P}(S)$ assumes the value ξ, where ξ is as above. It will be proved that Φ is the required isomorphism.

Obviously

(i) aQa;

(ii) if aQb, then bQa;

(iii) if $\alpha \neq 0$, then the relation $aQ(a+\alpha)$ does not hold;

(iv) if aQb, then $(a+\alpha)Q(b+\alpha)$;

(v) if $a \subseteq b \subseteq d$, $a \subseteq c \subseteq d$ and aQd, then bQc;

(vi) if $a \subseteq b \subseteq c$, aQd and cQd, then bQd;

(vii) if $\lim a_n = b$ and $a_n Q b$ for $n = 1, 2, \ldots$, then aQb.

We need some lemmas.

LEMMA 5.3. *If S is a* d*-lattice, $a, b \in S$, $a \neq b$, and if a, b are comparable, then there exists a real number $\alpha \neq 0$ such that the elements*

$$(a+\alpha) \cup b \quad \textit{and} \quad (a+\alpha) \cap b$$

are linked. Hence the elements $a+\alpha$ and b are contained in a bunch

$$\{x \in S\colon (a+\alpha) \cap b \subseteq x \subseteq (a+\alpha) \cup b\}.$$

Proof. From the assumptions we have

$$0 < d^+(a, b) < \infty \quad \text{or} \quad 0 < d^-(a, b) < \infty.$$

Let $0 < d^+(a, b) < \infty$. Then $b \subseteq a + d^+(a, b)$ and $b+\varepsilon \nsubseteq a+d^+(a, b)$ for any $\varepsilon > 0$. This means that $a + d^+(a, b)Qb$. Thus in this case the proof of Lemma 5.3 is complete since

$$\big(a+d^+(a, b)\big) \cup b = a+d^+(a, b)$$

and

$$\big(a+d^+(a, b)\big) \cap b = b.$$

Analogously one can prove the lemma for $0 < d^-(a, b) < \infty$.

Propositions (vi) and (vii) imply that the set $\{\alpha: (a+\alpha)Qb\}$ is a finite or infinite closed interval. It is not difficult to see that if $d^+(a, b) < \infty$, then this interval is bounded from above (if $d^-(a, b) < \infty$, it is bounded from below).

LEMMA 5.4. *If P is a pseudo-bunch in a metric* d*-lattice S and $a \in S$, then there exists a real number ξ such that $P \cup \{a+\xi\}$ is a pseudo-bunch The smallest of these numbers is*

$$\alpha = \sup\big\{\xi: \underset{x \in P}{\exists}(a+\xi \subseteq x)\big\}$$

and the largest one is

$$\beta = \inf\big\{\xi: \underset{x \in P}{\exists}(a+\xi \supseteq x)\big\}.$$

(Both sets within the braces are non-empty since S is a metric d*-lattice.)*

Proof. Let $y \in P$. For every $\varepsilon > 0$ there exists such an $x \in P$, that $a+\alpha \subseteq x+\varepsilon/2$. Since yQx, we have $y+\varepsilon/2 \nsubseteq x$ and $y+\varepsilon \nsubseteq a+\alpha$.

On the other hand, $a+\alpha+\varepsilon \nsubseteq y$. Hence $(a+\alpha)Qy$ for $y \in P$ and $P \cup \{a+\alpha\}$ is a pseudo-bunch. But for $\varepsilon > 0$ there exists an $x \in P$ such that $a+\alpha-\varepsilon/2 \subseteq x$. Thus the relation $(a+\alpha-\varepsilon)Qx$ does not hold and the set $P \cup \{a+\alpha-\varepsilon\}$ is not a pseudo-bunch. Hence α is minimal.

Similarly one proves the remaining part of the lemma.

The union of a monotonic family of pseudo-bunches (resp. bunches) is a pseudo-bunch (bunch). Therefore the Kuratowski-Zorn theorem is applicable and every pseudo-bunch (bunch) is contained in a certain maximal pseudo-bunch (bunch). For any maximal pseudo-bunch (bunch) condition (B3) is satisfied (use (vi) and (vii)). Any maximal bunch and any maximal pseudo-bunch are closed sets.

LEMMA 5.5. *If S is a metric* d*-lattice, then a pseudo-bunch P is maximal in S iff for every $x \in S$ there exists exactly one ξ such that $x-\xi \in P$.*

Proof. Let $x \in S$. From Lemma 5.4 we infer that if P is a maximal pseudo-bunch, then there exists such a number ξ that $P \cup \{x-\xi\}$ is a pseudo-bunch. Thus $x-\xi \in P$. Now the uniqueness of ξ and the implication in the opposite direction follow immediately from (iii).

The above lemma does not hold for non-metric d-lattices (see Theorem 5.8).

LEMMA 5.6. *A maximal pseudo-bunch P of a metric* d-*lattice S is a bunch iff the mapping $x \to \xi$, $x \in S$, given as in Lemma* 5.5, *is a functional on S.*

Proof. Let a maximal pseudo-bunch be a bunch and let $x, y \in S$, $\xi, \eta \in R$ and $x-\xi$, $y-\eta \in P$. Then

$$(x-\xi) \frown (y-\eta) \subseteq x \frown y - \min(\xi, \eta) \subseteq (x-\xi) \smile (y-\eta)$$

and

$$(x-\xi) \frown (y-\eta) \subseteq x \smile y - \max(\xi, \eta) \subseteq (x-\xi) \smile (y-\eta).$$

Since, P being a bunch,

$$(x-\xi) \frown (y-\eta),\ (x-\xi) \smile (y-\eta) \in P,$$

the elements

$$x \frown y - \min(\xi, \eta) \quad \text{and} \quad x \smile y - \max(\xi, \eta)$$

belong to P, i.e.

$$x \frown y \to \min(\xi, \eta) \quad \text{and} \quad x \smile y \to \max(\xi, \eta).$$

Now we shall show that if the mapping $x \to \xi$ is a functional, then a maximal pseudo-bunch P is a bunch.

Indeed, P is a counter-image of 0 under the mapping $x \to \xi$. The lemma is proved.

LEMMA 5.7. *In a metric distributive* d-*lattice S every maximal bunch P is a maximal pseudo-bunch.*

Proof. Let P be a maximal bunch in S and suppose that it is not a maximal pseudo-bunch. Then there exists an $a \in S$ such that $a+\xi \notin P$ for all ξ. Let α and β be the real numbers defined in Lemma 5.4. For all $\xi \in [\alpha, \beta]$ we have $(a+\xi)Qx$ for any $x \in P$.

Moreover, for $x, y \in P$ we have $((a+\alpha) \smile x)Qy$.

In fact, $(a+\alpha) \smile x+\varepsilon \nsubseteq y$ for $\varepsilon > 0$ since $x+\varepsilon \nsubseteq y$. On the other hand, $a+\alpha \subseteq z+\varepsilon/2$ for any $\varepsilon > 0$ and a certain $z = z(\varepsilon) \in P$. If we had

$$y+\varepsilon \subseteq (a+\alpha) \smile x \subseteq (z+\varepsilon/2) \smile x \subseteq z \smile x+\varepsilon/2,$$

then we would have $y+\varepsilon/2 \subseteq z \smile x \in P$ contrary to $yQz \smile x$ (P being a lattice). Thus $y+\varepsilon \nsubseteq (a+\alpha) \smile x$, whence $((a+\alpha) \smile x)Qy$.

Because of duality we also have $(a+\beta) \frown xQy$ for $x, y \in P$. We shall show that $(a+\gamma) \smile xQy$ for arbitrary $x, y \in P$, or $(a+\gamma) \frown xQy$ for any $x, y \in P$. We know that

$$(a+\gamma) \smile x+\varepsilon \nsubseteq y \quad \text{and} \quad y+\varepsilon \nsubseteq (a+\gamma) \frown x$$

for $x, y \in P$. If our assertion were not true, i.e. if there existed elements x_1, x_2, y_1, y_2 such that the relations $(a+\gamma) \smile x_1Qy_1$ and $(a+\gamma) \frown x_2Qy_2$ did not hold, then we would have

$$y_1+\varepsilon \subseteq (a+\gamma) \smile x_1 \quad \text{and} \quad (a+\gamma) \frown x_2+\varepsilon \subseteq y_2$$

for a certain $\varepsilon > 0$.

From these inclusions it follows that

$$(y_1 \frown x_2+\varepsilon) \frown (a+\gamma) \subseteq (x_2+\varepsilon) \frown (a+\gamma) \subseteq x_2 \frown (a+\gamma)+\varepsilon \subseteq y_2 \subseteq x_1 \smile y_2$$

and

$$x_1 \smile y_2 \smile (a+\gamma) \supseteq x_1 \smile (a+\gamma) \supseteq y_1+\varepsilon \supseteq y_1 \frown x_2+\varepsilon,$$

whence

$$(y_1 \frown x_2+\varepsilon) \frown (a+\gamma) \subseteq x_1 \smile y_2$$

and

$$y_1 \frown x_2+\varepsilon \subseteq (x_1 \smile y_2) \smile (a+\gamma).$$

It follows from the distributivity of S that $y_1 \frown x_2+\varepsilon \subseteq x_1 \smile y_2$; indeed if $p \frown q \subseteq r$ and $p \subseteq r \smile q$, then

$$p = p \frown (r \smile q) = (p \frown r) \smile (p \frown q) \subseteq (p \frown r) \smile r = r$$

contrary to $y_1 \frown x_2Qx_1 \smile y_2$ (for P is a lattice and a pseudo-bunch). Thus our auxiliary assertion is proved.

Proposition (vii) implies that the set P' of all elements $p \in S$ such that pQx for all $x \in P$ is closed. For $x \in S$ let φ_x: $R \to S$ be defined by $\varphi_x(\gamma) = (a+\gamma) \smile x$ and let ψ_x: $R \to S$ be defined by $\psi_x(\gamma) = (a+\gamma) \frown x$.

The mappings φ_x, ψ_x are continuous. Thus the set M of all γ such that $(a+\gamma) \smile x \in P'$ for all $x \in P$ is closed. Similarly, the set N of all γ such that $(a+\gamma) \frown x \in P'$ for all $x \in P$ is closed.

As we have just proved, every number belongs to one of these closed sets and each of them is non-empty (α belongs to the first, β — to the second one).

Thus there exists a $\gamma_0 \in [\alpha, \beta]$ such that $\gamma_0 \in M \frown N$.

Let us now consider the set T of all elements $z \in S$ for which there exist $x, y \in P$ with $x \subseteq z \subseteq (a+\gamma_0) \smile y$. This set contains P, it is also a lattice (if $x \subseteq z \subseteq (a+\gamma_0) \smile y$ and $x' \subseteq z' \subseteq (a+\gamma_0) \smile y'$, then $x \frown x' \subseteq z \frown z' \subseteq z \smile z' \subseteq (a+\gamma_0) \smile (y \smile y')$) and (by (v)) it is a pseudo-bunch.

Thus, by the maximality of P, it must coincide with P. But T contains all elements $(a+\gamma_0) \cup y$, and thus $(a+\gamma_0) \cup y \in P$ for $y \in P$.

Similarly, it can be shown that $(a+\gamma_0) \cap y \in P$ for $y \in P$. Thus we have proved that $P \cup \{a+\gamma_0\}$ is a lattice. It is also a pseudo-bunch for $a+a \subseteq a+\gamma_0 \subseteq a+\beta$. Consequently, it is a bunch. Thus $a+\gamma_0 \in P$, contrary to the assumption about a.

Proof of Theorem 5.1. The canonical isomorphism is defined in the following way: for $q \in S$ the map f_q: $\mathscr{P}(S) \to R$ is defined on the set $\mathscr{P}(S)$ of maximal bunches by:

$$\left.\begin{matrix} P \in \mathscr{P}(S) \\ q - a \in P \end{matrix}\right\} \Rightarrow f_q(P) = a.$$

Lemmas 5.5 and 5.7 imply that functions f_q are well-defined, real-valued and defined on the entire $\mathscr{P}(S)$.

Lemma 5.3 implies that for a pair of distinct (comparable) elements of S there exists a maximal bunch containing only one of them. Thus if q', $q'' \in S$ and $q' \neq q''$, then $f_{q'} \neq f_{q''}$. Thus the mapping $q \to f_q$ is 1-1. Lemma 5.6 implies that the transformation $q \to f_q$ is a homomorphism. The theorem is proved.

THEOREM 5.8. *If a* d-*lattice* S *is non-metric, then for every* $x \in S$ *there exists a maximal pseudo-bunch* P (*and also a maximal bunch* P) *such that* $x - a \notin P$ *for all* $a \in R$.

Proof. If a d-lattice S is non-metric, then there exist a, $b \in S$ such that $d(a, b) = \infty$.

Then for an arbitrary (but fixed) $x \in S$ we have

$$d(a, x) = \infty \quad \text{or} \quad d(x, b) = \infty;$$

we may assume that $d(x, a) = \infty$. Then $d^+(x, a) = \infty$ or $d^-(x, a) = \infty$. With no loss of generality we may assume that $d^+(x, a) = \infty$. Let $c = a \cup x$. Then $d^-(x, c) = 0$ and $d^+(x, c) = \infty$. Let $c_n = c \cup (x+n)$, $n = 1, 2, \ldots$ The sequence $\{c_n\}$ is a bunch for $c_k Q c_n$, k, $n = 1, 2, \ldots$

In fact, for $k \leqslant n$ we have $c_k \subseteq c_n$. Therefore, for no $\varepsilon > 0$, $c_n + \varepsilon \nsubseteq c_k$. On the other hand, $c_k + \varepsilon \nsupseteq c + \varepsilon$.

If we had $c + \varepsilon \subseteq c_n = c \cup (x+n)$, then by Lemma 1.3 we would have $c + \varepsilon \subseteq x + n$, contrary to the assumption $d^+(x, c) = \infty$. Thus, for no $\varepsilon > 0$, $c + \varepsilon \subseteq c_n$ and a fortiori $c_k + \varepsilon \subseteq c_n$.

Let a be arbitrary. Let n be such that $n \geqslant 1 - a$. Then $(x-a)+1 \subset c_n$, whence the relation $(x-a)Qc_n$ does not hold for such n. If P is an arbitrary maximal pseudo-bunch or bunch containing the sequence $\{c_n\}$, then obviously there is no a for which $x - a \in P$. The theorem is proved.

§ 6. In the present section we introduce the Tychonoff topology in the set $S^{\cdot}$ of all functionals defined in a metric distributive d-lattice S. If we divide the topological space $S^{\cdot}$ by a relation $\|$ (see Definition 1), we obtain a compact space $S^{\cdot}/\|$. The d-lattice S is isomorphic to the perfect d-lattice defined on $S^{\cdot}/\|$ (see Definition 2, cf. [13]). Next, to each homomorphism $F\colon S \to S_1$ of distributive metric d-lattices there corresponds a continuous map $F^{\cdot}/\|\colon S^{\cdot}/\| \to S^{\cdot}/\|$, so that we obtain a contravariant functor from the category of distributive metric d-lattices to the category of compact spaces (actually we know more than that, see Theorems 6.6 and 6.9).

For a metric functional d-lattice S defined on a set (a topological space) X there is a canonical map (continuous map) $({}^{\cdot}/\|)\colon X \to S^{\cdot}/\|$. If S is a perfect d-lattice, the canonical map turns out to be a homeomorphism into. In this way we can obtain all the compactifications of a space X (Theorem 6.8).

Let $S^{\cdot}$ be the set of all functionals on S, where S is a metric distributive d-lattice. It follows from the theorem that the function Ψ from S into the functional d-lattice of all real-valued function on $S^{\cdot}$ defined by

$$\big(\Psi(x)\big)(f) = f(x), \qquad x \in S, \qquad f \in S^{\cdot},$$

is an isomorphism of S onto $\Psi(S)$.

Thus we can introduce in $S^{\cdot}$ the weakest topology such that all functions $\Psi(x)$ are continuous, $x \in S$. The family of all subsets of $S^{\cdot}$ of the form

$$\bigcap_{i=1}^{n} \{f \in S^{\cdot}\colon\ a_i < f(x_i) < \beta_i\}$$

where $x_i \in S$, a_i, $\beta_i \in R$ for $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, is the base in this topology, which from now on we shall call the *weak topology* in $S^{\cdot}$. The weak topology is never compact, but we can replace $S^{\cdot}$ by a compact subspace X such that the restriction $\Psi(x)|X$, for any $x \in S$, gives an isomorphism of $\Psi(S)$ into the functional d-lattice of functions on X. A subspace X can be taken as the subspace $S^{\cdot}_a$ of all functionals $f \in S^{\cdot}$ such that $f(a) = 0$, where $a \in S$. The subspace $S^{\cdot}_a$ is compact as a closed subspace of the Tychonoff cube $\underset{x \in S}{P}\ [-d(a, x),\ d(a, x)]$.

