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SEMIMARTINGALE MEASURE IN THE INVESTIGATION OF STRATONOVICH-TYPE STOCHASTIC INTEGRALS AND INCLUSIONS

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Abstract

A random measure associated to a semimartingale is introduced. We use it to investigate properties of several types of stochastic integrals and properties of the solution set of Stratonovich-type stochastic inclusion.

Keywords: forward, backward and symmetric integral, time-reversible process, semimartingale measure, set-valued stochastic integral, Stratonovich inclusion.

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1. Introduction

Dynamical systems concerning practical applications, like these from optimal control, mathematical finance etc., are complex and usually not determined uniquely by the state of the system. That is why, differential equations (or stochastic differential equations) which usually try to describe these systems, need to be replaced by more general objects.

One of them are deterministic or stochastic differential inclusions, which were the subject of several studies (see e.g.: N.U. Ahmed [2, 3, 4], J.P. Aubin, A. Cellina [5], J.P. Aubin, H. Frankowska [6], J.P. Aubin, G. Da Prato [7], G. Da Prato, H. Frankowska [10], A. Fryszkowski [13], F. Hiai, H. Umegaki [15], M. Kisielewicz [16, 17, 18, 19], M. Kisielewicz, M. Michta, J. Motyl [20, 21], M. Michta, J. Motyl [23, 24], J. Motyl [25, 26, 27], D. Repovš, P.V. Semenov [31]).

The authors consider in their books and papers how to describe above mentioned problems in set-valued way. The objects of the study become set-valued processes, set-valued integrals and the problem of existence of solutions of differential inclusions. The deterministic case was the aim of investigations in [4, 5, 6, 13, 15, 19, 31], while the stochastic case in [2, 3, 10, 14, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27] and [28, 29, 34]. Most of them consider Itô-type stochastic integral or inclusion. Only in [14, 23, 24] and [29] the authors analyze Stratonovich-type set-valued stochastic integral and inclusion.

This work contains two main results: properties of backward stochastic integral and some properties of the solution set of the Stratonovich-type stochastic inclusion driven by a semimartingale.

First, we show the connection between backward stochastic integral defined by F. Russo and P. Vallois in [32] and well known Itô stochastic integral.

Then we present some properties of the solution set of the Stratonovich-type stochastic inclusion driven by a semimartingale. We show non-emptiness and closedness of above mentioned set.

To obtain the second mentioned result we use a semimartingale measure based on the Doléans-Dade measure, see [22, 28]. Its property (SMP-property) was used in [34] to investigate the Itô-type stochastic inclusion.

In Section 2 we introduce basic definitions and notations used in the paper. Section 3 contains the definition of a semimartingale measure, its properties and application to set-valued analysis. In Section 4 we define Stratonovich-type single- and set-valued stochastic integrals and present some properties of defined integrals. Section 5 contains the existence and closedness theorems for Stratonovich-type stochastic inclusion.

2. Preliminaries

Throughout the paper let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a complete filtered probability space, where $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq 1}$ denotes a filtration satisfying the usual hypothesis, i.e. it contains all P-null sets and it is right continuous.

By a stochastic process x on $(\Omega, \mathcal{F}, P)$ we mean a collection $x = (x_t)_{t\in[0,1]}$ of n-dimensional random variables $x_t : \Omega \to \mathbb{R}^n$, $t \in [0, 1]$. The process x is $\mathbf{F}$-adapted if x_t is $\mathcal{F}_t$-measurable for each $t \in [0, 1]$. A stochastic process x is càdlàg (càglàd) if it has right continuous sample paths with left limits (left continuous sample paths with right limits). A stochastic process x is called RV-càdlàg (RV-càglàd) if it is càdlàg (càglàd) and continuous at $t = 0$ and $t = 1$ (see e.g. [12]). A stochastic $\mathbf{F}$-adapted process x is an FV-process if it has paths of finite variation on compacts.

