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On generalized recurrent Kaehlerian manifolds of second order II *

by S. S. SINGH (Gorakhpur, India)**

Abstract. This paper is a continuation of paper [2], where properties of generalized Kaehler manifolds admitting some recurrent tensors were discussed.

1. Introduction. We shall consider a Kaehlerian manifold satisfying

$$R_{hijk,lm} = R_{hijk,l}\beta_m + \gamma_{lm} R_{hijk}; \tag{1.1}$$

$$R_{hk,lm} = R_{hk,l}\beta_m + \gamma_{lm} R_{hk}; \tag{1.2}$$

$$P_{hijk,lm} = P_{hijk,l}\beta_m + \gamma_{lm} P_{hijk}; \quad \text{and} \tag{1.3}$$

$$B_{hijk,lm} = B_{hijk,l}\beta_m + \gamma_{lm} B_{hijk} \tag{1.4}$$

for non-zero β_m and γ_{lm}, where the comma followed by an index denotes covariant differentiation with respect to the metric tensor g_{ij}. We call G2 *recurrent*, G2 *Ricci-recurrent*, G2 *H-projective recurrent*, and G2 *Bochner-recurrent Kaehlerian manifolds* satisfying (1.1), (1.2), (1.3) and (1.4), respectively, the H-projective curvature tensor and the Bochner curvature tensor being given by

$$P_{hijk} = R_{hijk} - \frac{1}{n+2}(R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk}), \tag{1.5}$$

where $H_{ij} = R_{ia}J^a{}_j$, and

$$\begin{aligned} B_{hijk} = R_{hijk} - \frac{1}{n+4}(R_{ij}g_{hk} - R_{hj}g_{ik} + H_{ij}J_{hk} - H_{hj}J_{ik} - 2H_{hi}J_{jk} + \\ + R_{hk}g_{ij} - R_{ik}g_{hj} + H_{hk}J_{ij} - H_{ik}J_{hj} - 2H_{jk}J_{hi}) + \\ + \frac{R}{(n+2)(n+4)}(g_{ij}g_{hk} - g_{hj}g_{ik} + J_{ij}J_{hk} - J_{hj}J_{ik} - 2J_{hi}J_{jk}). \end{aligned} \tag{1.6}$$

respectively.

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** Present address: Faculty of Mechanics and Mathematics, Moscow State University, Moscow, V-234, U.S.S.R.

2. Generalized recurrent Kaehler manifolds of second order.

DEFINITION 2.1. A Kaehler manifold is called *generalized recurrent of second order* (or briefly, G2 *recurrent*) if

$$R_{hijk,lm} = \gamma_{lm} R_{hijk} + R_{hijk,l}\beta_m \tag{2.1}$$

and it is called *generalized Ricci-recurrent Kaehler manifold of second order* (or briefly, G2 *Ricci-recurrent Kaehler manifold*) if

$$R_{ij,lm} = R_{ij,l}\beta_m + \gamma_{lm} R_{ij}, \tag{2.2}$$

or equivalently

$$H_{ij,lm} = H_{ij,l}\beta_m + \gamma_{lm} H_{ij}, \tag{2.3}$$

where β_m and γ_{lm} are not both zero.

Multiplying (2.2) by g^{ij}, we get

$$R_{,lm} = R_{,l}\beta_m + \gamma_{lm} R. \tag{2.4}$$

Remark 2.1. From (2.1) and (2.2) it follows that every G2 recurrent Kaehler manifold is G2 Ricci-recurrent but the converse is not necessarily true.

DEFINITION 2.2. A Kaehler manifold is called *generalized H-projective recurrent of second order* (or briefly, G2 *H-projective recurrent*) if its holomorphically projective curvature tensor satisfies the relation

$$P_{hijk,lm} = P_{hijk,l}\beta_m + \gamma_{lm} P_{hijk}, \tag{2.5}$$

where β_m and γ_{lm} are not both zero.

THEOREM 2.2. *The necessary and sufficient condition for a Kaehler manifold to be generalized* G2 *Ricci-recurrent is that*

$$P_{hijk,lm} - P_{hijk,l}\beta_m - \gamma_{lm} P_{hijk} = R_{hijk,lm} - R_{hijk,l}\beta_m - \gamma_{lm} R_{hijk} \tag{2.6}$$

holds.