Indeed,

$$S^{\cdot}_a \subseteq \underset{x \in S}{P}\,[-d(a, x), d(a, x)]$$

since functionals are metric functions and from $f(a) = 0$, $f \in S^{\cdot}$, we obtain

$$-d(a, x) \leqslant f(x) \leqslant d(a, x), \qquad x \in S.$$

Next, if $g \in \bar{S}^{\cdot}_a$, then $g(a) = 0$ and for every $x, y \in S$, $a \in R$ and $\varepsilon > 0$ there exists an $f \in S^{\cdot}_a$ such that

$$|g(x)-f(x)| < \varepsilon, \quad |g(y)-f(y)| < \varepsilon,$$

$$|g(x+a)-f(x+a)| < \varepsilon,$$

$$|g(x \cup y)-f(x \cup y)| < \varepsilon, \quad |g(x \cap y)-f(x \cap y)| < \varepsilon.$$

Hence for example

$$|g(x+a)-g(x)-a| \leqslant |g(x+a)-f(x+a)|+|f(x)-g(x)| \leqslant 2\varepsilon,$$

whence $g(x+a) = g(x)+a$.

Similarly,

$$g(x \cup y) = g(x) \cup g(y) \quad \text{and} \quad g(x \cap y) = g(x) \cap g(y),$$

and thus $g \in S^{\cdot}_a$.

We shall show that a mapping $x \to \Psi(x)|S^{\cdot}_a$ is an isomorphic imbedding, $x \in S$.

Indeed, if $x, y \in S$ and $x \neq y$, then there exists an $f \in S^{\cdot}$ such that

$$f(x) = \big(\Psi(x)\big)(f) \neq \big(\Psi(y)\big)(f) = f(y).$$

Then $f-f(a) \in S^{\cdot}_a$ and

$$\big(\Psi(x)\big)\big(f-f(a)\big) = f(x)-f(a) \neq f(y)-f(a) = \big(\Psi(y)\big)\big(f-f(a)\big).$$

Thus we obtain the isomorphism $\Psi^S_a = \Psi_a\colon S \to \Psi(S)|S^{\cdot}_a$ of S onto $\Psi(S)|S^{\cdot}_a = \{\Psi(x)|S^{\cdot}_a\}_{x \in S}$ where $\Psi_a(x) = \Psi(x)|S^{\cdot}_a$.

DEFINITION 1. We put $f\|g$ for $f, g \in S^{\cdot}$, if there exists $a \in R$ such that $f+a = g$, where

$$(f+a)(x) = f(x)+a, \quad x \in S.$$

It is obvious that the relation $\|$ is an equivalence and we can consider the canonical projection $p\colon S^{\cdot} \to S^{\cdot}/\|$.

It is obvious that for every $u \in S^{\cdot}/\|$ there exists exactly one $f \in S^{\cdot}_a$ such that $p(f) = u$, whence we obtain a 1-1 correspondence $p_a\colon S^{\cdot}_a \to S^{\cdot}/\|$, where $p_a = p|S^{\cdot}_a$, and a 1-1 correspondence $\omega_{ab}\colon S^{\cdot}_a \to S^{\cdot}_b$, where $\omega_{ab} = p_b^{-1}p_a$, $a, b \in S$.

It is easy to see that ω_{ab} is a homeomorphism, as $\omega_{ab}(f) = f-f(b) = f-\big(\Psi(b)\big)(f)$ for $f \in S^{\cdot}_a$. Thus we may introduce in $S^{\cdot}/\|$ a topology such that p_a is a homeomorphism for every $a \in S$. This is the quotient topology of the weak topology in $S^{\cdot}$; we shall call it the *weak quotient topology*. The mapping $(u, a) \to p_a^{-1}(u)+a$ is a homeomorphism between $S^{\cdot}/\| \times R$ and $S^{\cdot}$ $(a \in S)$.

For a homomorphism (dual homomorphism) $F\colon S\to S_1$ of a metric distributive d-lattice S into a metric distributive d-lattice S_1 let the mappings $F^{\cdot}\colon S_1^{\cdot}\to S^{\cdot}$ and $F^{\cdot}/\|\colon S_1^{\cdot}/\|\to S^{\cdot}/\|$ be defined as follows:

(1) $$F^{\cdot}f = f\circ F \quad (F^{\cdot}f = -f\circ F) \quad \text{for} \quad f\in S_1^{\cdot},$$

(2) $$(F^{\cdot}/\|)\circ p_1(f) = p\circ F^{\cdot}(f) \quad \text{for} \quad f\in S_1^{\cdot},$$

where $p_1\colon S_1^{\cdot}\to S_1^{\cdot}/\|$ is a canonical projection.

It is obvious that the following theorem is true:

THEOREM 6.1. *If $F\colon S\to S_1$ is an isomorphism or a dual isomorphism of the metric distributive* d*-lattices S and S_1 (in particular if $S_1 = S^0$ and $F(f) = f$ for $f\in S$), then the mapping $F^{\cdot}/\|$ is a homeomorphism of the spaces $S_1^{\cdot}/\|$ and $S^{\cdot}/\|$ in the weak quotient topologies of this spaces.*

THEOREM 6.2. *Let $F\colon S\to S_1$ be a homomorphism (dual homomorphism) of a metric distributive* d*-lattice S into a metric distributive* d*-lattice S_1. Then $F^{\cdot}/\|$ is a continuous mapping of $S_1^{\cdot}/\|$ into $S^{\cdot}/\|$.*

Proof. First let us remark that the mapping $F^{\cdot}$ is continuous.

Indeed, the counter-image of a set

$$\{f\in S^{\cdot}\colon\ a < f(x) < \beta\},$$

where $x\in S$, a, $\beta\in R$, under the mapping $F^{\cdot}$, is a set

$$\{g\in S_1^{\cdot}\colon\ a < g\circ F(x) < \beta\},$$

i.e. counter-images of the sets from the subbase of the weak topology of $S^{\cdot}$ are open subsets of $S_1^{\cdot}$, whence $F^{\cdot}$ is continuous.

Hence $p\circ F^{\cdot}\colon S_1^{\cdot}\to S^{\cdot}/\|$ is also a continuous mapping. Thus from (2) it follows that $F^{\cdot}/\|$ is continuous, since $S_1^{\cdot}/\|$ is considered with quotient topology. (The case of a dual homomorphism follows from Theorem 6.1.)

THEOREM 6.3. *Let $F\colon S\to S_1$ be a (dual) homomorphism of a metric distributive* d*-lattice S onto a* d*-lattice S_1. Then $F^{\cdot}/\|$ is a homeomorphic imbedding of $S_1^{\cdot}/\|$ into $S^{\cdot}/\|$.*

Proof. Let f, $g\in S_1^{\cdot}$. If the relation $f\|g$ does not hold, then the relation $f\circ F\|g\circ F$ does not hold either, since F is onto. Thus from (1) and (2) it follows that

$$(F^{\cdot}/\|)(u)\neq(F^{\cdot}/\|)(v) \quad \text{for} \quad u\neq v,\ u,\ v\in S^{\cdot}/\|.$$

THEOREM 6.4. *Let $F\colon S\to S_1$ be an isomorphic (a dual isomorphic) imbedding of a metric distributive* d*-lattice S into a metric distributive* d*-lattice S_1. Then $F^{\cdot}/\|$ is a continuous mapping of $S_1^{\cdot}/\|$ onto $S^{\cdot}/\|$.*

Proof. It is sufficient to prove that the mapping $F^{\cdot}\colon S_1^{\cdot}\to S^{\cdot}$ is onto. Thus Theorem 6.4 immediately follows from

THEOREM 6.5. *The* d-*lattice* R *has the following extension property: if* S *is a* d-*sublattice of a metric distributive lattice* S_1 *and* $f\colon S \to R$ *is a functional, then there exists a functional* $g\colon S_1 \to R$ *such that* $g|S = f$.

Proof. The set $f^{-1}(0)$ is a bunch in a d-lattice S_1. Thus it is contained in a maximal bunch $P \in \mathscr{P}(S_1)$. A functional $g \in S_1^{\cdot}$ such that $g^{-1}(0) = P$ is an extension of f. Theorems 6.5 and 6.4 are proved.

Let $[{}^{\cdot}/||]$ be a "mapping" defined on the category of metric distributive d-lattices with homomorphisms as the morphisms, into the category of compact spaces, with continuous mappings as the morphisms, such that

$$[{}^{\cdot}/||]\colon S \to S^{\cdot}/|| \quad \text{and} \quad [{}^{\cdot}/||]\colon F \to F^{\cdot}/||,$$

where S is a metric distributive d-lattice and F is a homomorphism of such d-lattices.

Next, let C be a "mapping" defined on the category of the compact spaces into the category of metric distributive d-lattices such that

$$C\colon X \to C(X)$$

for a compact space, and such that for a continuous mapping $\varphi\colon X \to X_1$, of X into another compact space X_1, a homomorphism $F_1 = C(\varphi)$ of a d-lattice $C(X_1)$ into $C(X)$ is given by

$$(F_1 f)(x) = f \circ \varphi(x)$$

for any $f \in C(X_1)$, $x \in X$.

Finally, let S be a metric functional d-lattice defined on a set X. Then we shall denote by

$$({}^{\cdot})\colon X \to S^{\cdot} \quad \text{and} \quad ({}^{\cdot}/||)\colon X \to S^{\cdot}/||$$

the mappings given by

$$({}^{\cdot})(x) = x^{\cdot} \quad \text{where} \quad x^{\cdot}(f) = f(x)$$

and

$$({}^{\cdot}/||)(x) = p(x^{\cdot})$$

for any $x \in X$ and $f \in S$.

THEOREM 6.6. *The "mappings"* $[{}^{\cdot}/||]$ *and* C *are contravariant functors such that for any compact spaces* X, X_1 *and any continuous mapping* $\varphi\colon X \to X_1$ *and for* $S = C(X)$, $S_1 = C(X_1)$, *the mappings*

$$({}^{\cdot}/||)\colon X \to S^{\cdot}/|| \quad \textit{and} \quad ({}^{\cdot}/||)_1\colon X_1 \to S_1^{\cdot}/||$$

are homeomorphisms such that

$$({}^{\cdot}/||)_1 \circ \varphi = (F_1^{\cdot}/||) \circ ({}^{\cdot}/||), \tag{3}$$

where $F_1 = C(\varphi)$.

Proof. First we shall show that if S, S_1 and S_2 are metric distributive d-lattices and $F\colon S\to S_1$ and $F_1\colon S_1\to S_2$ are homomorphisms, then

$$(F_1\circ F)^{\cdot}/\|=(F^{\cdot}/\|)\circ(F_1^{\cdot}/\|).$$

Indeed, for any $u=p_2(x)\in S_2^{\cdot}/\|$, where $x\in S_2^{\cdot}$, we have

$$\begin{aligned}(F^{\cdot}/\|)\circ(F_1^{\cdot}/\|)(u) &= (F^{\cdot}/\|)\circ p_1\circ F_1^{\cdot}(x)=p\circ F^{\cdot}\circ F_1^{\cdot}(x)\\ &= p\circ(F_1\circ F)^{\cdot}(x)=(F_1\circ F)^{\cdot}/\|(u).\end{aligned}$$

Obviously if F is an identity homomorphism of S onto itself, then $F^{\cdot}/\|$ is the identity homeomorphism of $S^{\cdot}/\|$ onto itself. Hence $[^{\cdot}/\|]$ is a contravariant functor.

It is quite easy to see that C is a contravariant functor.

We shall prove equality (3). Let $x\in X$. Then

$$x^{\cdot}\circ F_1(f)=(F_1f)(x)=f\circ\varphi(x)=\big(\varphi(x)\big)^{\cdot}(f)$$

for any $f\in C(X_1)$, i.e. $x^{\cdot}\circ F_1=\big(\varphi(x)\big)^{\cdot}$, whence

$$\begin{aligned}(F_1^{\cdot}/\|)\circ(^{\cdot}/\|)(x) &= (F_1^{\cdot}/\|)\circ p(x^{\cdot})=p_1\circ F_1^{\cdot}(x^{\cdot})\\ &= p_1\circ x^{\cdot}\circ F_1=p_1\circ\big(\varphi(x)\big)^{\cdot}=(^{\cdot}/\|)_1\circ\varphi(x).\end{aligned}$$

Thus we need only prove that $(^{\cdot}/\|)$ is a homeomorphism. We shall show this for a more general case in Theorem 6.8.

LEMMA 6.7. *Let S be a metric functional* d-*lattice defined on a set X. Then the image of X under the mapping $(^{\cdot}/\|)\colon X\to S^{\cdot}/\|$ is a dense set in $S^{\cdot}/\|$.*

Proof. It is sufficient to show that for every non-empty open set $G\subseteq S^{\cdot}$, such that if $g\in G$, then $g+a\in G$ for any $a\in R$ there exists an $x\in X$ such that $x^{\cdot}\in G$. We have

$$G\supseteq G_0=\{g\in S^{\cdot}\colon a_i<g(f_i)<\beta_i \text{ for } i=1,2,\ldots,n\}\neq\emptyset$$

for some $f_i\in S$, a_i, $\beta_i\in R$, $i=1,2,\ldots,n$.

Let $g_0\in G_0$. Then $g_0(f_i')=0$, for $f_i'=f_i-g_0(f_i)$ where $i=1,2,\ldots,n$, whence

$$g_0\Big(\bigcup_{i=1}^{n}f_i'\Big)=g_0\Big(\bigcap_{i=1}^{n}f_i'\Big)=0.$$

It means that the elements $f^-=\bigcap_{i=1}^{n}f_i'$ and $f^+=\bigcup_{i=1}^{n}f_i'$ are linked. Thus for every $\varepsilon>0$ the set

$$A_\varepsilon=\{x\in X\colon f^-(x)+\varepsilon\geqslant f^+(x)\}$$

is non-empty. Let

$$2\varepsilon=\min_{i=1,\ldots,n}\ \min\big(g_0(f_i)-a_i,\ \beta_i-g_0(f_i)\big)$$

and let $x \in A_\varepsilon$. Then

$$f_1'(x) - \varepsilon \leqslant f_i'(x) \leqslant f_1'(x) + \varepsilon$$

and consequently

$$f_1(x) - g_0(f_1) - \varepsilon \leqslant f_i(x) - g_0(f_i) \leqslant f_1(x) - g_0(f_1) + \varepsilon.$$

Hence, it follows from the definition of ε that

$$f_1(x) - g_0(f_1) + a_i < f_i(x) < f_1(x) - g_0(f_1) + \beta_i$$

for $i = 1, 2, \ldots, n$. We put $\gamma = f_1(x) - g_0(f_1) \in R$. Then the last inequality implies

$$a_i < (x^{\cdot} - \gamma)(f_i) < \beta_i$$

for $i = 1, 2, \ldots, n$. Hence $x^{\cdot} - \gamma \in G_0$ and $x^{\cdot} \in G$. The lemma is proved.

DEFINITION 2. S is a *perfect* d-*lattice* defined on a topological space X, if S is a d-sublattice of the d-lattice $C(X)$, and $x^{\cdot} \| y^{\cdot}$ does not hold for any different $x, y \in X$, and S distinguishes points and closed sets of X in the following sense: if $x \in X$, $x \notin A = \bar{A} \subset X$, then there exists a finite sequence $f_1, f_2, \ldots, f_n \in S$ such that

$$f(x) \notin \overline{f(A)},$$

where $f: X \to R^n$ is given by

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

It is easy to see that for a metric functional d-lattice S, the d-lattice $\Psi(S)$ distinguishes points and closed sets of $S^{\cdot}$. Thus $\Psi(S)|S_a^{\cdot}$ is a perfect d-lattice defined on $S_a^{\cdot}$, for an arbitrary $a \in S$. Hence every metric distributive d-lattice is isomorphic to a perfect d-lattice on a certain compact space. Evidently every perfect d-lattice is metric.

THEOREM 6.8. *Let X be a Tychonoff space and let S be a perfect* d-*lattice defined on X. Then the mapping $(^{\cdot}/\|)$: $X \to S^{\cdot}/\|$ is a compactification of space X. Every compactification of X can be obtained in this way. In particular, if $S = C(X)$, then the mapping $(^{\cdot}/\|)$: $X \to S^{\cdot}/\|$ is the Čech-Stone β-compactification.*

Proof. By Lemma 6.7, the image of X under the mapping $(^{\cdot}/\|)$: $X \to S^{\cdot}/\|$ is a dense subset of $S^{\cdot}/\|$. We shall show that this mapping is a homeomorphic imbedding of X in $S^{\cdot}/\|$.

