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ON A CERTAIN CLASS OF LINEAR DIFFERENTIAL EQUATIONS

1. Linear differential equations of class E. We shall define a certain class of linear differential equations, which we shall denote by E. A linear equation will be included in class E if its coefficients satisfy identically a certain algebraic-differential relation. We shall give two groups of such conditions, corresponding to two kinds of Riccati equations of the n-th order which can be assigned to linear equations (see [3] and [4]).

As we shall see, equations of class E have a certain common property: the order of each of then may be effectively lowered by one. Thus, in particular, every second order equation of class E can always be solved by reduction to a first order equation. Therefore in the second part of the present paper we shall discuss in detail equations of the second order; that will lead to the solution of a certain number of equations whose solutions have so far been unknown. This will, at the same time, serve as an illustration of the application of equations E.

1.1. Suppose we are given a linear equation of the $(n+1)$-st order

$$L_{n+1}[y] \equiv a_{n+1,0}y + \sum_{i=1}^{n+1} a_{n+1,i}y^{(i)} = b_{n+1}. \tag{1.1}$$

We shall assume that the coefficients $a_{n+1,i}$ $(i = 0, 1, \ldots, n)$ belong to class C^i in the interval (a, b), $a_{n+1,n+1} = 1$, and that b_{n+1} belongs to class $C(a, b)$. As we know (see [3]), equation (1.1) corresponds to the Riccati equation R_n of the n-th order of the first kind:

$$R_n[v_n] \equiv l_{v_n}^n[a_{n+1,n} - v_n] + \sum_{i=1}^{n} (-1)^i l_{v_n}^{n-i}[a_{n+1,n-i}] = 0, \tag{1.2}$$

where the operator l_g^k (k is either zero or a positive integer) is defined by inductive formulas

$$l_g^0[f] = f, \tag{1.3}$$

$$l_g^k[f] = \frac{d}{dx} l_g^{k-1}[f] + g l_g^{k-1}[f],$$

where $g(x)$ is a fixed function, differentiable a suitable number of times in the interval under consideration, and $f(x)$ is an arbitrary suitably differentiable function.

DEFINITION I. We shall say that the *linear differential equation belongs to class* E_1, if there exists a number $\tilde{v}_n$ such that the coefficients of this equation satisfy identically with respect to the variable x one of the relations

$$R_n[\tilde{v}_n a_{n+1,i}] = 0 \qquad (i = 0, 1, \ldots, n). \tag{1.4}$$

We can easily show that the Euler equation satisfies relation (1.4) (for $i = n$); the same is true for equations with constant coefficients. In the case of these two simplest types of equations condition (1.4) reduces to the algebraic (characteristic) equation. However, class E_1 contains many more equations of a more general type.

We know (theorem 4 in [3], p. 18) that the knowledge of one particular solution of equation R_n which corresponds to a given linear equation allows to lower by one the order of the linear equation. Hence we have the following

CONCLUSION 1. *If an equation belongs to class* E_1, *then its order can be effectively lowered by one.*

In fact, if the equation in question belongs to class E_1, i.e. if it satisfies one of the relations (1.4), then one of the particular solutions of equation R_n (1.2) is

$$v_n = \tilde{v}_n a_{n+1,i}, \tag{1.5}$$

and the knowledge of a particular solution v_n suffices for lowering the order of the linear equation.

If the new equation belongs to class E_1, it is possible to lower the order further. Such a property holds for Euler equations.

To the equation of the second order

$$L_2[y] \equiv y'' + a_{21}y' + a_{20}y = b_2 \tag{1.6}$$

corresponds the equation R_1 of the first order, [3]:

$$v_1' = -v_1^2 + a_{21}v_1 + a_{21}' - a_{20}. \tag{1.7}$$

Substituting $v_1 = \tilde{v}_1 a_{21}$ or $v_1 = \tilde{v}_1 a_{20}$ in the last equation, we find the following two conditions (1.4) for the linear equation of the second order:

$$a_{21}^2\tilde{v}_1^2 + (a_{21}' - a_{21}^2)\tilde{v}_1 - a_{21}' + a_{20} \underset{x}{\equiv} 0 \qquad (v_1 = \tilde{v}_1 a_{21}), \tag{1.8}$$

or

$$a_{20}^2\tilde{v}_1^2 + (a_{20}' - a_{21}a_{20})\tilde{v}_1 - a_{21}' + a_{20} \underset{x}{\equiv} 0 \qquad (v_1 = \tilde{v}_1 a_{20}). \tag{1.9}$$

In these formulas, as well as in the sequel, symbols with $\sim$ denote numbers.

1.2. Let us assume that the coefficients $a_{n+1,i}$ of equation (1.1) belong to class $C(a, b)$. Under this assumption, to equation (1.1) corresponds the n-th order Riccati equation of the second kind (see [4])

$$\hat{R}_n[u_n] \equiv a_{n+1,0} + \sum_{i=1}^{n+1} a_{n+1,i} I^{i-1}[-u_n] = 0, \tag{1.10}$$

where I^k is a differential operator of the k-th order defined by operator l_f^k as follows:

$$I^k[f] = l_f^k f.$$

Definition II. *Linear equation* (1.1) *belongs to class* E_2 *if there exists* a number $\tilde{u}_n$ such that the coefficients of this equation satisfy identically with respect to x one of the following relations:

$$\hat{R}_n[\tilde{u}_n a_{n+1,i}] = 0 \qquad (i = 0, 1, \ldots, n). \tag{1.11}$$

The coefficient appearing in the above definition is additionally assumed to belong to class $C^n(a, b)$.

As before, we can easily show, using the corresponding theorems from the theory of $\hat{R}$ equations (see [4]), that equations of class E_2 with coefficients satisfying (1.11) have the property formulated in Conclusion 1.