Let $\mathcal{P}(\mathbf{F})$ denote the smallest σ-algebra on $[0,1] \times \Omega$ generated by $\mathbf{F}$-adapted càglàd processes. It is generated by a class of all subsets of $[0,1] \times \Omega$ of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $0 \le s < t \le 1$. If a stochastic process x is $\mathcal{P}(\mathbf{F})$-measurable, it is called $\mathbf{F}$-predictable.

We say that a càdlàg process x is an $\mathbf{F}$-semimartingale if it can be expressed as a sum $x = N + A$, where N is a local $\mathbf{F}$-martingale and A is an FV-process.

Let $1 \le p \le \infty$. We denote by $|\cdot|$ an Euclidean norm on $I\!R^n$. Other norms are denoted with respect to a space on which they are defined, e.g.: $\|\cdot\|_{L^p(\Omega)}$ for the norm in $L^p(\Omega)$.

By $L^p(\Omega)$ we denote the space $L^p(\Omega, \mathcal{F}, P; I\!R^n)$.

Let S^p denote a space of all $\mathbf{F}$-adapted càdlàg processes x with finite S^p norm, where $\|x\|_{S^p} = \|\sup_{t\in[0,1]} |x_t|\|_{L^p(\Omega)}$. For an $\mathbf{F}$-semimartingale $Z = N + A$ we define $j_p(N,A) = \|[N,N]_1^{1/2} + \int_0^1 |dA_s|\|_{L^p(\Omega)}$, where $[N,N]_t$ is a quadratic variation process of a local $\mathbf{F}$-martingale N and $|dA_s(\omega)|$ denotes the total variation measure on $[0,1]$ induced by $s \mapsto A_s(\omega)$. Let $\mathcal{H}^p$ denote a space of all $\mathbf{F}$-semimartingales Z with finite $\mathcal{H}^p$ norm, where $\|Z\|_{\mathcal{H}^p} = \inf_{Z=N+A} j_p(N,A)$, and infimum is taken over all possible decompositions $Z = N+A$. $\mathcal{H}^p$ is a Banach space (see e.g. [30]).

For a Banach space X, by $cl(X)$ and $cc(X)$ we denote the spaces of all nonempty closed, compact and convex, respectively, subsets of X. By $\mathrm{dist}(a,A)$ we denote the distance of $a \in X$ to the set $A \in cl(X)$. For $A, B \in cl(X)$ let $\overline{h}(A,B) = \sup_{a\in A} \mathrm{dist}(a,B)$ and $H(A,B) = \max\{\overline{h}(A,B),\ \overline{h}(B,A)\}$.

For a set A the function $1\!\!1_A$ denotes the indicator function i.e., $1\!\!1_A(t) = 1$ for $t \in A$ and $1\!\!1_A(t) = 0$ otherwise.

3. Properties of semimartingale measure

In this section we recall definition of a measure for an $\mathbf{F}$-semimartingale $Z \in \mathcal{H}^2$, some of its properties and an application to the set-valued analysis. We use them in the next sections.

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be as before.

Let $\mathcal{H}_n^2$ denote a space of n-dimensional $\mathbf{F}$-semimartingales $Z = (Z^1, \ldots, Z^n)$, $Z^i \in \mathcal{H}^2$, $i = 1, \ldots, n$, with a norm

$$\|Z\|_{\mathcal{H}_n^2} = \Big(\sum_{i=1}^n \|Z^i\|_{\mathcal{H}^2}^2 \Big)^{1/2}.$$

For an $\mathbf{F}$-measurable $Z \in \mathcal{H}^2$, $Z_0 = 0$ we introduce a measure μ_Z called $\mathbf{F}$-semimartingale measure, (see e.g. [34]).

Its definition is based on a Doléans-Dade measure μ_N for a local $\mathbf{F}$-martingale N, (see e.g. [9]), and a measure ν_A for the FV-process A. The measure ν_A on $\mathcal{P}(\mathbf{F})$ we define as follows.

Let $\alpha(\omega, dt)$ denote a kernel of a random measure defined on $[0,1]$ by

$$\alpha(\omega, dt) := c_A(\omega)|dA_t(\omega)|,$$

where $c_A(\omega) = \int_0^1 |dA_t(\omega)|$ denotes the total variation of a random measure $|dA_t(\omega)|$ induced by the paths of the process A.

