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ON MATRICES WITH POSITIVE INVERSES

In his English summary of [4] (reprinted in the Mathematical Reviews, v. 23 (1962), No. A 2429) T. Kaczorek claimed that the following conditions are necessary for a real symmetric matrix $A = [a_{ij}]$ to have an inverse with positive elements:

$$a_{tt} \geqslant \sum_{\substack{i=1 \\ i \neq t}}^{n} |a_{ti}|, \qquad t = 1, \ldots, n, \tag{1}$$

$$a_{it} \leqslant 0 \quad \text{for} \quad i \neq t, \tag{2}$$

(3) at least two elements in every row (column) are different from 0.

Clearly these conditions are not necessary. For example, the matrix

$$\begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 6 \end{bmatrix}$$

contravenes both (1) and (2) and yet all the elements of its inverse are positive. The conditions are not sufficient either. For example,

$$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

satisfies all the conditions but it has no inverse.

A study of [4] itself revealed that the above conditions were in fact meant to be sufficient („dostateczne”) and that the condition (3) was apparently intended to be stronger than it appears in the English summary of [4], namely:

(3′) every leading principal submatrix of A, of order greater than 1, has at least two non-zero elements in every row (column).

However, in the example above, the matrix B satisfies conditions (1), (2) and (3′) and yet it is singular. Thus the theorem in [4] is false.

In this paper I give a correct and stronger version of Kaczorek's theorem applicable to matrices which are not necessarily symmetric.

Call a matrix *positive* (*non-negative*) if all its elements are positive (non-negative). An n-square matrix A is said to be *decomposable* if there exists a permutation matrix P such that $PAP^{-1} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ where X and Y are square submatrices and 0 denotes a zero submatrix. Otherwise A is said to be *indecomposable*. An alternative definition, often useful in the actual determination whether a given matrix is decomposable or not, is the following. Let e_j denote the n-vector with 1 as its j'th coordinate and 0 for its remaining coordinates. Then A is decomposable if and only if $Ae_{j_1}, Ae_{j_2}, \ldots, Ae_{j_k}$ belong to the space generated by e_{j_1}, $e_{j_2}, \ldots, e_{j_k}$, for some k subscripts $1 \leqslant j_1 < j_2 \ldots < j_k \leqslant n, k < n$, i.e. if and only if the corresponding linear operator has a k-dimensional invariant coordinate subspace with $k < n$. For symmetric matrices Kaczorek's condition (3′) implies indecomposability. Not all indecomposable symmetric matrices, however, satisfy (3′). An n-square matrix A is *partly decomposable* if there exist permutation matrices P and Q such that $PAQ = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$; otherwise it is *fully indecomposable*. In other words, an n-square matrix is partly decomposable if and only if it contains an $s \times (n-s)$ zero submatrix for some s, $1 \leqslant s \leqslant n-1$.

THEOREM 1. *A necessary condition that a matrix A have a positive inverse is that A be fully indecomposable.*

For if A is a non-singular partly decomposable matrix then $PAQ = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, where P and Q are permutation matrices, and $A^{-1} = Q\begin{pmatrix} X^{-1} & -X^{-1}YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix}P$ contains a zero submatrix and thus is not positive.

THEOREM 2. *An n-square indecomposable matrix $A = [a_{ij}]$ has a positive inverse if $a_{ij} \leqslant 0$, $i, j = 1, \ldots, n$, $i \neq j$, and all its row sums $\sum_{j=1}^{n} a_{ij}$ (or all its column sums $\sum_{i=1}^{n} a_{ij}$) are non-negative and not all equal to 0.*

Proof. Let $m = \max_i [a_{ii}]$ and let $C = [c_{ij}] = mI_n - A$, where I_n denotes the $n \times n$ identity matrix. Then C is non-negative and indecomposable since A is indecomposable and A and C have their non-zero off-diagonal elements in the same positions. Hence, by the Frobenius theorem (see [1], also [3], p. 63) the maximal positive characteristic root r of C (see [1], [2] or [3], p. 53) satisfies $r \leqslant \max_i \left(\sum_{j=1}^{n} c_{ij}\right)$, with equality if and only if all the row sums of C are equal, i.e. if and only if the row sums of A are equal and positive. Therefore

$$r \leqslant \max_i \Big(\sum_{j=1}^{n} c_{ij}\Big) = m - \min_i \Big(\sum_{j=1}^{n} a_{ij}\Big) \leqslant m$$

where at least one of the two inequalities is strict. Thus $r < m$. We now invoke the classical result of Frobenius ([2], also [3], p. 69, (3)) stating that if C is a non-negative indecomposable matrix with maximal characteristic root r then $(\lambda I_n - C)^{-1}$ exists and is positive for $\lambda > r$. Hence $A^{-1} = (mI_n - C)^{-1}$ exists and is positive.

A similar result for matrices with non-negative inverses was obtained by Ostrowski [6].

Theorem 2 can be improved as follows. If $A = [a_{ij}]$ is an n-square matrix let $P_i = \sum_{\substack{t=1\\ t\neq i}}^{n} |a_{it}|$ and $Q_i = \sum_{\substack{t=1\\ t\neq i}}^{n} |a_{ti}|$.