Indeed, from the perfectness of d-lattice S it follows that for any different elements $x, x_1 \in X$ the elements $p(x^{\cdot}), p(x_1^{\cdot}) \in S^{\cdot}/\|$ are different.

Next, it follows from the definition of weak topology in $S^{\cdot}$ that the mapping $(^{\cdot})$: $X \to S^{\cdot}$ is continuous, because the set $\{x \in X: a < x^{\cdot}(f) < \beta\}$ is open in X for any $f \in S$ and $a, \beta \in R$, since it is equal to the counter-image of the open segment (a, β) under the continuous real function f. Hence the mapping $(^{\cdot}/\|) = p \circ (^{\cdot})$ is also continuous.

In order to prove that the mapping $(\dot{}/||)\colon X \to S^{\cdot}/||$ is a homeomorphic imbedding, we must prove that if $A = \bar{A} \subseteq X$ and $x \in X \setminus A$, then

$$p(x^{\cdot}) \notin \overline{(\dot{}/||)(A)}.$$

It follows from the perfectness of S that for such x and A there exist real functions $f_1, f_2, \ldots, f_n \in S$ such that

$$f(x) \notin \overline{f(A)}$$

for the mapping $f = (f_1, f_2, \ldots, f_n)\colon X \to R^n$. Then

$$\varepsilon = \inf_{y \in A} \max_{i=1,2,\ldots,n} |f_i(x) - f_i(y)|$$

is a positive real number, whence

$$\bigcap_{i=1}^{n} \{u \in S^{\cdot}\colon f_i(x) - \varepsilon < u(f_i) < f_i(x) + \varepsilon\}$$

is an open neighbourhood of $x^{\cdot}$ in $S^{\cdot}$, which is disjoint with a set $(\dot{})(A)$. Thus

$$x^{\cdot} \notin \overline{(\dot{})(A)}.$$

This means that $(\dot{})\colon X \to S^{\cdot}$ is a homeomorphic imbedding.

Let $a \in S$ be an arbitrary fixed element. Then also a mapping

$$x \to x^{\cdot} - a(x) = x^{\cdot} - \big(\Psi(a)\big)(x^{\cdot})$$

of X into $S^{\cdot}_a$ is a homeomorphic imbedding, since a mapping

$$\varphi\colon u \to u - \big(\Psi(a)\big)(u), \qquad u \in S^{\cdot},$$

is a homeomorphism of $S^{\cdot}$ onto itself. Since $p_a\colon S^{\cdot}_a \to S^{\cdot}/||$ is a homeomorphism, we see that

$$(\dot{}/||) = p_a \circ \varphi \circ (\dot{})\colon X \to S^{\cdot}/||$$

is a homeomorphic imbedding. Thus $(\dot{}/||)\colon X \to S^{\cdot}/||$ is a compactification of X.

In particular, if X is a compact space, then $(\dot{}/||)\colon X \to S^{\cdot}/||$ is a homeomorphism of X onto $S^{\cdot}/||$. This completes the proof of the preceding Theorem 6.6.

Now we shall show that any compactification (with respect to the equivalents of compactifications) of a Tychonoff space X can be obtained in this way.

Indeed, let $a\colon X \to aX$ be the compactification of a Tychonoff space X and let S be a set of all the real functions $f \in C(X)$, which has the extension $af \in C(aX)$, where

$$(af)\big(a(x)\big) = f(x)$$

for $x \in X$. Then S is a functional metric d-lattice defined on X and the mapping

$$A: f \to af, \quad f \in S,$$

is an isomorphism of the d-lattice S and $S_1 = C(aX)$. Thus it is easy to see that S is a perfect d-lattice. Hence $(^{\cdot}/\|): X \to S^{\cdot}/\|$ is a compactification of X. We shall show that this compactification is equivalent to the compactification $a: X \to aX$.

As a matter of fact, it follows from Theorems 6.1 and 6.6 that the mappings

$$A^{\cdot}/\|: S_1^{\cdot}/\| \to S^{\cdot}/\| \quad \text{and} \quad (^{\cdot}/\|)_1: aX \to S_1^{\cdot}/\|$$

are homeomorphisms and consequently

$$(A^{\cdot}/\|) \circ (^{\cdot}/\|)_1: aX \to S^{\cdot}/\|$$

is a homeomorphism. We need only prove that

$$(A^{\cdot}/\|) \circ (^{\cdot}/\|)_1 \circ a(x) = (^{\cdot}/\|)(x)$$

for $x \in X$. In order to prove this let us remark that

$$A^{\cdot}\big((a(x))^{\cdot}\big) = x^{\cdot}$$

since

$$A^{\cdot}\big((a(x))^{\cdot}\big)(f) = (a(x))^{\cdot}(A(f)) = (a(x))^{\cdot}(af) = x^{\cdot}(f)$$

for any $f \in S$. Hence

$$(A^{\cdot}/\|) \circ (^{\cdot}/\|)_1 \circ a(x) = (A^{\cdot}/\|) \circ p_1((a(x)^{\cdot}) = p \circ A^{\cdot}\big((a(x))^{\cdot}\big)$$
$$= p \circ x^{\cdot} = (^{\cdot}/\|)(x).$$

Finally, let $S = C(X)$. Then S is isomorphic to the d-lattice $S_1 = C(\beta X)$, where the isomorphism $B: S \to S_1$ is given by

$$B(f) = \beta f.$$

Then the compactification $(^{\cdot}/\|): X \to S^{\cdot}/\|$ is equivalent to a β-compactification. The theorem is proved.

THEOREM 6.9. *Let S be the perfect* d-*lattice defined on a compact space X, let $S_1 \subseteq C(X_1)$ be a functional* d-*lattice defined on a compact space X_1, and let $F: S \to S_1$ be a homomorphism (a dual homomorphism) of these* d-*lattices. Then there exist a continuous mapping $\varphi: X_1 \to X$ and a real function $b \in C(X_1)$ such that*

(4) $$Ff = f \circ \varphi + b \quad (\textit{resp. } Ff = -f \circ \varphi + b) \quad \textit{for any } f \in S.$$

The mapping φ and the real function b are unique.

Furthermore

(i) *if S_1 is a perfect* d*-lattice and F is an isomorphism (a dual isomorphism) of S and S_1, then φ is a homeomorphism of X_1 and X;*

(ii) *if S_1 is a perfect* d*-lattice and F is a homomorphism (a dual homomorphism) of S onto S_1, then φ is a homeomorphic imbedding of X_1 into X;*

(iii) *if F is an isomorphic (a dual isomorphic) imbedding of S into S_1, then φ is a continuous mapping of X_1 onto X.*

Proof. We put

$$\varphi = (\dot{}/\|)^{-1} \circ F^{\cdot}/\| \circ (\dot{}/\|)_1 \tag{5}$$

and

$$b = Ff_0 - f_0 \circ \varphi \quad (\text{resp. } b = Ff_0 + f_0 \circ \varphi)$$

for an arbitrarily chosen $f_0 \in S$. It follows from Theorems 6.2 and 6.8 that φ is a continuous mapping (the mapping $(\dot{}/\|)$: $X \to S^{\cdot}/\|$ is onto, since X is a compact space). Hence $f_0 \circ \varphi$ is a continuous real function and consequently $b \in C(X_1)$.

Let us remark that

$$[(\dot{}/\|)^{-1} \circ p \circ F^{\cdot}(x^{\cdot})]^{\cdot} \| F^{\cdot}(x^{\cdot}), \qquad x \in X_1,$$

and

$$\begin{aligned}\varphi(x) &= (\dot{}/\|)^{-1} \circ F^{\cdot}/\| \circ (\dot{}/\|)_1(x) \\ &= (\dot{}/\|)^{-1} \circ F^{\cdot}/\| \circ p_1(x^{\cdot}) = (\dot{}/\|)^{-1} \circ p \circ F^{\cdot}(x^{\cdot}),\end{aligned}$$

whence

$$(\varphi(x))^{\cdot} \| F^{\cdot}(x^{\cdot}).$$

Thus

$$(F^{\cdot}(x^{\cdot}))(f) - (\varphi(x))^{\cdot}(f) = (F^{\cdot}(x^{\cdot}))(f_0) - (\varphi(x))^{\cdot}(f_0)$$

for any $f \in S$. Since

$$(F^{\cdot}(x^{\cdot}))(f) - (\varphi(x))^{\cdot}(f) = (Ff)(x) - f \circ \varphi(x)$$

(respectively

$$(F^{\cdot}(x^{\cdot}))(f) - (\varphi(x))^{\cdot}(f) = -(Ff)(x) - f \circ \varphi(x)),$$

we have

$$Ff - f \circ \varphi = Ff_0 - f_0 \circ \varphi = b$$

(respectively

$$Ff + f \circ \varphi = Ff_0 + f_0 \circ \varphi = b)$$

for any $f \in S$. This completes the proof of equality (4).

In order to prove the uniqueness of φ and b let us assume that also for a mapping φ': $X_1 \to X$ (not necessarily continuous), and for a real function b' (not necessarily continuous) defined on X_1 the equality

$$(6) \qquad Ff = f\circ\varphi' + b' \quad (\text{resp. } Ff = -f\circ\varphi' + b')$$

holds for any $f \in S$. Then, by (4) and (6),

$$(\varphi'(x))^{\cdot} \| (\varphi(x))^{\cdot},$$

whence

$$\varphi'(x) = (^{\cdot}/\|)^{-1} \circ p\left((\varphi'(x))^{\cdot}\right) = (^{\cdot}/\|)^{-1} \circ p\left((\varphi(x))^{\cdot}\right) = \varphi(x),$$

i.e.

$$\varphi' = \varphi.$$

Thus, by (4) and (6),

$$b' = b.$$

Next, if $F^{\cdot}/\|$ is a homeomorphism or a homeomorphic embedding and $(^{\cdot}/\|)_1$ is a homeomorphism, or if $F^{\cdot}/\|$ is a continuous mapping "onto", then it follows from (5) that φ has the same property as $F^{\cdot}/\|$. Hence from Theorems 6.1, 6.3, 6.4 and 6.8 follow assertions (i), (ii) and (iii). The theorem is proved.

COROLLARY. *If under the assumption from Theorem* 6.9 X *is a subspace of* X_1, *and*

$$(7) \qquad F(f)|X = f$$

for every $f \in S$, *then the mapping* φ *from Theorem* 6.9 *is a continuous retraction of* X_1 *onto* X.

Proof. It immediately follows from (4) and (7) that $x^{\cdot} \| (\varphi(x))^{\cdot}$ for $x \in X$. Hence $\varphi(x) = x$ for $x \in X$, as S is a perfect d-lattice.

Remark 1. For the perfectness of a functional d-lattice $S \subseteq C(X)$, where X is a compact space, the following condition is sufficient:

$$x^{\cdot} \| y^{\cdot} \quad \text{iff} \quad x = y, \ x, y \in X.$$

EXAMPLE. Let $X = I^2$, where $I = [0,1] \subseteq R$, and let S be the set of all functions $f \in C(X)$ such that

$$f(-1, 0) \leqslant f(0, 0) \leqslant f(1, 0).$$

Then S is a perfect d-lattice on I^2, but there exists no function $f \in S$, which distinguishes a point (0, 0) and $\dot{I}^2$ (boundary of I^2), i.e. for which $f(0,0) \notin f(\dot{I}^2)$.

Remark 2. If S is a metric distributive d-lattice and $X \subseteq S^{\cdot}$ is a compact set such that

$$\bigvee_{a,b \in S} \, \underset{u \in X}{\exists} \left[(a \neq b) \Rightarrow \left(u(a) \neq u(b)\right)\right],$$

then

$$\bigvee_{v \in S^{\cdot}} \, \underset{u \in X}{\exists} (u \| v).$$

Indeed, for such a compact $X \subseteq S^{\cdot}$ the mapping $F\colon S \to C(X)$ given by

$$(Fa)(u) = u(a) \quad \text{for} \quad u \in X,\ a \in S$$

is an isomorphic embedding of S into $S_1 = C(X)$. Then the mapping $(F^{\cdot}/\|) \circ (^{\cdot}/\|)_1\colon X \to S^{\cdot}/\|$ is onto (see Lemma 6.7 and Theorem 6.4). But

$$(F^{\cdot}/\|) \circ (^{\cdot}/\|)_1(u) = (F^{\cdot}/\|) \circ p_1(u^{\cdot}) = p \circ F^{\cdot}(u^{\cdot}) = p(u^{\cdot} \circ F)$$

for $u \in X$ and

$$u^{\cdot} \circ F(a) = (Fa)(u) = u(a) \quad \text{for} \quad u \in X,\ a \in S,$$

i.e.

$$u^{\cdot} \circ F = u$$

and

$$(F^{\cdot}/\|) \circ (^{\cdot}/\|)_1(u) = p(u).$$

§ 7. The main result of this section is the representation theorem for arbitrary, not necessarily metric distributive d-lattices (see Theorem 7.5). It turns out that every such d-lattice is isomorphic to a certain d-lattice of continuous real functions which may also assume the values $+\infty$ or $-\infty$, and which are defined on a certain compact space. This theorem enables us to distinguish a rather large abstract class of distributive d-lattices isomorphic to functional d-lattices (Theorem 7.6), and also a class of distributive d-lattices which are not isomorphic to functional d-lattices. More than that, it is even impossible to define any functional on a d-lattice from the second class.

We shall consider the non-metric distributive d-lattices. Let

$$R' = \{-\infty\} \cup R \cup \{\infty\}$$

be a usual two-point compactification such that the mapping

$$\varphi\colon R' \to [-\pi/2, \pi/2]$$

defined by

$$\varphi(x) = \begin{cases} -\pi/2 & \text{for} \quad x = -\infty, \\ \operatorname{arctg} x & \text{for} \quad x \in R, \\ \pi/2 & \text{for} \quad x = \infty \end{cases}$$

is a homeomorphism of R' onto a closed segment $[-\pi/2, \pi/2]$. We put

$$\begin{aligned} \max(a, \infty) &= \max(\infty, a) = \infty, \\ \min(a, -\infty) &= \min(-\infty, a) = -\infty, \\ \max(a, -\infty) &= \max(-\infty, a) = \min(a, \infty) \\ &= \min(\infty, a) = a \end{aligned}$$

for any $a \in R'$, and

$$\infty + a = a + \infty = \infty, \qquad -\infty + a = a + (-\infty) = -\infty$$

for any $a \in R$.

Let $D(X)$, for a topological space X, denote the set of all continuous mappings $f\colon X \to R'$, such that $f^{-1}(R)$ is a dense subset of X. Then for every $f, f' \in D(X)$ and $a \in R$ we can define the mappings $f \cup f'$, $f \cap f'$ and $f + a$ of X into R' by

$$(1) \qquad (f \cup f')(x) = \max\big(f(x), f'(x)\big),$$

$$(2) \qquad (f \cap f')(x) = \min\big(f(x), f'(x)\big),$$

$$(3) \qquad (f + a)(x) = f(x) + a$$

for any $x \in X$. Since

$$(f \cup f')^{-1}(R) = \big(f^{-1}(\{-\infty\} \cup R) \cap (f')^{-1}(\{-\infty\} \cup R)\big) \setminus \big(f^{-1}(-\infty) \cap \\ \cap (f')^{-1}(-\infty)\big),$$

$$(f \cap f')^{-1}(R) = \big(f^{-1}(R \cup \{\infty\}) \cap (f')^{-1}(R \cup \{\infty\})\big) \setminus \big(f^{-1}(\infty) \cap (f')^{-1}(\infty)\big),$$

$$(f + a)^{-1}(R) = f^{-1}(R),$$

the sets $(f \cup f')^{-1}(R)$, $(f \cap f')^{-1}(R)$ and $(f + a)^{-1}(R)$ are the dense subsets of X. Thus $f \cup f'$, $f \cap f'$, $f + a \in D(X)$ (since, evidently, this mappings are continuous).

THEOREM 7.1. *Let X be a non-empty topological space. Then the quadruple $\big(D(X), \cup, \cap, +\big)$ is a distributive* d*-lattice* ($\cup$, $\cap$ *and* $+$ *are given by* (1), (2) *and* (3)).

Proof. Evidently a triplet $\big(D(X), \cup, \cap\big)$ is a distributive lattice and the axioms 1-5 from § 1 hold (axiom 4 holds because $f^{-1}(R) \neq \emptyset$ for $f \in D(X)$, and axiom 5 holds because if $f \supset g$, $f, g \in D(X)$, then $f(x) > g(x)$ for a certain $x \in f^{-1}(R) \cap g^{-1}(R)$). We need only prove that axiom 6 holds.

