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ON INVERTING THE LAPLACE TRANSFORMS CONNECTED WITH THE ERROR FUNCTION

In many problems of mathematical physics the method of Laplace transforms is widely applied. Let us define this transform as follows:

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt, \tag{1}$$

where $\bar{f}(p)$ is a transform of a function $f(t)$.

The difficulties encountered in solving numerous problems lie above all in finding the inverse transforms.

In the problems concerned with heat conduction equation, the following form of transforms frequently occurs

$$\frac{(\sqrt{p})^r e^{-\varrho\sqrt{p}}}{p^l (p-a)^m}, \tag{2}$$

where l, r are non-negative integers, m are natural integers, and ϱ, a denote parameters, $\varrho > 0$.

Decomposing (2) into common fractions, we can present it in a form of the sum of expressions

$$\frac{(\sqrt{p})^r e^{-\varrho\sqrt{p}}}{p^l} \tag{3}$$

and

$$\frac{(\sqrt{p})^r e^{-\varrho\sqrt{p}}}{(p-a)^m}. \tag{4}$$

If $a \to 0$, and $m = l$, expression (4) tends to (3); it is sufficient therefore to consider only (4) as a more general expression.

We shall denote by $L^{-1}\{\bar{f}(p)\}$ the inverse Laplace transform. Write down the known formula (see [1])

$$L^{-1}\left\{\frac{e^{-\varrho\sqrt{p}}}{p-a}\right\} = \frac{e^{at}}{2}\left[e^{-\varrho\sqrt{a}}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{t}} - \sqrt{at}\right) + e^{\varrho\sqrt{a}}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{t}} + \sqrt{at}\right)\right], \tag{5}$$

where

$$\operatorname{erfc} a = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-\xi^2} d\xi$$

is a complementary error function. Denoting the right side of (5) by $U(t, \varrho, a)$ we obviously have

$$(6) \qquad L^{-1}\left\{\frac{(\sqrt{p})^r e^{-\varrho\sqrt{p}}}{(p-a)^m}\right\} = (-1)^r \frac{1}{(m-1)!} \cdot \frac{\partial^r}{\partial \varrho^r} \cdot \frac{\partial^{m-1}}{\partial a^{m-1}} U(t, \varrho, a).$$

A multiple differentiation of the function $U(t, \varrho, a)$ with respect to ϱ and a leads to lengthy and tedious calculations, the results being so involved that even extensive tables of Laplace transforms (see [1], [2]) do not contain the relations (6) for $m > 1$.

In what follows we shall present certain observation which greatly simplifies the calculations of inverting the transforms of form (4), and consequently of form (2). Owing to this observation the final results can also be presented concisely.

Note the existence of formula (see [1])

$$(7) \quad L^{-1}\left\{\frac{\sqrt{p} e^{-\varrho\sqrt{p}}}{p(p-a)}\right\} = \frac{e^{at}}{2\sqrt{a}}\left[e^{-\varrho\sqrt{a}} \operatorname{erfc}\left(\frac{\varrho}{2\sqrt{t}} - \sqrt{at}\right) - e^{\varrho\sqrt{a}} \operatorname{erfc}\left(\frac{\varrho}{2\sqrt{t}} + \sqrt{at}\right)\right].$$

Denoting the right side of (7) by $V(t, \varrho, a)$, we find that the derivatives of the functions $U(t, \varrho, a)$ and $V(t, \varrho, a)$ with respect to ϱ and a can again be expressed in terms of $U(t, \varrho, a)$ and $V(t, \varrho, a)$. In fact, it is easy to demonstrate that the following relations hold

$$(8) \quad \frac{\partial}{\partial a} U(t, \varrho, a) = tU(t, \varrho, a) - \frac{\varrho}{2} V(t, \varrho, a),$$

$$(9) \quad \frac{\partial}{\partial \varrho} U(t, \varrho, a) = -aV(t, \varrho, a) - \frac{1}{t}\sqrt{\frac{t}{\pi}} e^{-\frac{\varrho^2}{4t}},$$

$$(10) \quad \frac{\partial}{\partial a} V(t, \varrho, a) = -\frac{\varrho}{2a} U(t, \varrho, a) - \left(\frac{1}{2a} - t\right) V(t, \varrho, a) + \frac{1}{a}\sqrt{\frac{t}{\pi}} e^{-\frac{\varrho^2}{4t}},$$

$$(11) \quad \frac{\partial}{\partial \varrho} V(t, \varrho, a) = -U(t, \varrho, a).$$

Equations (8)-(11) enable us to invert the transforms of form (4) by means of consecutive substitution and elementary differentiation.

There is no necessity to determine the higher derivatives of $U(t, \varrho, a)$ and $V(t, \varrho, a)$.