The sum of classes E_1 and E_2 will be called *class* E, or the *class of elementary decomposable equations.*

For equations of the second order ($n = 1$) from (1.11) we obtain the following two conditions:

$$a_{21}^2 \tilde{u}_1^2 - (a_{21}^2 + a_{21}')\tilde{u}_1 + a_{20} \underset{x}{\equiv} 0 \qquad (u_1 = \tilde{u}_1 a_{21}) \tag{1.12}$$

or

$$a_{20}^2 \tilde{u}_1^2 - (a_{21} a_{20} + a_{20}')\tilde{u}_1 + a_{20} \underset{x}{\equiv} 0 \qquad (u_1 = u_1 a_{20}). \tag{1.13}$$

Criteria (1.4) and (1.11) have a form convenient for applications, since it is easy to determine whether or not a given equation belongs to class E by substituting its coefficients in the corresponding condition.

There are many differential equations which we are not able to solve. One can say that only those equations have been solved whose coefficients are selected in a certain special way. Having the notion of elementary decomposability at our disposal, we can say that class E contains equations whose coefficients are selected according to one of the conditions (1.4) or (1.11).

2. An approximate method of solving differential equations based upon the choice of one coefficient.

Theorem. *To every differential equation (outside class E) one can assign an E equation which differs from it only in one coefficient.*

In fact, in order to obtain from equation (1.1), which does not, by assumption, belong to class E, an equation of class E we must, as follows from (1.11), take for the coefficient $a_{n+1,j}$ one of following functions:

$$(2.1)\quad a_{n+1,j} = -\{I^n[-\tilde{u}_n a_{n+1,i}] + a_{n+1,n} I^{n-1}[-\tilde{u}_n a_{n+1,i}] + \\ + \ldots + a_{n+1,j+1} I^j[-\tilde{u}_n a_{n+1,i}] + a_{n+1,j-1} I^{j-2}[-\tilde{u}_n a_{n+1,i}] + \\ + \ldots + a_{n+1,0}\} : I^{j-1}[-\tilde{u}_n a_{n+1,i}]$$

$$(j = 0, 1, 2, \ldots, i-1, i+1, \ldots, n;\ I^{-1}[f] = 1).$$

One can also change the coefficient $a_{n+1,i}$ appearing in $u_n = \tilde{u}_n a_{n+1,i}$, but such a change requires the solution of a non-linear differential equation.

It follows from (2.1) that the simplest way is to change the last coefficient, $a_{n+1,0}$.

We can formulate the corresponding theorem using the decomposability condition (1.4); however, in that case one has to assume a suitable multiple differentiability of the corresponding coefficients in the interval (a, b).

Let us notice that the choice of one coefficient of a linear equation from the condition of decomposability is not unique. This coefficient depends upon the parameter $\tilde{u}_n$, and possibly upon the constants of integration. The fact that the coefficient which we choose depends upon parameters may be used for approximate solution of certain problems, especially in physics or technology. In fact, in such cases the coefficients of equations usually have a certain physical interpretation, and often are determined experimentally. Using that fact we may sometimes succeed in selecting the values of the parameters in the coefficient chosen from (1.4) or (1.11) in such a way, that the equation obtained provides us with a good description of the phenomenon under investigation.

EXAMPLE. As we know, the equation of a beam with a varying section, under the axially compressing force $\tilde{P}$ and arbitrary transverse load $q(x)$ may be reduced to the system of two equations

$$(2.2)\qquad \begin{aligned} M'' + \frac{\tilde{P}}{B} M &= q, \\ By'' &= M, \end{aligned}$$

where $M(x)$ denotes the bending moment, $B(x) = EJ(x)$ denotes the varying bending stiffness, $q(x)$ is the vertical weight per unit of length, and $y(x)$ is the deflection of the beam.

System (2.2) or, equivalently, the corresponding equation of the fourth order is usually solved in an approximate way, for instance by expanding the coefficients into infinite power series.

Let us apply the method of choice of one coefficient in equation $(2.2)_1$. In the case of this equation ($a_{21} \equiv 0$) we change the coefficient $a_{20} = \tilde{P}/B$.

Applying the above theorem we select the coefficient a_{21} for a_{20} in such a way that equation $(2.2)_1$ will belong to the class E; we shall use the decomposability condition (1.13). In this case, if $a_{21} \equiv 0$, condition (1.13) takes the form

$$a_{20}^2\tilde{u}_1^2 - a_{20}'\tilde{u}_1 + a_{20} = 0 \qquad (u_1 = \tilde{u}_1 a_{20}).$$

This is a Riccati equation with constant coefficients and its general solution is of the form

$$a_{20} = \frac{e^{x/\tilde{u}_1}}{\tilde{u}_1^2(\tilde{c} - e^{x/\tilde{u}_1})}.$$

Hence

$$B = \frac{\tilde{P}\,\tilde{u}_1^2(\tilde{c} - e^{x/\tilde{u}_1})}{e^{x/\tilde{u}_1}}. \tag{2.3}$$

Thus, our problem can be solved exactly by giving the solution in the so-called closed form if the stiffness of the beam can be determined by (2.3), or in an approximate way if this function gives the stiffness in an approximate way. The basic form of expression (2.3) has, for certain constructions, some practical advantages, in particular when we consider the possibility of modulation, owing to the appearance of two parameters $\tilde{u}_1$ and $\tilde{c}$.