Indeed, let $f, g, h \in D(X)$. If

$$h \supseteq f \cap (g + a)$$

for any $a \in R$, then

$$h(x) \geqslant f(x) \quad \text{for every } x \in f^{-1}(R) \cap g^{-1}(R) \cap h^{-1}(R).$$

Thus $h \supseteq f$ as the set $f^{-1}(R) \frown g^{-1}(R) \frown h^{-1}(R)$ is a dense subset of X (an intersection of a finite number of open dense subsets is an open dense subset). Hence

$$f = \bigcup_{a \in R} \big(f \frown (g+a)\big)$$

and similarly we can prove that

$$f = \bigcap_{a \in R} \big(f \smile (g+a)\big).$$

The theorem is proved.

We shall show that every distributive d-lattice can be imbedded in a certain d-lattice $D(X)$ such that X is a compact space. But first we shall introduce some auxiliary notions.

Let S be a distributive d-lattice, $a \in S$ a fixed element and $S_0 \subseteq S$ the d-sublattice of all elements $x \in S$, for which $d(x, a) < \infty$. Then we define a mapping u_x: $R \times R \to S$ by

$$u_x(\lambda, \mu) = (a+\lambda) \smile \big(x \frown (a+\mu)\big),$$

for every $x \in S$. Since

$$(4) \qquad a+\lambda \subseteq u_x(\lambda, \mu) \subseteq a+\max(\lambda, \mu),$$

we have

$$u_x(\lambda, \mu) \in S_0 \quad \text{for any } x \in S \text{ and } \lambda, \mu \in R.$$

It is obvious that the equalities

$$(5) \qquad u_{x \smile y}(\lambda, \mu) = u_x(\lambda, \mu) \smile u_y(\lambda, \mu),$$

$$(6) \qquad u_{x \frown y}(\lambda, \mu) = u_x(\lambda, \mu) \frown u_y(\lambda, \mu),$$

$$(7) \qquad u_{x+\xi}(\lambda, \mu) = u_x(\lambda-\xi, \mu-\xi)+\xi$$

hold for every $x, y \in S$ and $\lambda, \mu, \xi \in R$.

We have also

$$(8) \qquad u_{u_x(\lambda', \mu')}(\lambda, \mu) = u_x(\lambda, \mu)$$

for every real $\lambda' \leqslant \lambda$ and $\mu \leqslant \mu'$, and for any $x \in S$.

LEMMA 7.2. *Let $x \in S$ and $f \in S_a^{\cdot}$. If*

$$(9) \qquad \lambda \neq f\big(u_x(\lambda, \mu)\big) \neq \mu$$

for the real numbers λ, μ, then

$$(10) \qquad \lambda < f\big(u_x(\lambda, \mu)\big) < \mu$$

and for any real numbers λ', μ' such that

$$(11) \qquad \lambda' \leqslant f\big(u_x(\lambda, \mu)\big) \leqslant \mu'$$

we have

$$f\big(u_x(\lambda', \mu')\big) = f\big(u_x(\lambda, \mu)\big).$$

Proof. It follows from (4) that

$$\lambda \leqslant f\big(u_x(\lambda, \mu)\big) \leqslant \max(\lambda, \mu).$$

Thus from (9) we obtain (10).

Now, let (11) hold for the real numbers λ', μ' (and we assume that (9), and consequently (10), hold). First we shall consider the case of

$$\lambda' \leqslant \lambda < f\big(u_x(\lambda, \mu)\big) < \mu \leqslant \mu'.$$

Then, using (8), we have

$$\begin{aligned} f\big(u_x(\lambda, \mu)\big) &= f\big(u_{u_x(\lambda',\mu')}(\lambda, \mu)\big) \\ &= \max\Big(\lambda, \min\big(f\big(u_x(\lambda', \mu')\big), \mu\big)\Big) \\ &= f\big(u_x(\lambda', \mu')\big) \end{aligned}$$

(the last equality follows from (9)).

Next, in the general case of (11), we find that for

$$\lambda'' = \min(\lambda, \lambda') \qquad \text{and} \qquad \mu'' = \max(\mu, \mu')$$

we have

$$\begin{aligned} f\big(u_x(\lambda, \mu)\big) &= f\big(u_x(\lambda'', \mu'')\big) \\ &= \max\Big(\lambda', \min\big(f\big(u_x(\lambda'', \mu'')\big), \mu'\big)\Big) \\ &= f\big(u_{u_x(\lambda'',\mu'')}(\lambda', \mu')\big) = f\big(u_x(\lambda', \mu')\big). \end{aligned}$$

The lemma is proved.

LEMMA 7.3. *Let $x \in S$ and $f \in S_a^{\cdot}$. If*

$$f\big(u_x(\lambda, \mu)\big) = \lambda \quad \text{or} \quad f\big(u_x(\lambda, \mu)\big) = \mu$$

for any λ, $\mu \in R$, then

(12) $$f\big(u_x(\lambda, \mu)\big) = \lambda \quad \text{for any } \lambda < \mu \quad (\lambda, \mu \in R)$$

or

(13) $$f\big(u_x(\lambda, \mu)\big) = \mu \quad \text{for any } \lambda < \mu \quad (\lambda, \mu \in R).$$

Proof. Let $\lambda' \leqslant \lambda < \mu \leqslant \mu'$. Then, as in Lemma 7.2, we have

$$f\big(u_x(\lambda, \mu)\big) = \max\Big(\lambda, \min\big(f\big(u_x(\lambda', \mu')\big), \mu\big)\Big).$$

Thus if $f\big(u_x(\lambda', \mu')\big) = \lambda'$, then $f\big(u_x(\lambda, \mu)\big) = \lambda$ and if $f\big(u_x(\lambda', \mu')\big) = \mu'$, then $f\big(u_x(\lambda, \mu)\big) = \mu$.

Now let us assume that

$$f\big(u_x(\xi, \eta)\big) = \xi \quad \text{or} \quad f\big(u_x(\xi, \eta)\big) = \eta$$

for any $\xi, \eta \in R$. Then for any real numbers $\lambda, \mu, \lambda', \mu'$ such that $\lambda < \mu$, $\lambda' < \mu'$ we have $f(u_x(\lambda, \mu)) = \lambda$ iff

$$f\big(u_x(\min(\lambda, \lambda'), \max(\mu, \mu'))\big) = \min(\lambda, \lambda'),$$

and the last equality holds iff $f(u_x(\lambda', \mu')) = \lambda'$ (by the alternative assumption of the lemma), i.e. $f(u_x(\lambda, \mu)) = \lambda$ iff $f(u_x(\lambda', \mu')) = \lambda'$.

Analogously, $f(u_x(\lambda, \mu)) = \mu$ iff $f(u_x(\lambda', \mu')) = \mu'$. The lemma is proved.

LEMMA 7.4. *Let S be a distributive metric* d-*lattice, and let $U \subseteq S_a^{\cdot}$ be an open neighbourhood of a functional $f \in S_a^{\cdot}$ for an element $a \in S$. Then there exist elements $b, c \in S$ such that*

$$(14) \qquad f(b) > f(c) \quad \textit{and} \quad g(b) < g(c)$$

for each $g \in S_a^{\cdot} \setminus U$.

Proof. For any $g \in S_a^{\cdot} \setminus U$ there exist $y, z \in S$ such that

$$g(y) - g(z) \neq f(y) - f(z).$$

For such y, z we have

$$g(y) - g(z) < f(y) - f(z) \quad \text{or} \quad g(z) - g(y) < f(z) - f(y).$$

Consequently, for every $g \in S_a^{\cdot} \setminus U$ there exist $y, z \in S$ and a neighbourhood V of g such that

$$\sup_{g' \in V} \big(g'(y) - g'(z)\big) < f(y) - f(z).$$

Since $S_a^{\cdot}$ is compact, there exists a finite open cover $\{V_1, V_2, \ldots, V_n\}$ of $S_a^{\cdot} \setminus U$ and sequences $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in S$ such that

$$\beta_i = \big(f(y_i) - f(z_i)\big) - \sup_{g \in V_i} \big(g(y_i) - g(z_i)\big) > 0$$

for $i = 1, 2, \ldots, n$.

Now let

$$b_0 = \bigcap_{i=1}^{n} \big(y_i - f(y_i)\big) \quad \text{and} \quad c = \bigcup_{i=1}^{n} \big(z_i - f(z_i)\big).$$

Then

$$\begin{aligned}
&\sup_{g \in S_a^{\cdot} \setminus U} \big(g(b_0) - g(c)\big) \\
&\quad \leqslant \sup_{g \in \bigcup_{i=1}^{n} V_i} \big(g(b_0) - g(c)\big) = \max_{i = 1, 2, \ldots, n} \sup_{g \in V_i} \big(g(b_0) - g(c)\big) \\
&\quad \leqslant \max_{i = 1, 2, \ldots, n} \sup_{g \in V_i} \Big(\big(g(y_i) - f(y_i)\big) - \big(g(z_i) - f(z_i)\big)\Big) \\
&\quad = \max_{i = 1, 2, \ldots, n} \Big[-\big(f(y_i) - f(z_i)\big) + \sup_{g \in V_i} \big(g(y_i) - g(z_i)\big)\Big] = \max_{i = 1, 2, \ldots, n} (-\beta_i) < 0.
\end{aligned}$$

Hence

$$f(b_0) = f(c) = 0 > \sup_{g \in S_a^{\cdot} \setminus U} \big(g(b_0) - g(c)\big).$$

Thus for

$$b = b_0 - \tfrac{1}{2} \sup_{g \in S_a^{\cdot} \setminus U} \big(g(b) - g(c)\big)$$

we have

$$f(b) > f(c) \quad \text{and} \quad g(b) < g(c)$$

for every $g \in S_a^{\cdot} \setminus U$. The lemma is proved.

THEOREM 7.5. *If S is a distributive* d-*lattice and $S_0 \subseteq S$ is a maximal metric* d-*sublattice of S, then S can be isomorphically imbedded in $D(X)$, where $X = (S_0)_a^{\cdot}$ (see § 6) for any arbitrarily fixed element $a \in S_0$.*

Proof. We put

$$i_x(f) = \begin{cases} f\big(u_x(\lambda, \mu)\big) & \text{for } \lambda, \mu \text{ such that (9) holds,} \\ -\infty & \text{if (12) holds,} \\ \infty & \text{if (13) holds} \end{cases}$$

for any $x \in S$ and $f \in X = (S_0)_a^{\cdot}$.

First we shall show that $i_x \in D(X)$ for $x \in S$.

Indeed, let $\alpha, \beta \in R$, $\alpha < \beta$, and $(\alpha, \beta) \subset R$ be an open segment. Then it follows from Lemma 7.2 that

$$i_x^{-1}\big((\alpha, \beta)\big) = \big\{f \in X\colon\ \alpha < f\big(u_x(\alpha, \beta)\big) < \beta\big\}.$$

Hence $i_x^{-1}\big((\alpha, \beta)\big)$ is an open subset of $X = (S_0)_a^{\cdot}$.

Next from Lemmas 7.2 and 7.3 we obtain

$$i_x^{-1}\big([-\infty, \alpha)\big) = \big\{f \in X\colon\ f\big(u_x(\xi, \alpha)\big) < \alpha \text{ for some } \xi \in R\big\}$$

and

$$i_x^{-1}\big((\alpha, \infty]\big) = \big\{f \in X\colon\ f\big(u_x(\alpha, \xi)\big) > \alpha \text{ for some } \xi \in R\big\}$$

for any $\alpha \in R$. Hence the sets $i_x^{-1}\big([-\infty, \alpha)\big)$ and $i_x^{-1}\big((\alpha, \infty]\big)$ are also open. Thus we have proved that $i_x\colon X \to R$ is a continuous mapping. We need only prove that $i_x^{-1}(R)$ is a dense subset of X.

Let V be a non-empty open subset of X. Then

$$U = V \setminus i_x^{-1}(-\infty) \quad \text{or} \quad U' = V \setminus i_x^{-1}(\infty)$$

is also a non-empty open subset of X. We can assume that $U \neq \emptyset$ (the case of $V \setminus i_x^{-1}(\infty) \neq \emptyset$ is analogous). Let $f \in U$. Then it follows from Lemma 7.4 that there exist the elements $b, c \in S_0$ such that (14) holds for each $g \in (S_0)_a^{\cdot} \setminus U = X \setminus U$. Since

$$c \subseteq \big(c \cup (x + a)\big) \cap \big(c + d(b, c)\big) \in S_0$$

for any $a \in R$, for each $g \in X \setminus U$ we have

(15) $$g\left(\left(c \cup (x+a)\right) \cap \left(c+d(b,c)\right)\right) \geqslant g(b) \quad \text{for any } a \in R.$$

Now let $g \in X$ and $i_x(g) = \infty$. Then condition (15) holds for such a g. Indeed, for

$$\lambda \leqslant -d(a, c-a) \quad \text{and} \quad \mu \geqslant d\left(a, c+d(b,c)-a\right), \quad a \in R,$$

we obtain

$$\left((c-a) \cup x\right) \cap \left(c+d(b,c)-a\right) = \left((c-a) \cup u_x(\lambda,\mu)\right) \cap \left(c+d(b,c)-a\right)$$

and

$$\mu \geqslant d\left(a, c+d(b,c)-a\right) \geqslant g\left(c+d(b,c)-a\right)$$

(as $g(a) = 0$), whence

$$\begin{aligned} g\left(\left(c \cup (x+a)\right) \cap \left(c+d(b,c)\right)\right) &= g\left(\left((c-a) \cup x\right) \cap \left(c+d(b,c)-a\right)\right) + a \\ &= g\left(\left((c-a) \cup u_x(\lambda,\mu)\right) \cap \left(c+d(b,c)-a\right)\right) + a \\ &= \min\left(\max\left(g(c)-a, \mu\right), g(c)+d(b,c)-a\right) + a \\ &= g(c)+d(b,c) \geqslant g(b). \end{aligned}$$

It follows from axiom 1 from § 1 that

$$c = \bigcap_{a \in R} \left(\left(c \cup (x+a)\right) \cap \left(c+d(b,c)\right)\right),$$

and from (14) we have $b \nsubseteq c$. Hence there exists an $a_0 \in R$ such that

$$b \nsubseteq \left(c \cup (x+a_0)\right) \cap \left(c+d(b,c)\right).$$

Consequently, there exists a $g \in U$ such that

$$g\left(\left(c \cup (x+a_0)\right) \cap \left(c+d(b,c)\right)\right) < g(b)$$

(since for $g \in X \setminus U$ we have (15)). Then $-\infty < i_x(g) < \infty$ for such $g \in U$. Thus we have proved that $i_x^{-1}(R)$ is a dense subset of X.

We shall show that a mapping $i\colon S \to D(X)$ given by

$$i(x) = i_x, \quad x \in S,$$

is an isomorphic imbedding of a d-lattice S into $D(X)$.

Indeed, let $x, y \in S$, $f \in i_x^{-1}(R) \cap i_y^{-1}(R)$, $\xi \in R$ and

$$-\infty < \lambda < \min\left(i_x(f), i_y(f), i_{x\cup y}(f), i_{x\cap y}(f), i_{x+\xi}(f), i_x(f)+\xi\right),$$

$$\infty > \mu > \max\left(i_x(f), i_y(f), i_{x\cup y}(f), i_{x\cap y}(f), i_{x+\xi}(f), i_x(f)+\xi\right).$$

Then, from equality (5), Lemma 7.2 and the definition of i_z, $z \in X$, we have

$$\begin{aligned} i_{x\cup y}(f) &= f(u_{x\cup y}(\lambda, \mu)) \\ &= f(u_x(\lambda, \mu) \cup u_y(\lambda, \mu)) \\ &= \max(f(u_x(\lambda, \mu)), f(u_y(\lambda, \mu))) \\ &= \max(i_x(f), i_y(f)). \end{aligned}$$

Since i_z is a continuous mapping for any $z \in S$, we have

(16) $$i_{x\cup y}(f) = \max(i_x(f), i_y(f)) \quad \text{for any } f \in X.$$

Analogously we can prove

(17) $$i_{x\cap y}(f) = \min(i_x(f), i_y(f)), \quad \text{for any } f \in X.$$

Next, from equality (7), Lemma 7.2 and the definition of i_z, $z \in X$, we obtain

$$\begin{aligned} i_{x+\xi}(f) &= f(u_{x+\xi}(\lambda, \mu)) \\ &= f(u_x(\lambda-\xi, \mu-\xi))+\xi \\ &= i_x(f)+\xi \end{aligned}$$

for $f \in i_x^{-1}(R) = i_{x+\xi}^{-1}(R)$. Thus

(18) $$i_{x+\xi}(f) = i_x(f)+\xi \quad \text{for any } f \in X.$$

From (15), (16) and (17) we infer that $i\colon X \to D(X)$ is an isomorphic imbedding. Thus the proof of the theorem is complete.

THEOREM 7.6. *If a distributive* d-*lattice S contains at most a countable number of different maximal metric* d-*sublattices, then S is isomorphic to a certain functional* d-*lattice of continuous functions defined on an absolute G_δ-space Y* [5].