Let D be an $\mathbf{F}$-predictable subset of $[0,1] \times \Omega$. A measure ν_A is defined by

$$\nu_A(D) = \int_\Omega \int_0^1 \mathbb{I}_D(\omega, t)\alpha(\omega, dt)P(d\omega).$$

For an $\mathbf{F}$-measurable $Z \in \mathcal{H}^2$ we define a measure $\mu_Z = \mu_N + \nu_A$.

Let $Z \in \mathcal{H}^2$ and $f : [0,1] \times \Omega \to I\!\!R^n$. We define a space

$$L^2_{\mu_Z} = \left\{ f \in \mathcal{P}(\mathbf{F}) : \int_{\Omega \times [0,1]} |f|^2 d\mu_Z < \infty \right\}.$$

$L^2_{\mu_Z}$ endowed with a norm

$$\|f\|_{L^2_{\mu_Z}} = \left(\int_{\Omega \times [0,1]} |f|^2 d\mu_Z \right)^{\frac{1}{2}}$$

is a Banach space.

In the following Lemma we recall a property of the above measure, (SMP-property, see [34]), which allows to get similar properties of set-valued stochastic integrals driven by an $\mathbf{F}$-semimartingale as those of set-valued stochastic integrals driven by a square-integrable $\mathbf{F}$-martingale, (see e.g. [21]).

Lemma 1 ([34]). *Let $Z \in \mathcal{H}^2$ and $f \in L^2_{\mu_Z}$. Then we have*

(i)
$$\left\| \int f_\tau dZ_\tau \right\|^2_{\mathcal{H}^2_n} \leq 2\|f\|^2_{L^2_{\mu_Z}}.$$

(ii)
$$\left\| \int_s^t f_\tau dZ_\tau \right\|^2_{L^2(\Omega)} \leq 2 \int_{\Omega \times (s,t]} |f|^2 d\mu_Z, \quad \text{for } s, t \in [0,1],\ s < t.$$

We recall now definition and some properties of the set $\mathcal{S}_{\mu_Z}(G)$ and definitions of set-valued stochastic integrals driven by an $\mathbf{F}$-semimartingale Z, which are used in the next sections.

Definition 2 ([28])**.** For an $\mathbf{F}$-measurable $Z \in \mathcal{H}^2$, $Z_0 = 0$, and an $\mathbf{F}$-predictable set-valued process G, we define a set $\mathcal{S}_{\mu_Z}(G)$ by

$$\mathcal{S}_{\mu_Z}(G) \;:= \left\{ f \in Sel(G) \;:\; f \in L^2_{\mu_Z} \right\},$$

where $Sel(G)$ denotes the set of μ_Z-measurable selections.

An $\mathbf{F}$-predictable set-valued process G is integrable with respect to an $\mathbf{F}$-semimartingale measure μ_Z, if the set $\mathcal{S}_{\mu_Z}(G)$ is nonempty.

G is μ_Z-squareintegrably bounded if there exists a process $m \in L^2_{\mu_Z}$ such that $H(G, \{0\}) \leq m \;\mu_Z$-a.e.

Lemma 3 ([34])**.** *For an $\mathbf{F}$-measurable $Z \in \mathcal{H}^2$, $Z_0 = 0$ and an $\mathbf{F}$-predictable μ_Z-squareintegrably bounded set-valued process G we get*

(i) *the set $\mathcal{S}_{\mu_Z}(G)$ is a nonempty closed and bounded subset of $L^2_{\mu_Z}$,*

(ii) *if G takes on convex values, $\mathcal{S}_{\mu_Z}(G)$ is convex and weakly compact in $L^2_{\mu_Z}$.*

Definition 4 ([28])**.** Let Z be an $\mathbf{F}$-semimartingale from $\mathcal{H}^2$, $Z_0 = 0$. Let G be an $\mathbf{F}$-predictable μ_Z-squareintegrably bounded set-valued process. A set-valued stochastic integral $\int G_\tau dZ_\tau$ of G with respect to Z is defined by $\int G_\tau dZ_\tau = \{\int g_\tau dZ_\tau \;: g \in \mathcal{S}_{\mu_Z}(G)\}$. For fixed $0 \leq s < t \leq 1$ we also define $\int_s^t G_\tau dZ_\tau = \{\int_s^t g_\tau dZ_\tau \;: g \in \mathcal{S}_{\mu_Z}(G)\}$.