Equation (2.2) after substituting (2.3) belongs to class E ($u_1 = \tilde{u}_1 a_{20}$), and its solution can be obtained from formulas (3.26) derived below (see page 70):

$$M = \tilde{A}_1(\tilde{c} - e^{x/\tilde{u}_1}) + \tilde{A}_2\left[\frac{\tilde{c} - e^{x/\tilde{u}_1}}{\tilde{c}^2}\ln\left|\frac{e^{x/\tilde{u}_1}}{\tilde{c} - e^{x/\tilde{u}_1}}\right| + \frac{1}{\tilde{c}}\right] + S_m,$$

where

$$S_m = (\tilde{c} - e^{x/\tilde{u}_1})\int(\tilde{c} - e^{x/\tilde{u}_1})^{-2}\int q(\tilde{c} - e^{x/\tilde{u}_1})\,dx^2.$$

After using $(2.2)_2$ we find the general solution

$$y = \tilde{A}_4 + \tilde{A}_3 x + \frac{1}{\tilde{P}}\left\{\tilde{A}_2 e^{x/\tilde{u}_1} + \tilde{A}_1\left[\frac{e^{x\tilde{u}_1}}{\tilde{c}^2}\ln\left|\frac{e^{x/\tilde{u}_1}}{\tilde{c} - e^{x/\tilde{u}_1}}\right| + \right.\right.$$
$$\left.\left. + \frac{1}{\tilde{c}}\ln\left|\tilde{c} - e^{x/\tilde{u}_1}\right|\right]\right\} + S, \tag{2.4}$$

where S is the so-called weight function, and

$$S = \frac{1}{\tilde{u}_1^2 \tilde{P}} \iint e^{x/\tilde{u}_1} \int (\tilde{c} - e^{x/\tilde{u}_1})^{-2} \int q(\tilde{c} - e^{x/\tilde{u}_1}) dx^4. \tag{2.5}$$

The weight function for our problem has been written only for a continuous weight q. In the case of forces concentrated at one point and continuously distributed moments we can obtain the function S by using the well-known procedure (passage to the limit) for function q.

Having the general solution (2.4) we can solve every boundary problem for a certain class of beams, and the result can be thoroughly discussed before the technical realization.

3. The second order linear differential equation of class *E*. Now we shall systematically investigate the second order equations of class E. This discussion will be particularly useful in view of the fact that, as follows from the general properties of equations E, every second order E equation can be solved.

It follows from § 1 that an equation of the second order belongs to class E if its coefficients a_{20} and a_{21} satisfy one of the relations (1.8), (1.9), (1.11) or (1.13). It is easy to see that by putting $\tilde{u}_1 = 1 - \tilde{v}_1$ we reduce (1.12) to (1.8); thus it suffices to consider only three relations: (1.8), (1.9) and (1.13). They are either algebraic equations, linear differential equations or equations of Riccati with respect to the coefficients a_{20} or a_{21}. In the first two cases we easily express one coefficient by the other, in the last case this is possible only after solving the Riccati equation. The determination of a_{20} by a_{21} (or conversely) leads to the determination of a certain second order equation of class E.

3.1. Let us first consider an equation from class E whose coefficients satisfy relation (1.8). Thus, suppose that we are given an equation (1.6). Under the assumptions formulated earlier, the Riccati equation of the first order (1.7) corresponds to the equation (1.6). The following conclusion holds:

I. *A linear second order equation of class E whose coefficients satisfy relation* (1.8) *has the general solution* $y = \tilde{C}_1 y_1 + \tilde{C}_2 y_2 + Y_2$, *where*

$$y_1 = \exp[(\tilde{v}_1 - 1) A_{21}],$$

$$y_2 = \exp[(\tilde{v}_1 - 1) A_{21}] \int \exp[(1 - 2\tilde{v}_1) A_{21}] dx, \tag{3.1}$$

$$Y_2 = \exp[(\tilde{v}_1 - 1) A_{21}] \int \exp[(1 - 2\tilde{v}_1) A_{21}] \int b_2 \exp[\tilde{v}_1 A_{21}] dx^2.$$

The capital letters in these formulas (and in the further parts of the paper) will denote indefinite integrals.

Proof. In fact, let v_1 be a particular solution of equation (1.7), defined in the interval (a, b). The general solution of a linear equation (given in [3], pp. 20 and 21) after using the relation

$$a_{10} = a_{21} - v_1 \tag{3.2}$$

takes the form

$$\begin{aligned} y_1 &= \exp(V_1 - A_{21}), \\ y_2 &= \exp(V_1 - A_{21}) \int \exp(A_{21} - 2V_1)\,dx, \\ Y_2 &= \exp(V_1 - A_{21}) \int \exp(A_{21} - 2V_1) \int b_2 \exp(V_1)\,dx^2. \end{aligned} \tag{3.3}$$

Thus, the general solution of a second order linear equation satisfying the conditions formulated above is determined by one particular solution v_1 of the corresponding Riccati equation.

If equation (1.6) belongs to class E and its coefficients satisfy (1.8), then the corresponding Riccati equation (1.7) has a particular solution $v_1 = \tilde{v}_1 a_{21}$.

In this particular case functions (3.3) coincide with (3.1) which was to be shown.

3.2. Determining the function a_{20} from formula (1.8) we come to the following conclusion:

II. *The linear equation of the second order*

$$y'' + a_{21} y' + (1 - \tilde{v}_1)(\tilde{v}_1 a_{21}^2 + a_{21}') y = b_2, \tag{3.4}$$

belongs to class E and its general solution is (3.1) *provided that* $\tilde{v}_1 \neq 1$, $\tilde{v}_1 \neq 0$ *and* a_{21} *belongs to class* $C^1(a, b)$.

In fact, we can apply formulas (3.1) since the coefficient a_{20} has been chosen according to (1.8).

Despite its special form, equation (3.4) contains one arbitrary function and comprises many particular cases important for applications. This equation is known; it was given by H. Görtler [1], with no indication, however, of the method of solution.

3.3. Let us treat the coefficient a_{21} in (1.8) as an unknown function. It satisfies the Riccati equation

$$a_{21}' = -\tilde{v}_1 a_{21}^2 - \frac{a_{20}}{\tilde{v}_1 - 1} \qquad (\tilde{v}_1 \neq 1). \tag{3.5}$$

Thus we have the conclusion:

III. *A linear equation of the second order belongs to class E if its coeffi-*

cient a_{20} *is an arbitrary function from* $C(a, b)$, *and* a_{21} *is an arbitrary solution of the Riccati equation* (3.5).