Proof. It is easy to see that if $x, y \in S$ are comparable, then, under the notions for the previous theorem, $i_x^{-1}(R) = i_y^{-1}(R)$. Thus

$$Y = \bigcap_{x \in S} i_x^{-1}(R)$$

is a dense G_δ-subset of $(S_0)_a^{\cdot}$ (from Baire's theorem). The mapping $x \to i_x|Y$, $x \in S$, is an isomorphic imbedding of S into a functional d-lattice $S_1 = \{i_x|Y\}_{x \in S}$. The theorem is proved.

On the other hand, we shall prove

THEOREM 7.7. *If a compact space X satisfies the first axiom of countability and if X is dense in itself, then there exists no functional defined*

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[5] A topological space Y is an *absolute* G_δ if Y can be homeomorphically imbedded in a compact space as a dense G_δ-subset.

on the d-*lattice* $D(X)$. *Consequently* $D(X)$ *is not isomorphic to any functional* d-*lattice.*

Proof. Let $u\colon C(X)\to R$ be a functional defined on the d-lattice $C(X)\subseteq D(X)$. We must show that there exists no functional $v\colon D(X)\to R$ such that $v|D(X) = u$. One can assume $u(0) = 0$.

We know that there exists a point $x\in X$ such that $u = x^{\cdot}$ (i.e. $u(f) = f(x)$ for $f\in C(X)$). Let a real function $f\in C(X)$ be such that $f(x) = 0$ and $f(y) > 0$ for $y\in X\setminus\{x\}$. We put $g(y) = 1/f(y)$ for such points y, and $g(x) = \infty$. Then the function g belongs to $D(X)$. Let $c_a\in C(X)$, $a\in R$, be a constant function given by $c_a(x) = a$ for $x\in X$, and let $g_a = c_a \cap g$. Then $g_a \subseteq g$ and $u(g_a) = g_a(x) = a$. Hence if $v\colon D(X)\to R$ is a functional such that $v|C(X) = u$, then $v(g) \geqslant a$ for all $a\in R$. But this is impossible. Thus there exists no functional defined on $D(X)$. The theorem is proved.

Let us notice that the space X appearing in Theorem 7.5 depends on the choice of a maximal metric d-sublattice S_0 of the d-lattice S.

For example, let $S = S_0 \cup S_1$, where $S_0 = C(R)$ and S_1 is a set of all continuous real-valued functions f defined on R for which there exist

$$\lim_{x=-\infty}\big(f(x)-x^2\big) \quad \text{and} \quad \lim_{x=\infty}\big(f(x)-x^2\big).$$

Then S is a d-lattice and S_0 and S_1 are its maximal metric d-sublattices. But $S_0^{\cdot}/||$ is homeomorphic to $\beta(R)$, whereas $S_1^{\cdot}/||$ is homeomorphic to the closed segment $[0, 1]$.

§ 8. In this section we deal with the direct sum of d-lattices (and related notions characteristic for d-lattices). We give a condition which is an isometric invariant and which is necessary in order that a d-lattice decompose into a direct sum. A d-lattice which does not satisfy this condition, that is a d-lattice which satisfies the converse condition, is said to be *metrically simple*. For distributive metric d-lattices the notion of metric simplicity coincides with the notion of indecomposability into a direct sum. Moreover, every isometry of indecomposable distributive d-lattices is an isomorphism (see Theorem 8.5).

In this section, once again, we consider connections between the algebraic and the metric structures of d-lattices.

Definition 1. An equivalence relation $\sim$ defined in a d-lattice S is a *congruence* if the following conditions are satisfied:

(i) if $x\sim x'$ and $y\sim y'$, then $x\cup y\sim x'\cup y'$, $x\cap y\sim x'\cap y'$ and $x+a\sim x'+a$ for every $x, y, x', y'\in S$ and $a\in R$;

(ii) $S/\sim$ is a d-lattice with the operations $\cup$, $\cap$, $+$, defined as follows:

$$\tilde{x}\cup\tilde{y} = (x\cup y)^{\sim}, \quad \tilde{x}\cap\tilde{y} = (x\cap y)^{\sim}, \quad \tilde{x}+a = (x+a)^{\sim}$$

for $x, y\in S$ and $a\in R$.

In this paragraph $\tilde{x}$ and $(x)^{\sim}$ denote the *equivalence class* of the element $x \in S$ for congruence $\sim$.

Definition 2. The *direct sum* $S' \oplus S''$ of the d-lattice S' and S'' is a d-lattice $(S' \times S'', \subseteq, +)$, where $\subseteq$ and $+$ are defined as

$$(x', x'') \subseteq (y', y'')$$

iff

$$x' \subseteq y' \quad \text{and} \quad x'' \subseteq y'', \qquad (x', x'') + a = (x' + a, x'' + a)$$

for $x', y' \in S'$, $x'', y'' \in S''$, $a \in R$.

It is easy to see that $S' \oplus S''$ is a well defined d-lattice and that

$$(x', x'') \cup (y', y'') = (x' \cup y', x'' \cup y''),$$
$$(x', x'') \cap (y', y'') = (x' \cap y', x'' \cap y'').$$

Property 1. *The metric in a direct sum $S' \oplus S''$ is given by*

$$d\big((x', x''), (y', y'')\big) = \max\big(d(x', y'), d(x'', y'')\big).$$

Hence the d-*lattice $S' \oplus S''$ is metric iff the* d-*lattices S' and S'' are metric.*

Property 2. *The* d-*lattice $S' \oplus S''$ is distributive (modular) iff the* d-*lattices S' and S'' are distributive (modular).*

Property 3. *If $\sim$ and $\simeq$ are congruences in a* d-*lattice S such that from $x \sim y$ and $x \simeq y$ follows $x = y$, then a function $f\colon S \to S/\sim \oplus S/\simeq$ defined by $f(x) = (\tilde{x}, \tilde{\tilde{x}})$ is a monomorphism (isomorphic imbedding). Thus in this case*

$$d(x, y) = \max\big(d(\tilde{x}, \tilde{y}), d(\tilde{\tilde{x}}, \tilde{\tilde{y}})\big).$$

Definition 2'. A d-lattice S is said to be ***decomposable into a direct sum*** *of the* d-*lattices S' and S''* if S is isomorphic with $S' \oplus S''$. The isomorphism $f\colon S \to S' \oplus S''$ is called a *direct decomposition of* S onto the direct sum of S' and S''.

Definition 3. We say that a d-lattice $S \subseteq S' \oplus S''$ is a *right* (*left*) *direct half-sum if* $S \neq \emptyset$ and for every $(x', x'') \in S$ and $(y', y'') \in S' \oplus S''$ from $y' \subseteq x'$ and $y'' \supseteq x''$ ($y' \supseteq x'$ and $y'' \subseteq x''$) it follows that $(y', y'') \in S$.

Definition 4. We say that d-lattice S is *decomposable in a right* (*left*) *direct half-sum of* the d-lattices S' and S'' if there exists a monomorphism $f\colon S \to S' \oplus S''$ such that $f(S)$ is a right (left) direct half-sum of the d-lattice S' and S''. Such a monomorphism f is called a *right* (*left*) *direct decomposition of* S.

Property 4. *If a* d-*lattice $S \subseteq S' \oplus S''$ is a right and left direct half-sum of the* d-*lattices S' and S'', then $S = S' \oplus S''$. If $f\colon S \to S' \oplus S''$ is a right and left direct decomposition of S, then f is a direct decomposition of S.*

Remark 1. It is possible that there exist right and left direct decompositions f and g of a d-lattice S in the right and left sums of d-lattices S' and S'', and that there is no decomposition of the S in the sum of S' and S''. But if $f = g$, then, by Property 4, function f is a direct decomposition.

EXAMPLE 1. Let S_0 and S be subsets of d-lattices R^2 and R^3 (see § 1, Example 4) such that

$$(a_1, a_2,) \in S_0 \quad \text{if} \quad a_1 \leqslant a_2$$

and

$$(a_1, a_2, a_3) \in S \quad \text{if} \quad a_1 \leqslant a_2 \leqslant a_3.$$

Then S_0 and S are d-sublattices and a function

$$f\colon S \to R \oplus S_0 \text{ given by } f(a_1, a_2, a_3) = \big(a_1, (a_2, a_3)\big)$$

is a right direct decomposition and a function

$$g\colon S \to R \oplus S_0 \text{ given by } g(a_1, a_2, a_3) = \big(a_3, (a_1, a_2)\big)$$

is a direct left decomposition of S. But there is no direct decomposition of S in the direct sum of any d-lattices S' and S'', and especially of the d-lattices R and S_0.

PROPERTY 5. *If $f\colon S \to S' \oplus S''$, where $f(x) = \big(f'(x), f''(x)\big)$ is a right (left) decomposition of S in the right (left) direct half-sum of the* d*-lattices S' and S'', then the function*

$$g\colon S \to S'' \oplus S' \text{ given by } g(x) = \big(f''(x), f'(x)\big)$$

is the left (right) decomposition of S in the left (right) direct half-sum of S' and S''.

DEFINITION 5. We say that the *elements a, b of a* d*-lattice S are in a relation $I^{\cdot}$*, shortly $aI^{\cdot}b$, if aIb and for every $a > 0$ there exists a $c \in S$ such that aIc, bIc, $d(a, c) = d(a, b) + d(b, c)$ and $d(b, c) = a$. If $aI^{\cdot}b$ and $bI^{\cdot}a$, then we shall write $aI^{\cdot\cdot}b$ (for I see Definition 1 from § 4).

LEMMA 8.1. *If a* d*-lattice S is decomposable in a right direct half-sum of any* d*-lattices S' and S'', then there exist in S elements a, b such that $aI^{\cdot}b$. If S is decomposable in a direct sum of* d*-lattices S' and S'', then there exist elements a, $b \in S$ such that $aI^{\cdot\cdot}b$.*

Proof. Let $f\colon S \to S' \oplus S''$ be a right direct decomposition. We put $f(x) = \big(f'(x), f''(x)\big)$, $x \in S$, $f'(x) \in S'$, $f''(x) \in S''$. Let $a \in S$ be an arbitrary but fixed element and let $a' = f'(a)$, $a'' = f''(a)$. For every $a > 0$ there exists an element $a_a \in S$ such that

$$f'(a_a) = a' - a \quad \text{and} \quad f''(a_a) = a'' + a$$

(especially $a_0 = a$). If in addition f is a direct decomposition, then a_a exists for every $a \in R$.

If $\alpha \leqslant \beta \leqslant \gamma$, then

$$d(a_\alpha, a_\beta) + d(a_\beta, a_\gamma) = d(a_\alpha, a_\gamma)$$

since

$$d(a_\alpha, a_\beta) = d\big(f(a_\alpha), f(a_\beta)\big)$$
$$= d\big((a'-\alpha, a''+\alpha), (a'-\beta, a''+\beta)\big) = \beta - \alpha$$

and analogously

$$d(a_\beta, a_\gamma) = \gamma - \beta \quad \text{and} \quad d(a_\alpha, a_\gamma) = \gamma - \alpha.$$

Moreover, $a_\alpha I a_\beta$ for $\alpha \neq \beta$ since

$$(a'-\alpha, a''+\alpha)\, I\, (a'-\beta, a''+\beta) \quad \text{for} \quad \alpha \neq \beta.$$

Thus $aI^{\cdot}a_1$ and it is easy to see that if f is a direct decomposition, then $aI^{\cdot\cdot}a_1$. The lemma is proved.

THEOREM 8.2. *A metric distributive* d-*lattice S is decomposable in the right direct half-sum* (*resp. in the direct sum*) *of any* d-*lattices S' and S'' iff there exist elements $a, b \in S$ such that $aI^{\cdot}b$* (*resp. $aI^{\cdot\cdot}b$*).

Proof. The part "only if" follows from Lemma 8.1. We shall prove the remaining part of the theorem. Let S be a metric distributive d-lattice, $a, b \in S$ and $aI^{\cdot}b$. Then there exists an element b_a such that aIb_a, bIb_a, $d(a, b_a) = d(a, b) + d(b, b_a)$ and $d(b, b_a) = \alpha$ for every $\alpha > 0$.

We introduce two binary relations $\veebar$ and $\frown$:

$$x \veebar y \ \text{ iff } \ \underset{\alpha>0}{\exists}\ \underset{\xi\geqslant\alpha}{\forall} (x \cup b_\xi = y \cup b_\xi),$$
$$x \frown y \ \text{ iff } \ \underset{\alpha>0}{\exists}\ \underset{\xi\geqslant\alpha}{\forall} (x \cap b_\xi = y \cap b_\xi).$$

It is easy to see that $\veebar$ and $\frown$ are equivalence relations. Moreover, these relations are congruences. Indeed, let $x \veebar y$ and $u \veebar v$. Then there exists an $\alpha > 0$ such that for every $\xi \geqslant \alpha$ we have

$$x \cup b_\xi = y \cup b_\xi \quad \text{and} \quad u \cup b_\xi = v \cup b_\xi.$$

Hence

$$(x \cup u) \cup b_\xi = (y \cup v) \cup b_\xi$$

and by the distributivity of S

$$(x \cap u) \cup b_\xi = (y \cap v) \cup b_\xi.$$

Thus

$$x \cup u \veebar y \cup v \quad \text{and} \quad x \cap u \veebar y \cap v.$$

Further, we can assume that S is a functional d-lattice of functions on a certain set T. Then for $x \in S$ and $t \in T$ we find that $x(t)$ is a real number.

Let

$$A = \{t \in T:\ b(t) + d(a, b) = a(t)\},$$
$$(1)\qquad B = \{t \in T:\ a(t) + d(a, b) = b(t)\}.$$

Then $T = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and

$$b_\xi(t) = \begin{cases} b(t) - \xi & \text{for} \quad t \in A, \\ b(t) + \xi & \text{for} \quad t \in B. \end{cases}$$

Thus $x \underline{\smile} y$ (resp. $x \underline{\frown} y$) means that $x(t) = y(t)$ for $t \in A$ (resp. $t \in B$). Thus if $x \underline{\smile} y$, then $x + \beta \underline{\smile} y + \beta$. Furthermore, $S/\underline{\smile}$ is isomorphic with a functional d-lattice $S|A = \{f|A\}_{f \in S}$. Thus $\underline{\smile}$ is a congruence.

Similarly we can prove that $S/\underline{\frown}$ is isomorphic with $S|B$. Thus $\underline{\frown}$ is a congruence.

It is easy to see that if $x \underline{\smile} y$ and $x \underline{\frown} y$, then $x = y$, $x, y \in S$. Hence a map f: $S \to S/\underline{\smile} \oplus S/\underline{\frown}$ given by $f(x) = (\check{x}, \hat{x})$ is an isomorphic imbedding. We shall show that f is a right direct decomposition. Indeed, let $x, y, z \in S$, $\check{x} \supseteq \check{y}$, $\hat{x} \subseteq \hat{z}$. We must show that $(\check{y}, \hat{z}) \in f(S)$. Let $\alpha = \max\big(d(b, x), d(b, y), d(b, z)\big)$. Then

$$\check{b}_\alpha \subseteq \check{x} \cap \check{y} \cap \check{z} \qquad \text{and} \qquad \hat{b}_\alpha \supseteq \hat{x} \cup \hat{y} \cup \hat{z};$$

whence

$$f(y \cup b_\alpha) = (\check{y}, \hat{b}_\alpha), \qquad f\big((y \cup b_\alpha) \cap x\big) = (\check{y}, \hat{x}),$$
$$f(z \cap b_\alpha) = (\check{b}_\alpha, \hat{z}), \qquad f\Big(\big((y \cup b_\alpha) \cap x\big) \cup (z \cap b_\alpha)\Big) = (\check{y}, \hat{z}).$$

Thus $(\check{y}, \hat{z}) \in f(S)$.

If in addition $bI'a$, then there exists an element $a_\alpha \in S$ such that bIa_α, aIa_α, $d(b, a_\alpha) = d(b, a) + d(a, a_\alpha)$ and $d(a, a_\alpha) = \alpha$ for $\alpha > 0$. Then

$$a_\alpha(t) = \begin{cases} a(t) - \alpha & \text{for} \quad t \in B, \\ a(t) + \alpha & \text{for} \quad t \in A \end{cases}$$

and similarly we can prove that f is a left decomposition of S into a left direct half-sum of $S/\underline{\smile}$ and $S/\underline{\frown}$, since we have

$$x \underline{\smile} y \ \text{ iff } \ \exists_{\alpha > 0} \forall_{\xi \geqslant \alpha} (x \cap a_\xi = y \cap a_\xi),$$
$$x \underline{\frown} y \ \text{ iff } \ \exists_{\alpha > 0} \forall_{\xi \geqslant \alpha} (x \cup a_\xi = y \cup a_\xi).$$

Thus, if $aI''b$, then f is a direct decomposition of S.