4. Stratonovich-type stochastic integrals

In this section we define a set-valued Stratonovich-type stochastic integral. We start with a single-valued case.

Our definition of Stratonovich-type stochastic integral is based on definition introduced by F. Russo and P. Vallois in [32] and modified in [33] and [12]. For RV-càdlàg processes g and Z they considered forward, backward and Stratonovich integrals, respectively, as ucp-limits, (ucp means uniformly on compacts in probability, see e.g. [30]), of the following sums

$$\begin{aligned}
I^-_{\tau_n}(g, dZ)(a) &= \sum_i g(t_i \wedge a)(Z(t_{i+1} \wedge a) - Z(t_i \wedge a)),\\
I^+_{\tau_n}(g, dZ)(a) &= \sum_i g(t_{i+1} \wedge a)(Z(t_{i+1} \wedge a) - Z(t_i \wedge a)),\\
I^o_{\tau_n}(g, dZ)(a) &= 1/2\,(I^+_{\tau_n}(g, dZ)(a) + I^-_{\tau_n}(g, dZ)(a)),
\end{aligned}$$

when $|\tau_n| = \sup_i(t_{i+1} - t_i)$ of a subdivision $\{\tau_n\}$ of $[0,1]$, $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, tends to 0 for $n \to \infty$.

These integrals are denoted by $\int_{(0,a]} gd^-Z$, $\int_{(0,a]} gd^+Z$, and $\int_{(0,a]} g \circ dZ$. Some basic properties of above defined integrals can be found in [12, 29, 33]. We recall them in concise manner and give some new.

Definition 5 ([30])**.** For a stochastic càdlàg process g we set

$$\tilde{g}_t = (g_t)^\sim = g_{(1-t)-},$$

which is called a time-reversed process.

Proposition 6 ([12, 33])**.** *For RV-càdlàg processes g, Z and $0 < a < b \leq 1$ we have:*

(i) $$\int_{(a,b]} gd^\pm Z = \int_{(0,b]} gd^\pm Z - \int_{(0,a]} gd^\pm Z,$$

(ii) $$\int_{(a,b]} g \circ dZ = 1/2 \cdot (\int_{(a,b]} gd^+Z + \int_{(a,b]} gd^-Z),$$

(iii) $$\int_{(0,a)} gd^\pm Z = (\int_{(0,\cdot]} gd^\pm Z)_{a-} = \lim_{t\uparrow a} \int_{(0,t]} gd^\pm Z,$$

(iv) $$\int_{[a,b)} gd^\pm Z = \int_{(0,b)} gd^\pm Z - \int_{(0,a)} gd^\pm Z,$$

(v) $$\int_{(0,a]} \tilde{g}d^\pm \tilde{Z} = -\int_{[1-a,1)} gd^\mp Z$$

(vi) *Let Z be an $\mathbf{F}$-semimartingale and let g be an $\mathbf{F}$-adapted process. Then*

$$\int_{(0,\cdot]} gd^-Z = \int_{(0,\cdot]} g_{\tau-}dZ_\tau,$$

where the right-hand side integral denotes the semimartingale stochastic integral (for its definition and properties see e.g.: [30]).