Proof. From equation (1.5) we have

$$\begin{aligned} (2.7)\quad & P_{hijk,lm} - P_{hijk,l}\beta_m - \gamma_{lm} P_{hijk} \\ &= R_{hijk,lm} - R_{hijk,l}\beta_m - \gamma_{lm} R_{hijk} - \frac{1}{n+2}\{g_{hk}(R_{ij,lm} - R_{ij,l}\beta_m - \gamma_{lm} R_{ij}) - \\ &- g_{ik}(R_{hj,lm} - R_{hj,l}\beta_m - \gamma_{lm} R_{hj}) + J_{hk}(H_{ij,lm} - H_{ij,l}\beta_m - \gamma_{lm} H_{ij}) - \\ &- J_{ik}(H_{hj,lm} - H_{hj,l}\beta_m - \gamma_{lm} H_{hj}) - 2J_{jk}(H_{hi,lm} - H_{hi,l}\beta_m - \gamma_{lm} H_{hi})\}. \end{aligned}$$

Let us suppose that the manifold is G2 Ricci-recurrent; then in view of (2.2) and (2.3) we have condition (2.6).

Conversely, if (2.6) holds, we have from (2.7) the relation

(2.8) $$g_{hk}(R_{ij,lm}-R_{ij,l}\beta_m-\gamma_{lm}R_{ij})-g_{ik}(R_{hj,lm}-R_{hj,l}\beta_m-\gamma_{lm}R_{hj})+ +J_{hk}(H_{ij,lm}-H_{ij,l}\beta_m-\gamma_{lm}H_{ij})-J_{ik}(H_{hj,lm}-H_{hj,l}\beta_m-\gamma_{lm}H_{hj})- -2J_{jk}(H_{hi,lm}-H_{hi,l}\beta_m-\gamma_{lm}H_{hl})=0.$$

Now we have the relations

(2.9) $$J_{ik}g^{hk}H_{hj}=-g_{ia}J^a{}_k R_{jb}J^b{}_h g^{hk}=-R_{ij}.$$

and

(2.10) $$J_{jk}g^{hk}H_{hl}=-g_{ja}J^a{}_k g^{hk}R_{lb}J^b{}_h=-R_{ij}.$$

Using these relations in (2.9), we obtain

(2.11) $$R_{ij,lm}-R_{ij,l}\beta_m-\gamma_{lm}R_{ij}=0,$$

which shows that the Kaehlerian manifold is G2 Ricci-recurrent.

Now if the H-projective curvature tensor vanishes, then from equation (2.6) we get

(2.12) $$R_{hijk,lm}-R_{hijk,l}\beta_m-\gamma_{lm}R_{hijk}=0,$$

which shows that the manifold is G2 recurrent.

Following the pattern of proof of Theorem 2.2 we can establish the following

THEOREM. 2.3. *The necessary and sufficient condition for a Kaehler manifold to be generalized* G2 *Ricci-recurrent is that*

(2.13) $$B_{hijk,lm}-B_{hijk,l}\beta_m-\gamma_{lm}B_{hijk}=R_{hijk,lm}-R_{hijk,l}\beta_m-\gamma_{lm}R_{hijk}$$

holds, where B_{hijk} *is given by* (1.6).

In view ot Theorems 2.2 and 2.3 we can state

THEOREM. 2.4. *The necessary and sufficient condition for a Kaehler manifold to be Kaehlerian Ricci-recurrent is that*

(2.14) $$B_{hijk,lm}-B_{hijk,l}\beta_m-\gamma_{lm}B_{hijk}=P_{hljk,lm}-P_{hijk,l}\beta_m-\gamma_{lm}P_{hijk}$$

holds.

In view of Remark 2.1 and equation (2.7) we obtain

THEOREM 2.5. *The necessary and sufficient condition for a* G2 *Ricci-recurent Kaehler manifold to be* G2 *recurrent is that the manifold be* G2 *H-projective recurrent.*

From Remark 2.1 and Theorem 2.2 follows

THEOREM 2.6. *The necessary and sufficient conditions for a Kaehler manifold to be* G2 *recurrent are that the manifold be* G2 *H-projective recurrent and equation* (2.6) *be satisfied.*

In view of Remark 2.1 and Theorem 2.3 follows

THEOREM 2.7. *The necessary and sufficient conditions for a Kaehler manifold to be G2 recurrent are that the manifold be G2 Bochner recurrent and equation* (2.13) *be satisfied.*

In view of Theorems 2.4. and 2.5 we obtain

THEOREM 2.8. *If a Kaehler manifold satisfies any two of the following properties:*

1° *it is* G2 *Bochner recurrent;*

2° *it is* G2 *H-projective recurrent;*

3° *it is* G2 *Ricci-recurrent*

then it must also satisfy the third.

Moreover, we also have

THEOREM 2.9. *If a Kaehler manifold satisfies any two of the following properties:*

1° *it is* G2 *Bochner recurrent;*

2° *it is* G2 *recurrent;*

3° *it is* G2 *Ricci-recurrent;*

then it must also satisfy the third.

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DEPARTAMENT OF MATHEMATICS UNIVERSITY OF GORAKHPUR
GORAKHPUR, INDIA

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