Example. Let us invert the transform

$$\frac{p^2 e^{-\varrho\sqrt{p}}}{(p-a)^3}.$$

According to (6), we have

$$L^{-1}\left\{\frac{p^2 e^{-\varrho\sqrt{p}}}{(p-a)^3}\right\} = \frac{1}{2!}\cdot\frac{\partial^4}{\partial\varrho^4}\cdot\frac{\partial^2}{\partial a^2}\, U(t, \varrho, a).$$

Using (8) twice and (10) once, we obtain

$$L^{-1}\left\{\frac{p^2 e^{-\varrho\sqrt{p}}}{(p-a)^3}\right\}$$
$$= \frac{1}{2!}\cdot\frac{\partial^4}{\partial\varrho^4}\left[\left(t^2+\frac{\varrho^2}{4a}\right) U(t, \varrho, a) + \left(\frac{\varrho}{4a} - \varrho t\right) V(t, \varrho, a) - \frac{\varrho}{2a}\sqrt{\frac{t}{\pi}}\, e^{-\frac{\varrho^2}{4t}}\right].$$

Now using the formulae (9) and (11) four times, we finally get

$$L^{-1}\left\{\frac{p^2 e^{-\varrho\sqrt{p}}}{(p-a)^3}\right\}$$
$$= \left(\frac{a^2t^2}{2} + 2at + \frac{a\varrho^2}{8} + 1\right) U(t, \varrho, a) - \frac{\varrho}{2}\left(ta^2 + \frac{7}{4}a\right) V(t, \varrho, a) - \frac{\varrho a}{4}\sqrt{\frac{t}{\pi}}\, e^{-\frac{\varrho^2}{4t}}.$$

This result has been obtained without the necessity of computing the higher derivatives of $U(t, \varrho, a)$ and $V(t, \varrho, a)$, except for those which occur in (8)-(11).

To complete the analysis, we mention that in order to invert the transform (3), the following formula can also be applied ([3]):

$$L^{-1}\left\{\frac{e^{-\varrho\sqrt{p}}}{p^{l+1}}\right\} = (4t)^l i^{2l}\operatorname{erfc}\alpha, \tag{12}$$

where $\alpha = \dfrac{\varrho}{2\sqrt{t}}$ and

$$i^{2l}\operatorname{erfc}\alpha = \int\limits_{\alpha}^{\infty}\int\limits_{\alpha_{2l}}^{\infty}\int\limits_{\alpha_{2l-1}}^{\infty}\ldots\int\limits_{\alpha_3}^{\infty}\int\limits_{\alpha_2}^{\infty}\operatorname{erfc}\alpha_1\, d\alpha_1\, d\alpha_2\ldots d\alpha_{2l-2}\, d\alpha_{2l-1}\, d\alpha_{2l}. \tag{13}$$

For $l = 1, 2, 3, 4$ we receive ([4])

$$i^2 \operatorname{erfc} a = \tfrac{1}{4}[(1+2a^2)\operatorname{erfc} a - 2\pi^{-1/2} a \exp(-a^2)],$$

$$i^4 \operatorname{erfc} a = \frac{1}{8}\left[\left(\frac{1}{4}+a^2+\frac{a^4}{3}\right)\operatorname{erfc} a - \left(\frac{5}{6}a+\frac{a^3}{3}\right)\pi^{-1/2}\exp(-a^2)\right],$$

$$i^6 \operatorname{erfc} a = \frac{1}{12}\left[\left(\frac{1}{32}+\frac{3}{16}a^2+\frac{3}{24}a^4+\frac{a^6}{60}\right)\operatorname{erfc} a - \right.$$

$$\left. - \left(\frac{11}{80}a+\frac{7}{60}a^3+\frac{a^5}{60}\right)\pi^{-1/2}\exp(-a^2)\right],$$

$$i^8 \operatorname{erfc} a = \frac{1}{16}\left[\left(\frac{1}{384}+\frac{a^2}{48}+\frac{a^4}{48}+\frac{a^6}{180}+\frac{a^8}{2520}\right)\operatorname{erfc} a - \right.$$

$$\left. - \left(\frac{31}{2240}a+\frac{37}{2016}a^3+\frac{3}{560}a^5+\frac{a^7}{2520}\right)\pi^{-1/2}\exp(-a^2)\right].$$

Table 1 presents some transforms of form (2) for $m = 1, 2, 3$. Generally we write there U instead of $U(t, \varrho, a)$ and V instead of $V(t, \varrho, a)$.

For numerical computations it is necessary to know the values of functions $U(t, \varrho, a)$, $V(t, \varrho, a)$ for various values of the arguments. Owing to wide applications of $U(t, \varrho, a)$, $V(t, \varrho, a)$ they should be tabulated. In Table 2 some values of these functions are presented for $a = 1$, i.e. for the case most frequently occurring in practice. Tables 1 and 2 have been used in [5].