The general solution of such an equation is function (3.1), *where in place of* a_{21} *we put the corresponding solution of equation* (3.5).

Thus, one can construct equations of class E using functions a_{20} for which equation (3.5) is solved, or for which at least one particular solution is known.

It is convenient to formulate the above result in another way as follows:

IV. *The linear equation of the second order*

$$y''+\tilde{a}\frac{f'}{f}y'+a_{20}y = b_2 \tag{3.6}$$

belongs to class E *if* a_{20} *is an arbitrary function from* $C(a, b)$ *and* f *satisfies the equation*

$$f''+\frac{a_{20}}{\tilde{\beta}}f = 0 \quad (\tilde{\beta} \neq 0), \tag{3.7}$$

and if the numbers $\tilde{a}$ *and* $\tilde{\beta}$ *satisfy the relation* $\tilde{a}+\tilde{\beta} = 1$. *The solution of such a family of equations is*

$$\begin{aligned} y_1 &= f^{\tilde{\beta}}, \\ y_2 &= f^{\tilde{\beta}} \int f^{-(\tilde{\beta}+1)}dx, \\ Y_2 &= f^{\tilde{\beta}} \int f^{-(\tilde{\beta}+1)} \int b_2 f dx^2. \end{aligned} \tag{3.8}$$

Proof. As we know [4], the solution of the Riccati equation (3.5) can be determined by a solution of a certain linear equation; namely, if we put $a_{21} = -z/\tilde{v}_1$, then we get from (3.5)

$$z' = z'^2+\frac{\tilde{v}_1}{\tilde{v}_1-1}a_{20}, \tag{3.9}$$

and to equation (3.9) corresponds in turn the linear equation (3.7) where $\tilde{\beta} = (\tilde{v}_1-1)/\tilde{v}_1$. Thus, if the function $f = \tilde{C}_1f_1+\tilde{C}_2f_2$ is the general solution of (3.7), then, [4],

$$a_{21} = \frac{\tilde{a}f'}{f},$$

where $\tilde{a} = 1/\tilde{v}_1$. Hence the corresponding equation of class E satisfying (1.8) takes the form (3.6), provided that f is the solution of the equation E satisfying condition (3.7).

It remains to put

$$A_{21} = \tilde{a}\ln|f|$$

in formulas (3.1). This leads to (3.8) and completes the proof.

The result presented in Conclusion IV does not comprise the case $\tilde{\beta} = 1$. The solution for this particular case has been given by J. Zbornik, [7]. Thus the results of J. Zbornik and results IV complement each other. It should be stressed here that the solution of J. Zbornik for $\tilde{\beta} = 1$ is deeper and covers equations whose coefficients are defined in a certain special way and depend upon two arbitrary functions. Thus, we should expect the possibility of generalization of both results.

Using the known solutions of equations (3.7) one may construct a table of solutions of equations of form (3.6). Certain solutions are presented in Section 4.

3.4. In a similar way we define two families of equations E from condition (1.9).

Condition (1.9) is a differential equation with respect to both a_{20} and a_{21}.

It is easier to solve it treating a_{21} as an unknown (when a_{20} is given), since in this case we have a linear equation

$$a_{21}' + \tilde{v}_1 a_{20} a_{21} = \tilde{v}_1^2 a_{20}^2 + \tilde{v}_1 a_{20}' + a_{20}. \tag{3.10}$$

The general solution of this equation is

$$a_{21} = \tilde{c} e^{-\tilde{v}_1 A_{20}} + e^{-\tilde{v}_1 A_{20}} \Big(\tilde{v}_1^2 \int a_{20}^2 e^{\tilde{v}_1 A_{20}} dx + \tilde{v}_1 \int a_{20}' e^{\tilde{v}_1 A_{20}} dx + \int a_{20} e^{\tilde{v}_1 A_{20}} dx \Big).$$

We easily see that the integrals appearing in the above formula may be computed (by integrating by parts), and we get

$$a_{21} = \tilde{c} \exp(-\tilde{v}_1 A_{20}) + \tilde{v}_1 a_{20} + \frac{1}{\tilde{v}_1} \qquad (\tilde{v}_1 \neq 0). \tag{3.11}$$

The solution of the equation so defined can be found from (3.1) by putting $v_1 = \tilde{v}_1 a_{20}$. Hence

V. *The linear equation*

$$y'' + \Big(\tilde{c} e^{-\tilde{v}_1 A_{20}} + \tilde{v}_1 a_{20} + \frac{1}{\tilde{v}_1} \Big) y' + a_{20} y = b_2 \tag{3.12}$$

belongs to class E and its solution is

$$y_1 = \exp\Big(-\tilde{c} \int e^{-\tilde{v}_1 A_{20}} dx - \frac{x}{\tilde{v}_1} \Big),$$

$$y_2 = y_1 \int \exp\Big(\tilde{c} \int e^{-\tilde{v}_1 A_{20}} dx + \frac{x}{\tilde{v}_1} - \tilde{v}_1 A_{20} \Big) dx,$$

$$Y_2 = \exp\Big(-\tilde{c} \int e^{-\tilde{v}_1 A_{20}} dx - \frac{x}{\tilde{v}_1} \Big) \int \exp\Big(c^2 \int e^{-\tilde{v}_1 A_{20}} dx + \frac{x}{\tilde{v}_1} - \tilde{v}_1 A_{20} \Big) \int b_2 e^{\tilde{v}_1 A_{20}} dx^2.$$

The particular case of this equation, $\tilde{c} = 0$, leads to the equation of C. Olsson (see [6], p. 554, formula (2.77)) if we substitute $a_{20} = \tilde{c}f(x) + \tilde{d}$.