It follows from the proof of Theorem 8.2 that the maps $\check{x} \to x|A$ and $\hat{x} \to x|B$ are isomorphisms $S/\underline{\smile} \to S|A$ and $S/\underline{\frown} \to S|B$, where $S|A = \{x|A\}_{x \in S}$ and $S|B = \{x|B\}_{x \in S}$. Thus we obtain the following

COROLLARY 1. *If $f\colon S \to S' \oplus S''$, $f(x) = (f'(x), f''(x))$, is a right or left decomposition of S in a right or left direct sum of S' and S'', and S is a functional metric* d*-lattice of the functions on a certain set T, then there exists a unique pair of subsets $A, B \subseteq T$, $A \cup B = T$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$, such that there exist isomorphisms*

$$i'\colon S' \to S|A \quad \textit{and} \quad i''\colon S'' \to S|B$$

which satisfy

$$i' \circ f'(x) = x|A, \quad i'' \circ f''(x) = x|B$$

for $x \in S$. If in addition T is a topological space and the functions from S are continuous, then, as follows from formulas (1), *A and B are closed-open sets.*

THEOREM 8.3. *If a map $f\colon S \to S'$ is an isometry of the metric* d*-lattices S and S', and $f(a+\xi) = f(a)+\xi$ (resp. $f(a+\xi) = f(a)-\xi$) for some $a \in S$ and $\xi \neq 0$, then f is an isomorphism (resp. a dual isomorphism).*

Proof. First, under the conditions of the theorem we shall show that

(2) $$f(a+\alpha) = f(a)+\alpha \quad \text{for every } \alpha \in R.$$

We shall consider four cases (i)-(iv).

(i) $\xi > 0$ and $\alpha > 0$.

The set $S_1(f(a), f(a+\alpha))$ contains exactly one element, hence $f(a+\alpha) = f(a)-\alpha$, or $f(a+\alpha) I f(a)$, or $f(a+\alpha) = f(a)+\alpha$.

In the first two cases we have

$$d^-(f(a), f(a+\alpha)) = d(f(a), f(a+\alpha)) = \alpha$$

and

$$d^-(f(a+\xi), f(a+\alpha)) = d^-(f(a)+\xi, f(a+\alpha)) = \alpha+\xi.$$

But

$$d(f(a+\xi), f(a+\alpha)) = |\alpha-\xi| < \alpha+\xi.$$

This contradiction shows that if $\xi > 0$ and $\alpha > 0$, then

$$f(a+\alpha) = f(a)+\alpha.$$

(ii) $\xi > 0$ and $\alpha < \xi$.

If we put $a_0 = a+\xi$, $\alpha_0 = \xi-\alpha > 0$ and if $f_0\colon S^0 \to S'^0$ is given by $f_0(x) = f(x)$ for $x \in S$ (see § 1), then $f_0(a_0 +^0 \xi) = f_0(a_0) +^0 \xi$.

Thus from (i) we obtain $f_0(a_0 +^0 \alpha_0) = f_0(a_0) +^0 \alpha_0$; this means that

$$f(a+\alpha) = f(a)+\alpha$$

(as $a_0 +^0 \alpha_0 = (a+\xi)-(\xi-\alpha) = a+\alpha$ and $f_0(a_0) +^0 \alpha_0 = f(a+\xi)-\alpha_0 = f(a)+\xi-(\xi-\alpha) = f(a)+\alpha$).

(iii) $\xi < 0$ and $\alpha > \xi$.

This case is simply dual to (ii).

(iv) $\xi < 0$ and $\alpha < 0$.

This case is simply dual to (i). Thus formula (2) holds.

Now we shall prove that

(3) $a \subseteq b$ iff $f(a) \subseteq f(b)$, and $a \supseteq b$ iff $f(a) \supseteq f(b)$ for every $b \in S$.

Indeed, $a \subseteq b$ iff $d(a, b) \geqslant d(a+d(a, b), b)$. By (2) the last inequality is equivalent to

$$d(f(a), f(b)) \geqslant d(f(a)+d(f(a), f(b)), f(b)),$$

which is equivalent to $f(a) \subseteq f(b)$.

The proof of the second part of (3) is similar.

In order to prove (3) we used only (2). Thus if $f(c+\alpha) = f(c)+\alpha$ for every $c \in S$ and $\alpha \in R$, then our theorem holds. But this equality holds for $c = a+\beta$, $\beta \in R$. Hence (3) holds if we substitute $a+\beta$ for a. Now let $c \in S$ be arbitrary. Then

$$c \subseteq a+d(a, c) \quad \text{and} \quad a \subseteq c+d(a, c)+1,$$

whence

$$f(c) \subseteq f(a)+d(a, c) \quad \text{and} \quad f(a) \subseteq f(c+d(a, c)+1).$$

Then $f(c+d(a, c)+1) = f(c)-d(a, c)-1$, or $f(c+d(a, c)+1)\, I f(c)$, or $f(c+d(a, c)+1) = f(c)+d(a, c)+1$.

In the first two cases we have

$$d^{-}\big(f(c), f(c+d(a, c)+1)\big) = d(a, c)+1,$$

whence

$$f(c) \nsubseteq f(c+d(a, c)+1)+d(a, c),$$

whence

$$f(a)+d(a, c) \nsubseteq f(c+d(a, c)+1)+d(a, c)$$

in contradiction to $f(a) \subseteq f(c+d(a, c)+1)$. Thus $f(c+\xi') = f(c)+\xi'$ for $\xi' = d(a, c)+1 \neq 0$.

Now, the proof of the equality $f(c+\alpha) = f(c)+\alpha$, for every $\alpha \in R$, is analogous to the proof of (1). The theorem is proved.

LEMMA 8.4. *If a map $f\colon S \to S_1$ is an isometry, but neither an isomorphism nor a dual isomorphism, then in the lattices S and S_1 there are pairs of elements which are in the relation $I^{\cdot\cdot}$.*

Proof. Let $a \in S$. Then by Theorem 8.3

$$f(a+\alpha) \neq f(a+\beta)+(\alpha-\beta)$$

and

$$f(a+\alpha) \neq f(a+\beta)+(\beta-\alpha)$$

for every $\alpha \neq \beta$. Hence $f(a+\alpha) I f(a+\beta)$, α, $\beta \in R$.

Moreover, for every $\alpha \leqslant \beta \leqslant \gamma$ we have

$$d(f(a+\alpha), f(a+\beta)) + d(f(a+\beta), f(a+\gamma)) = d(f(a+\alpha), f(a+\gamma)).$$

Thus $f(a) I^{\cdot\cdot} f(a+1)$.

The mapping f^{-1} is also an isometry and not an isomorphism, whence $f^{-1}(a_1) I^{\cdot\cdot} f^{-1}(a_1+1)$ for $a_1 \in S_1$. The lemma is proved.

COROLLARY 2. *If a function f: $S \to S_1$ is an isometry of the metric distributive* d*-lattices S and S_1, and if S or S_1 is not decomposable in a direct sum, then f is an isomorphism or a dual isomorphism.*

Proof. If under the conditions of the corollary f is not an isomorphism or a dual isomorphism, then by Lemma 8.4 there are pairs of elements from S and S_1 which are in the relation $I^{\cdot\cdot}$. Hence the assumption that S or S_1 are not decomposable in a direct sum contradicts Theorem 8.2. Thus our corollary is proved.

From Lemma 8.4 we can obtain a generalization of the above assertion for non-distributive d-lattices. First we shall introduce the following definition:

DEFINITION 6. A d-lattice S is called *metrically simple* if $a I^{\cdot\cdot} b$ does not hold for any $a, b \in S$ and S is called *metrically be-simple* if $a I^{\cdot} b$ does not hold for any a, $b \in S$.

THEOREM 8.5. *If a function f: $S \to S_1$ is an isometry of the metric* d*-lattices S and S_1, and S or S_1 is metrically simple, then f is an isomorphism or a dual isomorphism.*

Remark 2. Theorem 8.5 shows that the properties of metric simplicity and metric be-simplicity are invariant under isometries.

Remark 3. Theorem 8.2 shows that a metric distributive d-lattice S is metrically simple (be-simple) iff it is not decomposable in a direct sum (half-sum). In general, this equivalence does not hold – the d-lattices from Examples 6 and 7 from § 1 are not decomposable in a direct half-sum and they are not metrically simple.

Now we shall give examples of some classes of metrically be-simple d-lattices.

Let S be a d-lattice. We define a function ν: $Q \to R$ (where $Q \subseteq S \times S$ is a relation defined in § 4, Definition 1),

$$\nu(a, b) = \sup[\alpha \in R: a \frown b + \alpha \subseteq a \smile b]$$

for $a, b \in S$, aQb.

If aIb, then aQb and $\nu(a, b) = d(a, b)$. In general, $\nu(a, b) \leqslant d(a, b)$.

EXAMPLE 2. The class of the d-lattices S such that

$$\sup_{aQb, a, b \in S} \nu(a, b) < \infty$$

is a subclass of the class of metrically be-simple d-lattices.

EXAMPLE 3. We say that a d-lattice S is *connected* if aQb implies $a \frown bQa \smile b$ for $a, b \in S$, i.e. if

$$\sup_{a, b \in S} \nu(a, b) = 0.$$

One can prove that if S is a metric distributive d-lattice, then S is connected iff the compact space $S^*/\|$ is topologically connected.

EXAMPLE 4. Let S be a metric d-lattice and let $p: S \to S/\|$ (see the definition from § 6) be a canonical mapping. We put

$$d(u, v) = \inf_{a \in u, b \in v} d(a, b) \quad \text{for} \quad u, v \in S/\|.$$

Then

$$d(p(a), p(b)) = \inf_{\beta \in R} d(a, b+\beta)$$

and one can prove that there exists a β such that $d(p(a), p(b)) = p(a, b+\beta)$ for $a, b \in S$. Thus for $a, b, c \in S$ there exist β and γ such that

$$d(p(a), p(b)) = d(a, b+\beta)$$

and

$$d(p(b), p(c)) = d(b+\beta, c+\gamma),$$

whence

$$d(p(a), p(b)) + d(p(b), p(c)) \geqslant d(a, c+\gamma) \geqslant d(p(a), p(c)).$$

It is easy to verify the first two axioms of the metric. Thus $(S/\|, d)$ is a metric space. We say that a metric d-lattice S is *bounded* if $S/\|$ is bounded. We shall prove that every bounded d-lattice is metrically be-simple.

Indeed, let S be a bounded d-lattice. Then

$$d(a, b) = d(p(a), p(b))$$

iff

$$d^-(a, b) = d^+(a, b), \qquad a, b \in S.$$

We shall show that aIb implies $d(a, b) = d(p(a), p(b))$. Otherwise we could assume that $d^+(a, b) - d^-(a, b) > 0$, i.e. that $d(a, b) = d^+(a, b) = d^-(a, b) + \varepsilon$ for some $\varepsilon > 0$. Then

$$b \supseteq a - d^-(a, b) = a - d(a, b) + \varepsilon$$

and

$$a \supseteq a - d^-(a, b) = a - d(a, b) + \varepsilon,$$

whence

$$a \cap b \supseteq a - d(a, b) + \varepsilon \quad \text{and} \quad a \subseteq a \cap b + d(a, b) - \varepsilon.$$

But the assumption aIb implies $a \cap b + d(a, b) = a \cup b$, thus $a \subseteq a \cup b - \varepsilon$ and from Lemma 1.3, we obtain $a \subseteq b$. Then $a = a \cap b$, $b = a \cup b$ and $a \| b$ in contradiction to aIb. Thus if aIb, then

$$d(a, b) = d\big(p(a), p(b)\big) \leqslant \operatorname{diam} S/\|.$$

Let X be a bounded metric space. Then the set $\operatorname{Met} X$ of all real-valued metric functions on X is a concrete example of a bounded functional d-lattice.

"Met" is a contravariant functor from the category of bounded metric spaces, with metric mappings as morphisms, into the category of d-lattices. This functor can be extended preserving its important properties to a functor from the entire category of metric spaces. Such an extension can be achieved in several ways. The functor "Met" has a number of fundamental properties of the functor C (see § 6). Functor "Met" was considered in [12].

THEOREM 8.6. *If $f\colon S \to S' \oplus S''$ is a direct (left direct, right direct) decomposition of a* d*-lattice, and S_0 is a maximal metric* d*-sublattice of S, then the map $f|S_0\colon S_0 \to f'(S_0) \oplus f''(S_0)$, where $f(x) = \big(f'(x), f''(x)\big)$ for $x \in S$, is a direct (left direct, right direct) decomposition of S_0.*

Proof. Let $f\colon S \to S' \oplus S''$ be a right decomposition of S. Then $f|S_0$ is a monomorphism. Let $y' \subseteq f'(x)$ and $f''(x) \subseteq y''$ for an arbitrarily chosen $x \in S_0$, $y' \in f'(S_0)$, $y'' \in f''(S_0)$. Then there exists an $a \in S$ such that $y' = f'(a)$, $y'' = f''(a)$. Let us put

$$b = \Big(\big[x - d\big(y', f'(x)\big)\big] \cup a\Big) \cap \big[x + d\big(y'', f''(x)\big)\big].$$

Then $b \in S_0$ and $f'(b) = y'$, $f''(b) = y''$.

The proof for a left decomposition is analogous.

The remaining part of the theorem immediately follows from Property 4.

EXAMPLE 5. Let S be a set of all real continuous functions $f\colon \{-1\} \cup [0, \infty) \to R$ such that

$$f(-1) \neq f(0) \Rightarrow \lim_{x=\infty} f(x) = \infty.$$

Then $C\big(\{-1\} \cup [0, \infty)\big) \cap S$ is a maximal metric d-sublattice of S and it is not decomposable even in a direct half-sum. On the other hand, the maximal metrical d-sublattice which contains the real function $f(x) = x^2$ is decomposable in a direct sum.

§ 9. The results of this section are a continuation of the results of § 6 and § 8. First we prove that every isometry of distributive metric d-lattices which is not an isomorphism is associated with a decomposition into a direct sum. Next, we prove three theorems: 9.4, 9.5 and 9.7. Theorem 9.4 says that any isometry of perfect d-lattices defined on compact spaces induces a homeomorphism of those spaces. It is a generalization of the classical Banach-Stone Theorem (see [5]).

Theorem 9.5 says that an isometric mapping of a perfect d-lattice onto a d-sublattice of a functional d-lattice (both lattices being defined on compact spaces X, X_1 respectively) induces a continuous map of X_1 onto X.

Finally, Theorem 9.7, the most essential one here, says that even if no additional assumptions are made on an isometric embedding of a perfect d-lattice in a functional d-lattice, the embedding will induce a continuous map of a certain closed subset of the space X_1 onto the space X. It is more general than the corresponding theorems contained in [2], [6], [7] (even if we restrict our theorem to the case of d-lattices $C(X)$, $C(X_1)$ because we assume nothing on the isometric embedding, no linearity condition. The isometric image of $C(X)$ in $C(X_1)$ need not even contain a triple of algebraically collinear elements in $C(X_1)$, i.e. elements $f, g, h \in \operatorname{Im} C(X)$ such that $h = (1-t)f+tg$ for some $t \in R$).

Incidentally, we prove (see Theorem 9.6 and the corollary to that theorem) that for any (not necessarily linear) isometric embedding of one space of continuous real functions in another one there exists a linear (metric) mapping of this other space into a third space such that a composition of these mappings is an affine isometric embedding of the first space into the third space. Thus the following questions are natural (the first was asked by J. Lindenstrauss during a talk I had with him):

Is it true that for an isometric mapping of one Banach space into another there exists a linear mapping of this other space into a third Banach space such that the composition of these mappings is an affine isometric embedding? Does there exist such a metric linear mapping of the second Banach space into a third space? ([6]).

THEOREM 9.1. *Let S and S_1 be metric* d*-lattices, let S_2 be a metrically simple* d*-lattice. Let $\varphi\colon S_1 \to S_2$ be a homomorphism (or a dual homomorphism) and let $f\colon S \to S_1$ be an isometry. Then $\varphi \circ f\colon S \to S_2$ is a homomorphism or a dual homomorphism.*

First we shall prove the following lemma:

([6]) The latter question, and consequently also the former one, has recently been affirmatively solved by Figiel [6], and later also by the author of this paper (see [11]).

LEMMA 9.2. *Let S be a* d-*lattice. If aIb, then*

$$d^-(a, b) = d^+(a, b) = d(a, b) \tag{1}$$

and if, in addition, the sets $S_1(a, c)$ and $S_1(b, c)$ are one-element sets, then aIc and bIc for any different $a, b, c \in S$.