TABLE 1

	$\tilde{f}(p)$	$f(t)$
1	$\dfrac{e^{-\varrho\sqrt{p}}}{p-a}$	U
2	$\dfrac{e^{-\varrho\sqrt{p}}}{p(p-a)}$	$\dfrac{1}{a}\left(U - \operatorname{erfc}\dfrac{\varrho}{2\sqrt{t}}\right)$
3	$\dfrac{\sqrt{p}e^{-\varrho\sqrt{p}}}{p(p-a)}$	V
4	$\dfrac{pe^{-\varrho\sqrt{p}}}{p-a}$	$aU + \dfrac{\varrho}{2t^{2}}\sqrt{\dfrac{t}{\pi}}\,e^{-\frac{\varrho^2}{4t}}$
5	$\dfrac{\sqrt{p}e^{-\varrho\sqrt{p}}}{p-a}$	$aV + \dfrac{1}{t}\sqrt{\dfrac{t}{\pi}}\,e^{-\frac{\varrho^2}{4t}}$

	$\tilde{f}(p)$	$f(t)$
6	$\frac{e^{-\varrho\sqrt{p}}}{p^2(p-a)}$	$\frac{1}{a^2}U-\frac{1}{a}\left(t+\frac{\varrho^2}{2}+\frac{1}{a}\right)\operatorname{erfc}\frac{\varrho}{2\sqrt{t}}+\frac{\varrho}{a}\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
7	$\frac{e^{-\varrho\sqrt{p}}}{(p-a)^2}$	$tU-\frac{\varrho}{2}V$
8	$\frac{\sqrt{p}e^{-\varrho\sqrt{p}}}{(p-a)^2}$	$-\frac{\varrho}{2}U+\left(ta+\frac{1}{2}\right)V+\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
9	$\frac{pe^{-\varrho\sqrt{p}}}{(p-a)^2}$	$(ta+1)U-\frac{a\varrho}{2}V$
10	$\frac{p\sqrt{p}e^{-\varrho\sqrt{p}}}{(p-a)^2}$	$a\left[-\frac{\varrho}{2}U+\left(ta+\frac{3}{2}\right)V+\left(1+\frac{1}{ta}\right)\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}\right]$
11	$\frac{e^{-\varrho\sqrt{p}}}{(p-a)^3}$	$\frac{1}{2}\left(t^2+\frac{\varrho^2}{4a}\right)U+\frac{1}{2}\left(-t\varrho+\frac{\varrho}{4a}\right)V-\frac{\varrho}{4a}\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
12	$\frac{e^{-\varrho\sqrt{p}}}{p(p-a)^3}$	$\frac{1}{a}\left(\frac{t^2}{2}-\frac{t}{a}+\frac{\varrho^2}{8a}+\frac{1}{a^2}\right)U+\frac{1}{a}\left(-\frac{t\varrho}{2}+\frac{5}{8}\cdot\frac{\varrho}{a}\right)V-$ $-\frac{1}{a^3}\operatorname{erfc}\frac{\varrho}{2\sqrt{t}}-\frac{\varrho}{4a^2}\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
13	$\frac{\sqrt{p}e^{-\varrho\sqrt{p}}}{(p-a)^3}$	$-\frac{1}{2}\left(t\varrho+\frac{\varrho}{4a}\right)U+\frac{1}{2}\left(t^2a+t+\frac{\varrho^2}{4}-\frac{1}{4a}\right)V+\frac{1}{4}\left(2t+\frac{1}{a}\right)\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
14	$\frac{\sqrt{p}e^{-\varrho\sqrt{p}}}{p(p-a)^3}$	$\frac{\varrho}{2a^2}\left(-ta+\frac{3}{4}\right)U+\frac{1}{2}\left(t^2-\frac{t}{a}+\frac{\varrho^2}{4a}+\frac{3}{4a^2}\right)V+\frac{1}{2a}\left(t-\frac{3}{2a}\right)\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
15	$\frac{pe^{-\varrho\sqrt{p}}}{(p-a)^3}$	$\left(\frac{t^2a}{2}+t+\frac{\varrho^2}{8}\right)U-\frac{\varrho}{2}\left(ta+\frac{3}{4}\right)V-\frac{\varrho}{4}\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}$
16	$\frac{p^2e^{-\varrho\sqrt{p}}}{(p-a)^3}$	$a\left[\left(\frac{t^2a}{2}+2t+\frac{\varrho^2}{8}+\frac{1}{a}\right)U-\frac{\varrho}{2}\left(ta+\frac{7}{4}\right)V-\frac{\varrho}{4}\sqrt{\frac{t}{\pi}}e^{-\frac{\varrho^2}{4t}}\right]$