If we treat a_{20} as an unknown in condition (1.9), then equation (1.6) may belongs to class E, only if a_{20} satisfies the Riccati equation

$$(3.14)\qquad a_{20}' = -\tilde{v}_1 a_{20}^2 + \left(a_{21} - \frac{1}{\tilde{v}_1}\right) a_{20} + \frac{a_{21}'}{\tilde{v}_1} \qquad (\tilde{v}_1 \neq 0).$$

This equation can be solved. The substitution

$$(3.15)\qquad a_{20} = -\frac{z}{\tilde{v}_1}$$

leads to the equation

$$z' = z^2 + \left(a_{21} - \frac{1}{\tilde{v}_1}\right) z - a_{21}',$$

to which, in the theory of $\hat{R}$ equations, corresponds the linear equation [4]:

$$(3.16)\qquad f_*'' - \left(a_{21} - \frac{1}{\tilde{v}_1}\right) f_*' - a_{21}' f_* = 0.$$

We find successively

$$f_*'' - \frac{d}{dx}(a_{21} f_*) + \frac{1}{\tilde{v}} f_*' = 0,$$

$$(3.17)\qquad f_* = \tilde{a}\exp\left(A_{21} - \frac{x}{\tilde{v}_1}\right) + \tilde{b}\exp\left(A_{21} - \frac{x}{\tilde{v}_1}\right) \int \exp\left(\frac{x}{\tilde{v}_1} - A_{21}\right) dx,$$

and, using (3.15) and (3.17), we find for the function a_{20}, after suitable transformations, the formula $a_{20} = f_*'/\tilde{v}_1 f_*$.

Thus we come to the conclusion that the equation from class E which satisfies relation (1.9) has the form

$$(3.18)\qquad y'' + a_{21} y' + \frac{1}{\tilde{v}_1} \cdot \frac{f_*'}{f_*} y = b_2,$$

where f_* is given by $(3.17)_2$.

Let us solve this equation. In order to do this we have to substitute

$$V_1 = \int v_1 dx = \tilde{v}_1 \int a_{20} dx = \ln|f_*|,$$

into formulas (3.1). That gives

$$(3.19)\qquad \begin{aligned} y_1 &= f_* \exp(-A_{21}), \\ y_2 &= f_* \exp(-A_{21}) \int f_*^{-2} \exp(A_{21}) dx, \\ Y_2 &= f_* \exp(-A_{21}) \int f_*^{-2} \exp(A_{21}) \int b_2 f_* dx^2. \end{aligned}$$

Let us reduce equation (3.18) to a simpler form. Putting

$$\exp\left(\frac{x}{\tilde{v}_1}-A_{21}\right)=f' \tag{3.20}$$

we get

$$\begin{gathered} a_{21}-\frac{1}{\tilde{v}_1}=-\frac{f''}{f'}, \qquad \int\exp\left(\frac{x}{\tilde{v}_1}-A_{21}\right)dx=f, \\ f_*=f'^{-1}(\tilde{c}+f), \qquad \exp(A_{21})=\exp\left(\frac{x}{\tilde{v}_1}\right)f'^{-1}. \end{gathered} \tag{3.21}$$

After substituting (3.21) in equation (3.18) and in its solutions (3.19) we obtain the following result:

VI. *The second order linear equation*

$$y''+\left(\frac{1}{\tilde{v}_1}-\frac{f''}{f'}\right)y'+\frac{1}{\tilde{v}_1}\left(\frac{f'}{\tilde{c}+f}-\frac{f''}{f'}\right)y=b_2, \tag{3.22}$$

where f belongs to class $C^2(a,b)$, belongs to class E, and its solution is

$$\begin{aligned} y_1&=e^{-x/\tilde{v}_1}(\tilde{c}+f), \\ y_2&=e^{-x/\tilde{v}_1}(\tilde{c}+f)\int e^{x/\tilde{v}_1}f'(\tilde{c}+f)^{-2}dx, \\ Y_2&=e^{-x/\tilde{v}_1}(\tilde{c}+f)\int e^{x/\tilde{v}_1}f'(\tilde{c}+f)^{-2}\int b_2f'^{-1}(\tilde{c}+f)dx^2. \end{aligned} \tag{3.23}$$

We assume here that the denominators in the above formulas do not vanish in the interval (a,b).

3.5. Suppose we are given the linear equation (1.6) of the second order. As we know, [4], to equation (1.6) corresponds the equation $\hat{R}_1$

$$u'=u^2-a_{21}u+a_{20}, \tag{3.24}$$

and, if u_1 is a particular solution of this equation, then the general solution of the linear equation (1.6) is

$$\begin{aligned} y_1&=\exp(-U_1), \\ y_2&=\exp(-U_1)\int\exp(U_1-A_{10})dx, \\ Y_2&=\exp(-U_1)\int\exp(U_2-A_{10})\int b_2\exp(A_{10})dx^2. \end{aligned} \tag{3.25}$$

Since $a_{10}=a_{21}-u_1$, we obtain the following result:

VII. *One particular solution of equation* (1.6) *of the second kind $\hat{R}_1$ determines the general solution of the corresponding linear non-homogeneous* *equation* (1.6), *and this solution has the form*

$$
(3.26)\qquad \begin{aligned}
y_1 &= \exp(-U_1),\\
y_2 &= \exp(-U_1)\int \exp(2U_1-A_{21})\,dx,\\
Y_2 &= \exp(-U_1)\int \exp(2U_1-A_{21})\int b_2\exp(A_{21}-U_1)\,dx^2.
\end{aligned}
$$

Let us consider equations E satisfying the last condition (1.13).