Proof. Let aIb. Then $a \nleq b$ and $b \nleq a$. Thus from Lemma 1.3 it follows that

$$a \nleq (a-\varepsilon) \cup b \quad \text{and} \quad b \nleq a \cup (b-\varepsilon)$$

for any $\varepsilon > 0$. Hence

$$a \nleq a \cup b - \varepsilon \quad \text{and} \quad b \nleq a \cup b - \varepsilon, \quad \varepsilon > 0,$$

and consequently, as $a \cap b = a \cup b - d(a, b)$, we have

$$a \cap b \ngeq a - d(a, b) + \varepsilon$$

and

$$a \cap b \ngeq b - d(a, b) + \varepsilon.$$

Thus equality (1) holds.

Next if, in addition, $a \neq c \neq b$ and for example $a \| c$, then the relation $b \| c$ does not hold, since aIb, the relation bIc does not hold either, since $d^-(b, c) < d^+(b, c)$ if $c = a + d(a, c)$, and $d^-(b, c) > d^+(b, c)$ if $c = a - d(a, c)$. Hence $S_1(b, c)$ is not a one-element set in contradiction to our assumptions. Hence aIc and, similarly, bIc. The lemma is proved.

Proof of Theorem 9.1. Let $x \in S$, $\alpha \neq 0$, $x \in R$. Then

$$f(x) \cap f(x+\alpha) + |\alpha| = f(x) \cup f(x+\alpha)$$

(since $S_1(f(x), f(x+\alpha))$ is a one-element set, see § 4) and

$$\varphi \circ f(x) \cap \varphi \circ f(x+\alpha) + |\alpha| = \varphi \circ f(x) \cup \varphi \circ f(x+\alpha). \tag{2}$$

Hence, as we shall show,

$$\varphi \circ f(x+\alpha) = \varphi \circ f(x) + \alpha \quad \text{or} \quad \varphi \circ f(x+\alpha) = \varphi \circ f(x) - \alpha. \tag{3}$$

Indeed, from (2) it follows that $S_1(\varphi \circ f(x+\alpha), \varphi \circ f(x+\beta))$ is a one-element set and

$$d(\varphi \circ f(x+\alpha), \varphi \circ f(x+\beta)) = |\alpha - \beta|$$

for every $\alpha, \beta \in R$. Since S_2 is a metrically simple d-lattice, it follows from Lemma 9.2 that the relation $\varphi \circ f(x) I \varphi \circ f(x+\alpha)$ does not hold.

Let us assume that for a certain arbitrary but hence forth fixed $\alpha \neq 0$ the first of the above-mentioned equalities from (3) holds. If sign α' = sign α, then also $\varphi \circ f(x+\alpha') = \varphi \circ f(x) + \alpha'$, since $\varphi \circ f$ is a metric function (i.e. $d(\varphi\circ f(y), \varphi\circ f(z)) \leqslant d(y, z)$ for $y, z \in S$). Now if sign a' $= -$ sign a, then putting $x' = x+a'$ we obtain

$$\varphi\circ f(x'-a') = \varphi\circ f(x')+\varepsilon(-a')$$

and

$$\varphi\circ f(x'-a'+a) = \varphi\circ f(x')+\varepsilon(-a'+a),$$

where $\varepsilon = -1$ or 1, as $\operatorname{sign}(-a') = \operatorname{sign}(-a'+a)$. But

$$\varphi\circ f(x'-a'+a) = \varphi\circ f(x+a) = \varphi\circ f(x)+a$$

and consequently

$$\varphi\circ f(x) = \varphi\circ f(x')+\varepsilon(-a'+a)-a.$$

Hence $\varepsilon = 1$, as

$$|\varepsilon(-a'+a)-a| = d(\varphi\circ f(x), \varphi\circ f(x')) \leqslant |a'|.$$

Thus if for a certain $x\in S$ and a certain $a \neq 0$ we have $\varphi\circ f(x+a)$ $= \varphi\circ f(x)+a$, then $\varphi\circ f(x+a') = \varphi\circ f(x)+a'$ for every $a'\in R$, and similarly, by duality, if for some $x\in S$ and $a \neq 0$ we have $\varphi\circ f(x+a)$ $= \varphi\circ f(x)-a$, then $\varphi\circ f(x+a') = \varphi\circ f(x)-a'$ for every $a'\in R$.

We shall consider only the first case, since the second one can be obtained the first case by substituting S_2^0 for S_2. Let $y\in S$. Then $\varphi\circ f(y+\beta)$ $=\varphi\circ f(y)+\beta$ for $\beta = d(x, y)+1 \neq 0$, since $\varphi\circ f(y+\beta) = \varphi\circ f(y)-\beta$ implies

$$d(\varphi\circ f(x+\beta), \varphi\circ f(y+\beta)) \geqslant d(x, y)+2 > d(x, y) = d(x+\beta, y+\beta).$$

Thus

$$\varphi\circ f(y+\beta') = \varphi\circ f(y)+\beta'$$

for every $y\in S$ and $\beta'\in R$ (in the second case we have $\varphi\circ f(y+\beta')$ $=\varphi\circ f(y)-\beta'$).

Now we shall show that $y \subseteq z$ implies $\varphi\circ f(y) \subseteq \varphi\circ f(z)$ (in the second case $\varphi\circ f(y) \supseteq \varphi\circ f(z)$).

Indeed, if $y \subseteq z$, then

$$d(y, z) \geqslant d(y+d(y, z), z) \geqslant d(\varphi\circ f(y)+d(y, z), \varphi\circ f(z)).$$

Thus $\varphi\circ f(y) \subseteq \varphi\circ f(z)$. The theorem is proved.

COROLLARY 1. *If $f\colon S \to S_1$ is an isometry of the metric* d-*lattices S and S_1, and if $\varphi\colon S_1 \to R$ is a functional on S_1, then $\varphi\circ f$ is a functional or a dual functional on S.*

THEOREM 9.3. *If a mapping $f\colon S \to S_1$ is an isometry of the metric distributive* d-*lattices S and S_1, then f is an isomorphism or a dual isomorphism or there exist* d-*lattices S' and S'' and isomorphisms $i\colon S \to S' \oplus S''^0$ and $i_1\colon S_1 \to S' \oplus S''$ such that $i_1' \circ f = i'$, $i_1'' \circ f = i''$, where*

$$i(x) = \big(i'(x), i''(x)\big) \in S' \oplus S''^0 \quad \textit{for} \quad x \in S$$

and

$$i_1(x) = \big(i_1'(x), i_1''(x)\big) \in S' \oplus S'' \quad \textit{for} \quad x \in S_1.$$

Proof. Let $f\colon S \to S_1$ be as in the theorem. We denote by A the set of all functionals $\varphi\colon S_1 \to R$ such that $\varphi \circ f$ is a functional on S and by B the set of all functionals $\psi\colon S_1 \to R$ such that $\psi \circ f$ is a dual functional on S. Then Corollary 1 implies that $A \cup B = S_1$. First we shall consider the case where A and B are non-empty.

Let the mappings i_1' and i_1'' map S_1 onto the functional d-lattices S' and S'', defined on A and B respectively, given by

$$\big(i_1'(x)\big)(a) = a(x), \qquad \big(i_1''(x)\big)(b) = b(x)$$

for $x \in S_1$, $a \in A$, $b \in B$. Then

$$i_1'\colon S_1 \to i_1'(S_1) = S' \quad \text{and} \quad i_1''\colon S_1 \to i_1''(S_1) = S''$$

are homomorphism of S_1 onto some d-lattices S' and S''. Thus, by the representation theorem, a mapping $i_1\colon S_1 \to S' \oplus S''$ given by $i_1(x) = \big(i_1'(x), i_1''(x)\big)$ for $x \in S_1$ is an isomorphic imbedding.

We put $i' = i_1' \circ f\colon S \to S'$, $i'' = i_1'' \circ f\colon S \to S''$. The first map is a homomorphism of S onto S', the second one is a dual homomorphism of S onto S''. Then the mapping $i\colon S \to S' \oplus S''^0$, where $i(x) = \big(i'(x), i''(x)\big)$ for $x \in S$, is an isomorphic imbedding. It only remains to show that $i\colon S \to S' \oplus S''^0$ maps S onto $S' \oplus S''^0$, i.e. that for every $x, y \in S$ there exists a $z \in S$ such that

$$i'(x) = i'(z) \quad \text{and} \quad i''(y) = i''(z).$$

Let $x, y \in S$. Then for

$$z = \big(x \cup f^{-1}(f(x) - d(x, y))\big) \cap \big(y \cup f^{-1}(f(y) + d(x, y))\big)$$

we have

$$\begin{aligned} i'(z) &= \big(i'(x) \cup i' \circ f^{-1}(f(x) - d(x, y))\big) \cap \big(i'(y) \cup i' \circ f^{-1}(f(y) + d(x, y))\big) \\ &= \big(i'(x) \cup (i_1' \circ f(x) - d(x, y))\big) \cap \big(i'(y) \cup (i_1' \circ f(y) + d(x, y))\big) \\ &= \big(i'(x) \cup (i'(x) - d(x, y))\big) \cap \big(i'(y) \cup (i'(y) + d(x, y))\big) \\ &= i'(x) \cap \big(i'(y) + d(x, y)\big) = i'(x), \end{aligned}$$

thus $i'(z) = i'(x)$, and analogously

$$\begin{aligned} i''(z) &= \big(i''(x) \frown i''\circ f^{-1}(f(x)-d(x,y))\big) \smile \big(i''(y) \frown i''\circ f^{-1}(f(y)+d(x,y))\big) \\ &= \big(i''(x) \frown (i_1''\circ f(x)-d(x,y))\big) \smile \big(i''(y) \frown (i_1''\circ f(y)+d(x,y))\big) \\ &= \big(i''(x) \frown (i''(x)-d(x,y))\big) \smile \big(i''(y) \frown (i''(y)+d(x,y))\big) \\ &= \big(i''(x)-d(x,y)\big) \smile i''(y) = i''(y); \end{aligned}$$

thus $i''(z) = i''(y)$.

(In the last equalities of the above two sequences of equalities we used the fact that the homomorphisms i' and i'' are metric.)

If the set B (respectively A) is empty, then the homomorphisms i_1' and i' (resp. a homomorphism i_1'' and a dual homomorphism i'') are isomorphisms (resp. an isomorphism and a dual isomorphism), thus the isometry $f = (i_1')^{-1}i'$ (resp. $f = (i_1'')^{-1}i''$) is an isomorphism (resp. a dual isomorphism). The theorem is proved.

From the above theorem and Theorem 4.17 we obtain, as a corollary, the following generalization of Theorem 4.17:

COROLLARY 2. *If S and S_1 are functional metric* d-*lattices defined on the sets X and X_1 respectively and $f\colon S \to S_1$ is an isometric mapping of S onto S_1, then there exists an affine mapping $F\colon L \to L_1$ such that $F|S = f$ (where, as in Theorem* 4.17, *L and L_1 are the linear spaces of all real functions on X and X_1 respectively*).

Now we can give a full generalization of the Banach-Stone theorem:

THEOREM 9.4. *If $S \subseteq C(X)$ and $S_1 \subseteq C(X_1)$ are isometric perfect* d-*lattices on the compact spaces X and X_1, then X and X_1 are homeomorphic.*

Proof. If the d-lattices S and S_1 are isomorphic (resp. dual isomorphic), then by Theorem 6.9 there exists a homeomorphism $\varphi\colon X_1 \to X$ such that $Ff = f\varphi + b$ (resp. $Ff = -f\varphi + b$) for a certain $b \in C(X_1)$ and for every $f \in S$.

Now if $F\colon S \to S_1$ is an isometry mapping, but not an isomorphism or a dual isomorphism, then by Theorem 9.3 there exist d-lattices S' and S'' such that S is isomorphic to $S' \oplus S''^{0}$ and S_1 to $S' \oplus S''$. Then there exist pairs of closed-open sets $A, B \subseteq X$ and $A_1, B_1 \subseteq X_1$ such that $B = X \setminus A$, $B_1 = X \setminus A_1$, $S|A$ is isomorphic to $S_1|A_1$ and $S|B$ is dually isomorphic to $S_1|B_1$. Then, by Theorem 6.9, X and X_1 are homeomorphic spaces.

Moreover, there exists a canonically associated homeomorphism $\varphi\colon X_1 \to X$ to F such that

$$Ff(x) = \begin{cases} f\varphi(x) + b(x) & \text{if} \quad x \in A_1, \\ -f\varphi(x) + b(x) & \text{if} \quad x \in B_1 \end{cases} \tag{4}$$

for a certain $b \in C(X_1)$ and any $f \in S$. Formula (4) also holds if F is a homomorphism or a dual homomorphism then respectively B_1 or A_1 is the empty set.

THEOREM 9.5. *Let $F\colon S \to S_1$ be an isometric imbedding of a perfect functional* d-*lattice $S \subseteq C(X)$ into a functional* d-*lattice $S_1 \subseteq C(X_1)$, where X and X_1 are compact spaces, and let $F(S)$ be a* d-*sublattice of S. Then there exists a continuous function $\varphi\colon X_1 \to X$ of X_1 onto X such that formula* (4) *holds for a certain closed-open subset A_1 of X_1, $B_1 = X_1 \setminus A_1$, $b \in C(X_1)$ and for any $f \in C(X)$.*

Proof. Let $E\colon F(S) \to S_0$ be an isomorphism of $F(S)$ onto a perfect d-lattice $S_0 \subseteq C(X_0)$ and let $F_0\colon S_0 \to S_1$ be an isomorphic imbedding such that $F_0 \circ E(f) = f$ for any $f \in F(S)$. Then $E \circ F\colon S \to S_0$ is an isometry of S onto S_0. Thus, by Theorem 9.4, there exists a homeomorphism $\varphi_0\colon X_0 \to X$ of X_0 onto X such that formula (4) holds for $E \circ F$ instead of F, and for a closed-open subset A_0 of X_0, $B_0 = X_0 \setminus A_0$, $b_0 \in C(X_0)$ and for any $f \in S$. Next, it follows from Theorem 6.9 that there exists a continuous mapping $\varphi_1\colon X_1 \to X_0$ of X_1 onto X_0 and $b_1 \in C(X_1)$ such that $F_0 f = f \circ \varphi_1 + b_1$. Hence formula (4) holds for

$$F = F_0 \circ E \circ F, \quad \varphi = \varphi_0 \circ \varphi_1, \quad A_1 = \varphi_1^{-1}(A_0),$$
$$B_1 = \varphi_1^{-1}(B_0) = X_1 \setminus A_1, \quad b = b_0 \circ \varphi + b_1 \in C(X_1)$$

and for any $f \in S$. The theorem is proved.

THEOREM 9.6. *If a mapping $\Phi\colon S \to S_1$ is an isometric imbedding of a metric distributive* d-*lattice S into a functional* d-*lattice $S_1 \subseteq C(X_1)$, where X_1 is a compact space, then there exist closed subsets A' and A'' of the space X_1 such that the mappings*

$$\Psi'\colon S \to S_1|A', \quad \Psi''\colon S \to S_1|A'' \quad \text{and} \quad \Psi\colon S \to S_1|A,$$

where $A = A' \cup A''$, given by

$$\Psi'(f) = (\Phi f)|A', \quad \Psi''(f) = (\Phi f)|A'', \quad \Psi(f) = (\Phi f)|A,$$

are a homomorphism, a dual homomorphism and an isometric imbedding respectively.

Obviously, the sets A' and A'' are disjoint.