THEOREM 3. *If* $0 \leqslant a \leqslant 1$ *and* $A = [a_{ij}]$ *is an* n*-square indecomposable matrix such that* $a_{ij} \leqslant 0$, $i, j = 1, \ldots, n$, $i \neq j$, *and*

$$a_{ii} \geqslant P_i^a Q_i^{1-a}, \quad i = 1, \ldots, n,$$

with strict inequality for at least one i, then A^{-1} *exists and is positive.*

Proof. Define $m = \max_i [a_{ii}]$ and $C = [c_{ij}] = mI_n - A$, as in the proof of Theorem 2. Then C is non-negative and indecomposable. Moreover, the P_i and the Q_i are the same for C as for A. Therefore, if r is the maximal positive characteristic root of C, we have, by a theorem of Ostrowski ([5], Theorem IV):

$$r \leqslant \max_i (c_{ii} + P_i^a Q_i^{1-a}) = \max_i \big(m - (a_{ii} - P_i^a Q_i^{1-a})\big)$$

$$= m - \min_i (a_{ii} - P_i^a Q_i^{1-a}) \leqslant m.$$

But, by another theorem of Ostrowski ([5], Theorem VI), the matrix A is non-singular and thus $C = mI_n - A$ cannot have a characteristic root equal to m. Hence $r < m$ and, by the Frobenius result quoted in the proof of Theorem 2, $A^{-1} = (mI_n - C)^{-1}$ exists and is positive.

COROLLARY. *If* $0 \leqslant a \leqslant 1$ *and* $A = [a_{ij}]$ *is an* n*-square indecomposable matrix such that* $a_{ij} \leqslant 0$, $i, j = 1, \ldots, n$, $i \neq j$, *and*

$$a_{ii} \geqslant aP_i + (1-a)Q_i, \quad i = 1, \ldots, n,$$

with strict inequality for at least one i, then A^{-1} *exists and is positive.*

Proof. It is easily checked that $x^a - 1 \leqslant a(x-1)$ for any $x \geqslant 0$ and $0 \leqslant a \leqslant 1$. For $x = P_i/Q_i$ the inequality becomes $P_i^a Q_i^{1-a} \leqslant aP_i + (1-a)Q_i$. Hence, using the notation of the proof of the preceding theorem, $r \leqslant \max_i(c_{ii}+aP_i+(1-a)Q_i)$ and therefore $r < m$ as in Theorem 3. The result follows.

THEOREM 4. *If* $0 \leqslant a \leqslant 1$ *and* $A = [a_{ij}]$ *is an n-square indecomposable matrix such that* $a_{ij} \leqslant 0, i, j = 1, \ldots, n, i \neq j, a_{ii} > 0, i = 1, \ldots, n,$ *and*

$$a_{ii}a_{jj} > (P_iP_j)^a(Q_iQ_j)^{1-a}, \quad i, j = 1, \ldots, n, \quad i \neq j,$$

then A^{-1} *exists and is positive.*

Proof. Let r be the maximal positive characteristic root of $C = [c_{ij}] = mI_n - A$ where $m = \max_i[a_{ii}]$ as before. Then by a theorem of Ostrowski ([5], Theorem V):

$$(r-c_{ii})(r-c_{jj}) \leqslant (P_iP_j)^a(Q_iQ_j)^{1-a} \quad \text{for some } i, j \ (i \neq j),$$

i.e.

$$(r-m+a_{ii})(r-m+a_{jj}) - (P_iP_j)^a(Q_iQ_j)^{1-a} \leqslant 0$$

and therefore

$$(r-m)^2 + (r-m)(a_{ii}+a_{jj}) + \left(a_{ii}a_{jj} - (P_iP_j)^a(Q_iQ_j)^{1-a}\right) \leqslant 0$$

for some i, j $(i \neq j)$. Now, $a_{ii}+a_{jj} > 0$ and $a_{ii}a_{jj} - (P_iP_j)^a(Q_iQ_j)^{1-a} > 0$ for all i and j $(i \neq j)$. Therefore $r-m < 0$ and the theorem holds by virtue of the Frobenius result quoted in the proof of Theorem 2.

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O MACIERZACH, KTÓRYCH MACIERZE ODWROTNE MAJĄ ELEMENTY DODATNIE

STRESZCZENIE

Nota koryguje usterki w pracy T. Kaczorka pod identycznym tytułem, ogłoszonej w Zastosowaniach Matematyki 5 (1960), str. 141-148 oraz zawiera wzmocnienie podanych tam twierdzeń.

Х. МИНЦ (Геинсвилль)

О МАТРИЦАХ, У КОТОРЫХ ОБРАТНЫЕ МАТРИЦЫ СОСТОЯТ ИЗ ПОЛОЖИТЕЛЬНЫХ ЭЛЕМЕНТОВ

РЕЗЮМЕ

Заметка исправляет неточности допущенные в работе Т. Качорка, вышедшей под тем самым названием в Zastosowania Matematyki, в номере 5 (1960), стр. 141-148, и приводит усиление заключенных в ней теорем.