We assume first that the function a_{20} is given; then, from (1.13) we find for a_{21}:

$$
(3.27)\qquad a_{21} = \tilde{u}_1 a_{20} - \frac{a_{20}'}{a_{20}} + \frac{1}{\tilde{u}_1}.
$$

Since the function $u_1 = \tilde{u}_1 a_{20}$ is a particular solution of the Riccati equation (3.24), we find from (3.26) a solution of the linear equation whose one coefficient is an arbitrary function a_{20}, and the other — the function (3.27). Thus

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$$
(3.28)\qquad y'' + \left(\tilde{u}_1 a_{20} - \frac{a_{20}'}{a_{20}} + \frac{1}{\tilde{u}_1}\right) y' + a_{20} y = b_2,
$$

where a_{20} *belongs to the class* $C^1(a, b)$, $\tilde{u}_1 \neq 0$ *and* $a_{20} \neq 0$, *belongs to class* E *and its general solution is*

$$
(3.29)\qquad \begin{aligned}
y_1 &= \exp(-\tilde{u}_1 A_{20}),\\
y_2 &= \exp(-\tilde{u}_1 A_{20})\int a_{20}\exp\left(\tilde{u}_1 A_{20} - \frac{x}{\tilde{u}_1}\right) dx,\\
Y_2 &= \exp(-\tilde{u}_1 A_{20})\int a_{20}\exp\left(\tilde{u}_1 A_{20} - \frac{x}{\tilde{u}_1}\right)\int b_2 a_{20}^{-1}\exp\left(\frac{x}{\tilde{u}_1}\right) dx^2.
\end{aligned}
$$

3.6. Finally, we shall select the coefficient a_{20} from condition (1.13). The function a_{20} should satisfy the Riccati equation

$$
(3.30)\qquad a_{20}' = u_1 a_{20}^2 - \left(a_{21} - \frac{1}{\tilde{u}_1}\right) a_{20}.
$$

Substitution $a_{20} = z/\tilde{u}_1$ leads to the equation

$$
(3.31)\qquad z' = z^2 - \left(a_{21} - \frac{1}{\tilde{u}_1}\right) z,
$$

which can be solved. Indeed, to equation (3.31) corresponds the linear equation

$$
f_*'' + \left(a_{21} - \frac{1}{\tilde{u}_1}\right) f_*' = 0,
$$

whose solution is

$$f_* = -\tilde{c}_1 \int \exp\left(\frac{x}{\tilde{u}_1} - A_{21}\right) dx + \tilde{c}_2 .$$

It follows that the general solution of the equation (3.30) is

$$a_{20} = -\frac{1}{\tilde{u}_1} \cdot \frac{\tilde{c}_1 \exp\left(\frac{x}{\tilde{u}_1} - A_{21}\right)}{\tilde{c}_1 \int \exp\left(\frac{x}{\tilde{u}_1} - A_{21}\right) dx + \tilde{c}_2} .$$

Let us denote

$$f' = \exp\left(\frac{x}{\tilde{u}_1} - A_{21}\right).$$

We have in this notation

$$a_{21} = \frac{1}{\tilde{u}_1} - \frac{f''}{f'} .$$

Thus equation E whose coefficients satisfy (1.13) has the following form:

$$y'' + \left(\frac{1}{\tilde{u}_1} - \frac{f''}{f'}\right) y' - \frac{1}{\tilde{u}_1} \cdot \frac{\tilde{c}_1 f'}{\tilde{c}_1 f + \tilde{c}_2}\, y = b_2 .$$

The solution of this equation can be found also from formulas (3.26). Since the particular solution $u_1 = \tilde{u}_1 a_{20}$ has, in the case under consideration, the form

$$u_1 = -\frac{\tilde{c}_1 f'}{\tilde{c}_1 f + \tilde{c}_2},$$

we have

$$U_1 = -\ln|\tilde{c}_1 f + \tilde{c}_2| .$$

We have also

$$A_{21} = \frac{x}{\tilde{u}_1} - \ln|f'|$$

and we reach the following result:

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$$y'' + \left(\frac{1}{\tilde{u}_1} - \frac{f''}{f'}\right) y' - \frac{1}{\tilde{u}_1} \cdot \frac{\tilde{c}_1 f'}{\tilde{c}_1 f + \tilde{c}_2}\, y = b_2 \tag{3.32}$$

belongs to class E if f belongs to class $C^2(a, b)$. Its general solution is

$$\begin{aligned} y_1 &= \tilde{c}_1 f + \tilde{c}_2, \\ y_2 &= (\tilde{c}_1 f + \tilde{c}_2) \int \exp\left(-\frac{x}{\tilde{u}_1}\right) (\tilde{c}_1 f + \tilde{c}_2)^{-2} dx, \end{aligned} \tag{3.33}$$

$$Y_2 = (\tilde{c}_1 f + \tilde{c}_2) \int \exp\left(-\frac{x}{\tilde{u}_1}\right) (\tilde{c}_1 f + \tilde{c}_2)^{-2} \int b_2 \exp\left(\frac{x}{\tilde{u}_1}\right) f'^{-1} (\tilde{c}_1 f + \tilde{c}_2)\, dx^2$$

$$(\tilde{u}_1 \neq 0,\ f' \neq 0,\ \tilde{c}_1 f + \tilde{c}_2 \neq 0).$$

4. Finally we shall list a certain number of equations of class E and their solutions, using theorem IV.

Let us denote the general solution (3.8) of equation (3.6) by

$$(4.1)\qquad y \equiv A(\tilde{\beta},f) = \tilde{c}_1 f^{\tilde{\beta}} + \tilde{c}_2 f^{\tilde{\beta}} \int f^{-(\tilde{\beta}+1)} dx + f^{\tilde{\beta}} \int f^{-(\tilde{\beta}+1)} \int b_2 f dx^2 .$$

In equations of type (3.6) we have two numerical parameters. We assume that they satisfy the relation

$$(4.2)\qquad \tilde{\alpha} + \tilde{\beta} = 1 .$$

4.1. The general solution of the equation

$$f'' + \tilde{\lambda}^2 f = 0 ,$$

where $\tilde{\lambda}$ is a real constant, is the function

$$f = \tilde{c}_1 \sin \tilde{\lambda} x + \tilde{c}_2 \cos \tilde{\lambda} x .$$