Proof. It is easy to see that if

$$d(f', g') = d(f', f) + d(f, g) + d(g, g') \tag{5}$$

and

$$(\Phi f')(x) - (\Phi g')(x) = d(f', g'),$$

then

$$(\Phi f)(x) - (\Phi g)(x) = d(f, g),$$

where

$$f, f', g, g' \in S \quad \text{and} \quad x \in X_1.$$

Let $f_1 f_2, \ldots, f_n, g, h \in S$ and $a_i \geqslant 0$ for $i = 1, \ldots, n$. There exists a functional $u\colon S \to R$ such that

$$|u(g) - u(h)| = d(g, h).$$

We can assume that $u(g) \leqslant u(h)$. Let

$$\beta = \min\big(u(f_1), u(f_2), \ldots, u(f_n), u(g)\big),$$

$$\gamma = \max\big(u(f_1) + a_1, u(f_2) + a_2, \ldots, u(f_n) + a_n, u(h)\big),$$

$$\delta = \operatorname{diam}\{f_1, f_2, \ldots, f_n, f_1 + a_1, f_2 + a_2, \ldots, f_n + a_n, g, h\}.$$

Then for

$$\check{f} = \bigcup_{i=1}^{n} \big(f_i - u(f_i)\big) \smile \big(g - u(g)\big) + \beta - \delta$$

and

$$\hat{f} = \bigcap_{i=1}^{n} \big(f_i - u(f_i)\big) \frown \big(g - u(g)\big) + \gamma + \delta$$

we have

$$u(\check{f}) = \beta - \delta, \qquad u(\hat{f}) = \gamma + \delta$$

and

$$d(\check{f}, \hat{f}) \geqslant u(\hat{f}) - u(\check{f}) = 2\delta + \gamma - \beta \geqslant d(\check{f}, \hat{f}),$$

whence

$$d(\check{f}, \hat{f}) = 2\delta + \gamma - \beta$$

and

$$d(\check{f}, \hat{f}) = d(\check{f}, f_i) + d(f_i, f_i + a_i) + d(f_i + a_i, \hat{f})$$
$$= d(\check{f}, g) + d(g, h) + d(h, \hat{f}), \qquad \text{for } i = 1, 2, \ldots, n,$$

since

$$d(\check{f}, f_i) = u(f_i) - u(\check{f}),$$

$$d(f_i + a_i, \hat{f}) = u(\hat{f}) - u(f_i + a_i),$$

$$d(\check{f}, g) = u(g) - u(\check{f}),$$

$$d(h, \hat{f}) = u(\hat{f}) - u(h)$$

and, by the assumption,

$$d(g, h) = u(h) - u(g)$$

and also

$$d(f_i, f_i + a_i) = u(f_i + a_i) - u(f_i)$$

for $i = 1, 2, \ldots, n$.

Of course, there exists a point $x \in X_1$ such that

$$|(\Phi\check{f})(x) - (\Phi\hat{f})(x)| = d(\check{f}, \hat{f}).$$

Hence, by (5), for such an x we have

$$(6) \qquad \begin{aligned} &|(\Phi g)(x) - (\Phi h)(x)| = d(g, h), \\ &|(\Phi f_i)(x) - \big(\Phi(f_i + a_i)\big)(x)| = a_i \end{aligned}$$

for $i = 1, 2, \ldots, n$.

The set $A\big((f_i)_{1 \leqslant i \leqslant n}, (a_i)_{1 \leqslant i \leqslant n}, g, h\big)$, of all $x \in X_1$ such that (6) holds is a non-empty closed subset of the compact space X_1. The intersection of two such sets

$$A\big((f_i)_{1 \leqslant i \leqslant n}, (a_i)_{1 \leqslant i \leqslant n}, g, h\big) \cap A\big((f_i)_{n+1 \leqslant i \leqslant n+k}, (a_i)_{n+1 \leqslant i \leqslant n+k}, g, h\big) = A\big((f_i)_{1 \leqslant i \leqslant n+k}, (a_i)_{1 \leqslant i \leqslant n+k}, g, h\big)$$

is also a set of this form, and the intersection of the family of all such sets is non-empty. Thus

(7) there exists a point $x \in X_1$ such that

$$|(\Phi g)(x) - (\Phi h)(x)| = d(g, h)$$

and

$$|(\Phi f)(x) - \big(\Phi(f + a)\big)(x)| = |a|$$

for any $f \in S$ and $a \in R$.

Let A_1 be the set of all $x \in X_1$ such that

$$|(\Phi f)(x) - \big(\Phi(f + a)\big)(x)| = |a| \quad \text{for any } f \in S \text{ and } a \in R.$$

Then, by (7), the mapping Ψ_1: $S \to S_1|A_1$, given by $\Psi_1(f) = (\Phi f)|A_1$, is an isometric imbedding.

If

$$\big(\Phi(f + a_0)\big)(x) = (\Phi f)(x) + \varepsilon a_0,$$

where $f \in S$, $0 \neq a_0 \in R$, $x \in A_1$ and $\varepsilon = \pm 1$, then evidently

$$\big(\Phi(f + a)\big)(x) = (\Phi f)(x) + \varepsilon a$$

for every $a \in R$.

Furthermore, we have

$$\big(\Phi(g + a)\big)(x) = (\Phi g)(x) + \varepsilon a$$

for any $g \in S$ and $a \in R$.

Indeed, if

$$\big(\Phi(g + \beta_0)\big)(x) = (\Phi g)(x) - \varepsilon \beta_0$$

for some $\beta_0 \neq 0$, then

$$\big(\Phi(g+\beta)\big)(x) = (\Phi g)(x) - \varepsilon\beta$$

for every $\beta \in R$. Hence

$$\begin{aligned} d(f, g) &= d\Big(\Phi\big(f+2d(f, g)\big), \Phi\big(g+2d(f, g)\big)\Big) \\ &\geqslant |[(\Phi f)(x)+\varepsilon\cdot 2d(f, g)] - [(\Phi g)(x)-\varepsilon\cdot 2d(f, g)]| \\ &\geqslant 4d(f, g) - |(\Phi f)(x) - (\Phi g)(x)| \geqslant 3d(f, g) \end{aligned}$$

in contradiction to $f \neq g$.

We put

$$A_1' = \{x \in A_1 \colon \bigvee_{f \in S} \bigvee_{a \in R} \big(\Phi(f+a)\big)(x) = (\Phi f)(x) + a\}$$

and

$$A_1'' = \{x \in A_1 \colon \bigvee_{f \in S} \bigvee_{a \in R} \big(\Phi(f+a)\big)(x) = (\Phi f)(x) - a\}.$$

Then

$$A_1' \cup A_1'' = A_1 \quad \text{and} \quad A_1' \cap A_1'' = \emptyset.$$

We shall show that for the mappings

$$\Psi_1' \colon S \to S_1|A_1' \quad \text{and} \quad \Psi_1'' \colon S \to S_1|A_1''$$

given by

$$\Psi_1' f = (\Psi_1 f)|A_1' \quad \text{and} \quad \Psi_1'' f = (\Psi_1 f)|A_1''$$

we have

$$(8) \qquad f \subseteq g \quad \text{iff} \quad \Psi_1' f \subseteq \Psi_1' g \quad \text{and} \quad \Psi_1'' f \supseteq \Psi_1'' g$$

for any $f, g \in S$.

Indeed,

$$\Psi_1'(f+a) = \Psi_1' f + a$$

and

$$\Psi_1''(f+a) = \Psi_1'' f - a$$

for any $f \in S$, $a \in R$, and the mappings Ψ_1' and Ψ_1'' are metric. Hence from $f \subseteq g$ follows

$$\begin{aligned} d(f, g) &\geqslant d\big(f+d(f, g), g\big) \\ &\geqslant d\Big(\Psi_1'\big(f+d(f, g)\big), \Psi_1' g\Big) \\ &= d\big(\Psi_1' f + d(f, g), \Psi_1' g\big), \end{aligned}$$

whence $\Psi_1' f \subseteq \Psi_1' g$.

Analogously, since

$$d(f, g) \geqslant d\big(\Psi_1'' f - d(f, g), \Psi_1'' g\big),$$

we have $\Psi_1'' f \supseteq \Psi_1'' g$.

Now let us assume

$$\Psi_1' f \subseteq \Psi_1' g \quad \text{and} \quad \Psi_1'' f \supseteq \Psi_1'' g.$$

Then

$$\begin{aligned} d(f, g) &\geqslant \max\left[d\left(\Psi_1' f + d(f, g), \Psi_1' g\right), d\left(\Psi_1'' f - d(f, g), \Psi_1'' g\right)\right] \\ &= \max\left[d\left(\Psi_1'(f + d(f, g)), \Psi_1' g\right), d\left(\Psi_1''(f + d(f, g)), \Psi_1'' g\right)\right] \\ &= d\left(f + d(f, g), g\right), \end{aligned}$$

whence $f \subseteq g$.

(9) If fQg (i.e. $u(f) = u(g)$ for a certain functional $u\colon S \to R$) and $f \subseteq g$, then $\Psi_1 f Q \Psi_1 g$ (i.e. $(\Psi_1 f)(x) = (\Psi_1 g)(x)$ for a certain $x \in A_1$), for any $f, g \in S$.

Indeed, if $f \subseteq g$, then

$$\Psi_1' f \subseteq \Psi_1' g \quad \text{and} \quad \Psi_1'' f \supseteq \Psi_1'' g,$$

and from fQg follows

$$\Psi_1' f + \varepsilon \nsubseteq \Psi_1' g \quad \text{or} \quad \Psi_1'' f - \varepsilon \nsupseteq \Psi_1'' g$$

for arbitrary $\varepsilon > 0$ and consequently it is easy to prove

$$\Psi_1' f + \varepsilon \nsubseteq \Psi_1' g \quad \text{for any } \varepsilon > 0$$

or

$$\Psi_1'' f - \varepsilon \nsupseteq \Psi_1'' g \quad \text{for any } \varepsilon > 0.$$

Hence $\Psi_1 f Q \Psi_1 g$.

Now let $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n, g, h \in S$. Then $|u(g) - u(h)| = d(g, h)$ for a certain functional $u\colon S \to R$. We can assume that $u(g) \leqslant u(h)$. Let

$$\hat{f} = \bigcap_{i=1}^{n} \left(e_i - u(e_i)\right) \cap \bigcap_{i=1}^{n} \left(f_i - u(f_i)\right) \cap \left(g - u(g)\right) \cap \left(h - u(h)\right)$$

and

$$\check{f} = \bigcup_{i=1}^{n} \left(e_i - u(e_i)\right) \cup \bigcup_{i=1}^{n} \left(f_i - u(f_i)\right) \cup \left(g - u(g)\right) \cup \left(h - u(h)\right).$$

Then $u(\hat{f}) = u(\check{f}) = 0$ and $\hat{f} \subseteq \check{f}$ and for $a \geqslant d(\hat{f}, \check{f})$ such that $\hat{f} + a \supseteq h$ and $\check{f} - a \subseteq g$ we have

$$2a = d(\hat{f} + a, \check{f} - a) = d(f + a, h) + d(h, g) + d(g, f - a).$$

From (9) follows

$$(\Psi_1 \hat{f})(x) = (\Psi_1 \check{f})(x) \tag{10}$$

for a certain $x \in A_1$. We shall consider the case of $x \in A_1'$. Then

$$\left(\Psi_1(\hat{f} + a)\right)(x) - \left(\Psi_1(\check{f} - a)\right)(x) = 2a = d(\hat{f} + a, \check{f} - a)$$

and by (5)

$$(11) \qquad |(\Psi_1 g)(x) - (\Psi_1 h)(x)| = d(g, h).$$

We have also

$$\big(\Psi_1'(e_i \frown f_i)\big)(x) = \min\big((\Psi_1' e_i)(x), (\Psi_1' f_i)(x)\big)$$

and

$$\big(\Psi_1'(e_i \smile f_i)\big)(x) = \max\big((\Psi_1' e_i)(x), (\Psi_1' f_i)(x)\big)$$

for $i = 1, 2, \ldots, n$.

Indeed,

$$\hat{f} \subseteq (e_i - \beta_i) \frown (f_i - \beta_i) \begin{array}{l} \nearrow \subseteq e_i - \beta_i \\ \searrow \subseteq f_i - \beta_i \end{array}$$

and $e_i - \beta_i \subseteq \check{f}$ or $f_i - \beta_i \subseteq \check{f}$ for $\beta_i = \min\big(u(e_i), u(f_i)\big)$, hence from (8) and (10)

$$\begin{aligned}(\Psi_1'\check{f})(x) = (\Psi_1'\hat{f})(x) &\leqslant \big(\Psi_1'(e_i \frown f_i - \beta_i)\big)(x) \\ &\leqslant \min\Big(\big(\Psi_1'(e_i - \beta_i)\big)(x), \big(\Psi_1'(f_i - \beta_i)\big)(x)\Big) \\ &\leqslant (\Psi_1'\check{f})(x),\end{aligned}$$

i.e.

$$(12') \qquad \big(\Psi_1'(e_i \frown f_i)\big)(x) = \min\big((\Psi_1' e_i)(x), (\Psi_1' f_i)(x)\big)$$

and analogously we can prove

$$(13') \qquad \big(\Psi_1'(e_i \smile f_i)\big)(x) = \max\big((\Psi_1' e_i)(x), (\Psi_1' f_i)(x)\big)$$

for $i = 1, 2, \ldots, n$.

Similarly, if a point x such that (10) holds belongs to the set A_1'', then also

$$|(\Psi_1 g)(x) - (\Psi_1 h)(x)| = d(g, h)$$

and

$$(12'') \qquad \big(\Psi_1''(e_i \frown f_i)\big)(x) = \max\big((\Psi_1'' e_i)(x), (\Psi_1'' f_i)(x)\big),$$

$$(13'') \qquad \big(\Psi_1''(e_i \smile f_i)\big)(x) = \min\big((\Psi_1'' e_i)(x), (\Psi_1'' f_i)(x)\big).$$

Let

$$(14') \qquad A'\big((e_i)_{1 \leqslant i \leqslant n}, (f_i)_{1 \leqslant i \leqslant n}, g, h\big)$$

be the set of all $x \in A_1'$ such that the conditions (11), (12′) and (13′) hold, and let

$$(14'') \qquad A''\big((e_i)_{1 \leqslant i \leqslant n}, (f_i)_{1 \leqslant i \leqslant n}, g, h\big)$$

be the set of all $x \in A_1''$ such that conditions (11), (12″) and (13″) hold.

These sets are closed disjoint subsets of the compact space X_1 and their union is a non-empty set. Further, the intersection of two sets of form (14′) or (14″), where only g and h are fixed, is also a set of the form (14′) or (14″) respectively. Hence the intersection of all the sets of form (14′) or the intersection of all the sets of form (14″) is a non-empty set. Thus it is easy to see that the theorem holds for the sets

$$A' = \{x \in A_1' : x^{*} \circ \Psi_1' \text{ is a functional defined on } S\},$$

$$A'' = \{x \in A_1'' : x^{*} \circ \Psi_1'' \text{ is a dual functional defined on } S\}$$

and functions Ψ' and Ψ'' given by

$$\Psi' f = (\Psi_1' f)|A' = (\Psi f)|A'$$

and

$$\Psi'' f = (\Psi_1'' f)|A'' = (\Psi f)|A''.$$

The theorem is proved.

COROLLARY 3. *Let F: $C(X) \to C(Y)$ be an isometric imbedding, and X, Y — compact spaces such that $F(0) = 0$ (but F is not necessary linear). Then there exists a closed subset Y_0 in Y such that for the restriction mapping F_1: $C(Y) \to C(Y_0)$ given by $F_1 f = f|Y_0$ ($f \in C(Y)$) the composition $F_1 \circ F$ is a linear isometric imbedding.*

From Theorem 9.5 and Theorem 9.6 we obtain

THEOREM 9.7. *Let F: $S \to S_1$ be an isometric imbedding of a perfect* d-*lattice $S \subseteq C(X)$ into a functional* d-*lattice $S_1 \subseteq C(X_1)$, where X and X_1 are compact spaces. Then there exist closed disjoint subsets A', A'' of the space X_1 such that there exists a continuous function φ: $A \to X$ of $A = A' \cup A''$ onto X, such that formula* (4) *holds for a certain $b \in C(A)$ and any $f \in C(X)$.*

Remark 1. Theorem 9.6 can be formulated in the following manner:

If Φ: $S \to S_1$ is an isometric imbedding of a metric distributive d-lattice S in a metric distributive d-lattice S_1, then

(i) there exist a d-lattice S' and a homomorphism Φ': $S_1 \to S'$ of S_1 onto S' such that $\Psi = \Phi' \circ \Phi$: $S \to S'$ is a homomorphism or a dual homomorphism,

or

(ii) there exist d-lattices S', S'' and a homomorphism Φ': $S_1 \to S'$ and a dual homomorphism Φ'': $S_1 \to S''$ of S_1 onto S' and S'' respectively, such that the mapping Ψ: $S \to S' \oplus S''$ given by $\Psi(f) = \big(\Phi' \circ \Phi(f), \Phi'' \circ \Phi(f)\big)$ is an isometric imbedding.

This theorem, in general, is not true for the non-distributive d-lattices. If S is a non-distributive d-lattice, S_1 is a distributive d-lattice, $S \subseteq S_1$ and Φ: $S \to S_1$, given by $\Phi(f) = f$ for $f \in S$, is an isometric imbedding, then the theorem does not hold (for example: $S = S^3_{2,1}$ and $S_1 = R^3$, see § 1, Example 8 or 10 and Example 4).

Remark 2. Let us consider the category C of compact spaces, where $\mathrm{Hom}_C(X, Y)$ is defined as the set of all continuous mappings from any closed subsets of space X onto space Y, and the category D of metric distributive d-lattices, where $\mathrm{Hom}_D(S, S_1)$ is defined as the set of all isometric imbeddings of S into S_1. It easily follows from Theorem 9.7 that there exists a contravariant functor of category D into category C.

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