TABLE 2

t	ϱ	$[U(t, \varrho, a)]_{a=1}$	$[V(t, \varrho, a)]_{a=1}$
0,25	0,2	0,95103	0,44572
	0,25	0,87627	0,40005
	0,3	0,80504	0,35803
	0,5	0,55590	0,22290
	1,0	0,17240	0,05410
1,0	0,1	2,44597	2,03270
	0,2	2,19841	1,80068
	0,25	2,08322	1,69366
	0,5	1,58340	1,23784
	1,0	0,88548	0,63502
	1,5	0,46825	0,30588
	2,0	0,23092	0,13696
	3,0	0,04356	0,02134
	4,0	0,00555	0,00228
2,0	0,1	6,67962	6,35000
	0,2	6,03715	5,71470
	0,25	5,73904	5,42032
	0,5	4,45108	4,15323
	1,0	2,66141	2,41195
	1,5	1,57322	1,37585
	2,0	0,91505	0,76764
	2,5	0,52102	0,41713
	3,0	0,28888	0,21987
	3,5	0,15522	0,11196

t	ϱ	$[U(t, \varrho, a)]_{a=1}$	$[V(t, \varrho, a)]_{a=1}$
3,0	0,1	18,17031	17,88699
	0,2	16,43698	16,15807
	0,25	15,63308	15,35648
	0,5	12,16361	11,89992
	1,0	7,35323	7,12026
	1,5	4,43233	4,23439
	2,0	2,65997	2,49805
	2,5	1,58633	1,45879
	3,0	0,93779	0,84154
	3,5	0,54827	0,47832
	4,0	0,31612	0,26726
4,0	0,1	49,39977	49,14719
	0,2	44,69585	44,44631
	0,25	42,51442	42,26653
	0,5	33,10237	32,86328
	1,0	20,06025	19,84312
	1,5	12,14699	11,95539
	2,0	7,34593	7,18173
	2,5	4,43385	4,29724
	3,0	2,66866	2,55832
	3,5	1,59997	1,51345
	4,0	0,95414	0,88856
	4,5	0,56538	0,51672

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O OBLICZANIU ODWROTNYCH TRANSFORMAT LAPLACE'A PEWNYCH FUNKCJI ZWIĄZANYCH Z FUNKCJĄ BŁĘDU

STRESZCZENIE

Podano pewne spostrzeżenia ułatwiające odwracanie transformat Laplace'a typu (2). Wyrażenie (2) może być rozłożone na sumę wyrażeń typu (3) i (4), przy czym dla $a \to 0$, (4) dąży do (3). Korzystając z (5), odwracanie transformat typu (4) można przeprowadzać za pomocą wzoru (6), jednak obliczanie występujących w tym wzorze pochodnych funkcji $U(t, \varrho, a)$ jest bardzo pracochłonne. Dobrano więc taką zależność (7), z występującą w niej funkcją $V(t, \varrho, a)$, że pochodne funkcji U i V względem ϱ i a znów wyrażają się przez funkcje U i V (wzory (8)-(11)). Wykorzystanie wzorów (8)-(11) do odwracania transformat typu (4) uwalnia od konieczności obliczania pochodnych funkcji U i pozwala na krótki zapis wyników. Do odwracania transformat typu (3) może być wykorzystany również wzór (12).

Zestawiono tablicę niektórych transformat dla $m = 1, 2, 3$, oraz tablicę niektórych wartości funkcji U i V dla $a = 1$.

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О ВЫЧИСЛЕНИИ ОБРОТНЫХ ПРЕОБРАЗОВАНИЙ ЛАПЛАСА НЕКОТОРЫХ ФУНКЦИЙ, СВЯЗАННЫХ С ФУНКЦИЕЙ ОШИБКИ

РЕЗЮМЕ

Здесь указаны некоторые замечания облегающие обращение преобразований Лапласа типа (2). Выражение (2) можно разложить на сумму выражений типа (3) и (4), при чем при $a \to 0$, (4) стремится к (3). Пользуюсь (5) можно провести обращение преобразований типа (4) с помощью формулы (6), но вычисление производных функции $U(t, \varrho, a)$ выступающих в этой формуле является очень трудоемким. Ввиду этого подобрана так зависимость (7), с выступающей в ней функцией $V(t, \varrho, a)$ чтобы производные функций U и V по ϱ и a выражались опять через функции U и V (см. формулы (8)-(11)). Пользование формулами (8)-(11) для обращения преобразований типа (4) освобождает от необходимости вычисления производных функции U и позволяет записать коротко результат. Для обращения преобразований типа (3) можно также пользоваться формулой (12).

Приводится таблица некоторых преобразований при $m = 1, 2, 3$, а также таблица некоторых величин функций U и V для $a = 1$.