It follows from theorem IV that the equation

$$(4.3)\qquad y'' + \tilde{\alpha}\tilde{\lambda} \frac{\tilde{c}_1 \cos \tilde{\lambda} x - \tilde{c}_2 \sin \tilde{\lambda} x}{\tilde{c}_1 \sin \tilde{\lambda} x + \tilde{c}_2 \cos \tilde{\lambda} x} y' + \tilde{\beta}\tilde{\lambda}^2 y = b_2$$

has the following solution:

$$(4.4)\qquad y = A(\tilde{\beta}, c_1 \sin \tilde{\lambda} x + \tilde{c}_2 \cos \tilde{\lambda} x) .$$

The particular cases of this equation lead to the following two equations: if $\tilde{c}_1 = 0$,

$$(4.5)\qquad y'' - y' \tilde{\lambda} \tilde{\alpha} \operatorname{tg} \tilde{\lambda} x + \tilde{\beta} \tilde{\lambda}^2 y = b_2 ,$$

whose solution is

$$(4.6)\qquad y = A(\tilde{\beta}, \tilde{c}_2 \cos \tilde{\lambda} x) ;$$

if $\tilde{c}_2 = 0$,

$$(4.7)\qquad y'' + y' \tilde{\alpha} \tilde{\lambda} \operatorname{ctg} \tilde{\lambda} x + \tilde{\beta} \tilde{\lambda}^2 y = b_2 ,$$

with the solution

$$(4.8)\qquad y = A(\tilde{\beta}, \tilde{c}_1 \sin \tilde{\lambda} x) .$$

Equations (4.5) and (4.7) are discussed in book [5], p. 552. The solution of (4.5) has been given for the particular case $\tilde{\lambda} = 1$, $\tilde{\alpha} = -2$, $\tilde{\beta} = 3$. J. Halm, [2], has given the solution of (4.5) for $\tilde{\lambda} = 1$, $\tilde{\alpha} = 2$ and arbitrary $\tilde{\beta}$, hence solution (4.6) coincides with Halm's solution at one point.

For equation (4.7) Goldszejzer, [5], p. 552, has given solutions for values of parameters which do not coincide with (4.2).

4.2. The equation

$$f'' - \tilde{\lambda}^2 f' = 0$$

($\tilde{\lambda}$ is an arbitrary real number) has the solution

$$f = \tilde{c}_1 \operatorname{sh}\tilde{\lambda}x + \tilde{c}_2 \operatorname{ch}\tilde{\lambda}x.$$

It follows that the equation

$$y'' + \tilde{a}\tilde{\lambda}\frac{\tilde{c}_1 \operatorname{ch}\tilde{\lambda}x + \tilde{c}_2 \operatorname{sh}\tilde{\lambda}x}{\tilde{c}_1 \operatorname{sh}\tilde{\lambda}x + \tilde{c}_2 \operatorname{ch}\tilde{\lambda}x} y' - \tilde{\beta}\tilde{\lambda}^2 y = b_2 \tag{4.9}$$

has the solution

$$y = A(\tilde{\beta}, \tilde{c}_1 \operatorname{sh}\tilde{\lambda}x + \tilde{c}_2 \operatorname{ch}\tilde{\lambda}x). \tag{4.10}$$

If $\tilde{c}_1 = 0$, we get the equation

$$y'' + y'\tilde{a}\tilde{\lambda}\operatorname{th}\tilde{\lambda}x - \tilde{\beta}\tilde{\lambda}^2 y = b_2 \tag{4.11}$$

with the solution

$$y = A(\tilde{\beta}, \operatorname{ch}\tilde{\lambda}x). \tag{4.12}$$

If $\tilde{c}_2 = 0$, we have the equation

$$y'' + y'\tilde{a}\tilde{\lambda}\operatorname{ch}\tilde{\lambda}x - \tilde{\beta}\tilde{\lambda}^2 y = b_2 \tag{4.13}$$

with the solution

$$y = A(\tilde{\beta}, \operatorname{sh}\tilde{\lambda}x). \tag{4.14}$$

Also in this case J. Halm has given a solution of equation (4.11) (see [5], p. 550) for $\tilde{\lambda} = 1$, $\tilde{a} = 2$ and arbitrary $\tilde{\beta}$.

Solution (4.12) coincides with Halm's solution at one point $\tilde{\beta} = 1$.

4.3. The equation

$$f'' - (x^2 + 3)f = 0$$

has the solution (see [5], p. 528, formula (2.11a)):

$$f = \tilde{c}_1 x \exp(\tfrac{1}{2}x^2) + \tilde{c}_2 x \exp(\tfrac{1}{2}x^2)\int x^{-2}\exp(x^{-2})\,dx.$$

The simplest of the equations which can be solved (with coefficients formed from elementary functions) by the method based upon theorem IV) is the equation

$$y'' + \tilde{a}\left(x + \frac{1}{x}\right)y' - \tilde{\beta}(x^2 + 3)y = b_2. \tag{4.15}$$

The solution of (4.15) is

$$y = A[\tilde{\beta}, x\exp(\tfrac{1}{2}x^2)].$$

4.4. The equation

$$f'' - (\tilde{a}^2 x^2 + \tilde{a})f = 0$$

has the solution (see [5], p. 529, formula (2.13)):

$$f = \tilde{c}_1 \exp(\tfrac{1}{2}\tilde{a}x^2) + \tilde{c}_2 \exp(\tfrac{1}{2}\tilde{a}x^2)\int \exp(-\tilde{a}x^2)\,dx.$$

Thus, the second order equation formed from the first particular solution $\exp(\frac{1}{2}\tilde{a}x^2)$ is

$$y''+\tilde{\alpha}\tilde{a}xy'-\tilde{\beta}(\tilde{a}^2x^2+\tilde{a})y = b_2 \tag{4.16}$$

with the solution

$$y = A[\tilde{\beta}, \exp(\tfrac{1}{2}\tilde{a}x^2)]. \tag{4.17}$$

Equation (2.47) in [5], p. 547 (for $\tilde{a} = 2$, $\tilde{\alpha} = 2$, $\tilde{\beta} = -1$) is a particular case of this equation.

Equation (2.48) given in [5], p. 547,

$$y''-4xy'+(3x^2+2n-1)y = b_2, \tag{4.18}$$

(where n is an integer) can, in two cases, be solved directly, since it can be reduced to equation (4.16). In the first case ($n = 0$) this equation takes the form

$$y''-4xy'+(3x^2-1)y = b_2. \tag{4.19}$$

In fact, (4.16) reduces to (4.19) if $\tilde{a} = -3$, $\tilde{\alpha} = \frac{4}{3}$, $\tilde{\beta} = -\frac{1}{3}$, and its solution is

$$y = A[-\tfrac{1}{3}, \exp(-\tfrac{3}{2}x^2)]. \tag{4.20}$$

If $n = -1$, then (4.18) reduces to

$$y''-4xy'+(3x^2-3)y = b_2. \tag{4.21}$$

This is a particular case of (4.16) (for $\tilde{a} = -1$, $\tilde{\alpha} = 4$, $\tilde{\beta} = -3$) and its solution is

$$y = A[-3, \exp(-\tfrac{1}{2}x^2)]. \tag{4.22}$$

Equation (2.50) given in [6] on p. 547

$$y''-4xy'+(4x^2-2)y = b_2 \tag{4.23}$$

is also a particular case of equation (4.16) ($\tilde{a} = -2$, $\tilde{\alpha} = 2$, $\tilde{\beta} = -1$) and its solution is

$$y = A[-1, \exp(-x^2)]. \tag{4.24}$$

4.5. In book [5] on p. 539, formula (2.29) gives a particular solution

$$f_1 = \exp(H)$$

of the equation

$$f''-(h^2+h')f = 0. \tag{4.25}$$

It is easy to verify that the general solution of (4.25) is the function

$$f = \tilde{c}_1\exp(H)+\tilde{c}_2K\exp(H),$$

where $K = \int\exp(-2H)dx$.

It follows from theorem IV that the equation

$$y''+\tilde{a}\,\frac{\tilde{c}_1 h+\tilde{c}_2(hK+K')}{\tilde{c}_1+\tilde{c}_2K}\,y'-\tilde{\beta}(h^2+h')y = b_2 \tag{4.26}$$

has the general solution

$$y = A[\tilde{\beta},\, \tilde{c}_1\exp(H)+\tilde{c}_2K\exp(H)]. \tag{4.27}$$

If $\tilde{c}_1 = 0$, then we have

$$y''+\tilde{a}\left(h+\frac{K'}{K}\right)y'-\tilde{\beta}(h^2+h')y = b_2 \tag{4.28}$$

with the solution

$$y = A(\tilde{\beta},\, K\exp(H)). \tag{4.29}$$

If $c_2 = 0$, we have

$$y''+\tilde{a}hy'-\tilde{\beta}(h^2+h')y = b_2 \tag{4.30}$$

with the solution

$$y = A[\tilde{\beta},\, \exp(H)]. \tag{4.31}$$

4.6. It follows also from theorem IV that the equation

$$y''+\tilde{a}\,\frac{f'}{f}\,y'-\tilde{\beta}\,\frac{f''}{f}\,y = b_2 \tag{4.32}$$

has the solution

$$y = A(\tilde{\beta}, f). \tag{4.33}$$

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O PEWNEJ KLASIE LINIOWYCH RÓWNAŃ RÓŻNICZKOWYCH

STRESZCZENIE

Opierając się na definicji tzw. uogólnionych równań Riccatiego n-tego rzędu (por. prace [3] i [4] cytowane w artykule) autor określa pewną klasę różniczkowych równań liniowych (oznaczoną symbolem E). Równanie liniowe należy do klasy E, jeśli istnieje taka liczba, dla której współczynniki równania spełniają tożsamościowo ze względu na zmienną x jeden ze związków (1.4) lub (1.11). Szczególnym przypadkiem klasy E są równania liniowe o współczynnikach stałych oraz równanie Eulera (zwykłe i uogólnione). Dla tych ostatnich równań kryteria (1.4) i (1.11), występujące w definicji równań E, sprowadzają się do równań charakterystycznych odpowiednio dla równań liniowych o współczynnikach stałych lub dla równań liniowych Eulera.

Równania E posiadają następującą własność: rząd każdego równania E można efektywnie obniżyć o jedność. Wynika stąd, że wszystkie równania E rzędu drugiego można rozwiązać.

Z tego względu, w drugiej części artykułu autor zajmuje się równaniami E rzędu drugiego, podając pewną liczbę rozwiązań równań dotychczas nie rozwiązanych lub rozwiązanych tylko w szczególnych przypadkach.

Т. ИВИНЬСКИ (Варшава)

О НЕКОТОРОМ КЛАССЕ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

РЕЗЮМЕ

Исходя из определения так наз. обобщенных уравнений Рикати n-го порядка (ср. [3] и [4]), автор определяет некоторый класс дифференциальных линейных уравнений (обозначенный символом E). Линейные уравнение принадлежит к классу E, если существует такое число, для которого коэффициенты уравнения тождественно выполняют по отношению к переменной x одну из зависимостей (1.4) или (1.11). Особым случаем класса E являются линейные уравнения с постоянными коэффициентами и уравнение Эйлера (обыкновенное и обобщенное). Для этих последних уравнений выступающие в определении уравнений E критерии (1.4) и (1.11) сводятся к характеристическим уравнениям соответственно для линейных уравнений с постоянными коэффициентами или для линейных уравнений Эйлера.

Уравнения E обладают следующим свойством: порядок каждого уравнения E можно эффективно снизить на единицу. Отсюда следует, что все уравнения E второго порядка решимы.

В связи с этим во второй части статьи автор занимается уравнениями E второго порядка а дает некоторое количество решений уравнений, которые не были до сих пор решены или были решены в частных случаях.