Lemma 7 ([29])**.** *Let g be an $\mathbf{F}$-adapted RV-càdlàg process and let Z be an $\mathbf{F}$-semimartingale, $Z_0 = 0$. Let $0 < \alpha < \beta < 1$ be $\mathbf{F}$-stopping times. Then the forward integral satisfies*

(i) $$\int_{(0,\alpha]} gd^-Z = \int_{(0,1]} g\, \mathbb{1}_{[0,\alpha)}d^-Z = \int_{(0,1]} g_{\tau-}\, \mathbb{1}_{[0,\alpha]}(\tau)dZ_\tau,$$

(ii) $$\int_{(\alpha,\beta]} gd^-Z = \int_{(0,1]} g\, \mathbb{1}_{[\alpha,\beta)}d^-Z = \int_{(0,1]} g_{\tau-}\, \mathbb{1}_{(\alpha,\beta]}(\tau)dZ_\tau,$$

(iii) $$\int_{(\alpha,1]} gd^-Z = \int_{(0,1]} g\, \mathbb{1}_{[\alpha,1]}d^-Z = \int_{(0,1]} g_{\tau-}\, \mathbb{1}_{(\alpha,1]}(\tau)dZ_\tau.$$

Definition 8. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider on Ω two filtrations $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq 1}$ and $\mathbf{H} = (\mathcal{H}_t)_{0\leq t\leq 1}$ satisfying usual hypothesis. A càdlàg process x is $(\mathbf{F}, \mathbf{H})$-reversible if x is an $\mathbf{F}$-adapted process on $[0, 1]$ and $\tilde{x}$ is an $\mathbf{H}$-adapted process on $[0, 1]$, ([29]). A càdlàg process Z is an $(\mathbf{F}, \mathbf{H})$-reversible semimartingale, if Z is an $\mathbf{F}$-semimartingale on $[0, 1]$ and $\tilde{Z}$ is an $\mathbf{H}$-semimartingale on $[0, 1)$, ([30]).

Lemma 9. *Let g be an $(\mathbf{F},\mathbf{H})$-reversible RV-càdlàg process and let Z be an $(\mathbf{F},\mathbf{H})$-reversible semimartingale, $Z_0=0$. Then for all $0<a<b<1$ we get*

(i) $\int_{(0,a]} gd^+Z = -\int_{(0,1]} \tilde{g}\cdot \mathbb{I}_{[1-a,1)}d^-\tilde{Z} = -\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[1-a,1)}(\tau)d\tilde{Z}_\tau,$

(ii) $\int_{(a,b]} gd^+Z = -\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[1-b,1-a)}(\tau)d\tilde{Z}_\tau,$

(iii) $\int_{[a,1)} gd^+Z = -\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[0,1-a]}(\tau)d\tilde{Z}_\tau,$

where the right-hand side integral denotes the semimartingale stochastic integral (for its definition and properties see e.g.: [30]).

Proof. (i) Using Proposition 6(v), (iv), (iii) and (vi) we get

$$\begin{aligned}
J &= \int_{(0,a]} gd^+Z = -\int_{[1-a,1)} \tilde{g}d^-\tilde{Z} = -\Big(\int_{(0,1)} \tilde{g}d^-\tilde{Z} - \int_{(0,1-a)} \tilde{g}d^-\tilde{Z}\Big)\\
&= -\Big(\lim_{s\uparrow 1}\Big(\int_{(0,s]} \tilde{g}d^-\tilde{Z}\Big) - \lim_{\rho\uparrow 1-a}\Big(\int_{(0,\rho]} \tilde{g}d^-\tilde{Z}\Big)\Big)\\
&= -\Big(\lim_{s\uparrow 1}\Big(\int_{(0,s]} \tilde{g}_{\tau-}d\tilde{Z}_\tau\Big) - \lim_{\rho\uparrow 1-a}\Big(\int_{(0,\rho]} \tilde{g}_{\tau-}d\tilde{Z}_\tau\Big)\Big).
\end{aligned}$$

By the property of the limit operator and definition of Itô-type stochastic integral over $(\rho,s]$ we get

$$J = -\lim_{s\uparrow 1}\lim_{\rho\uparrow 1-a}\Big(\int_{(0,s]} \tilde{g}_{\tau-}d\tilde{Z}_\tau - \int_{(0,\rho]} \tilde{g}_{\tau-}d\tilde{Z}_\tau\Big) = -\lim_{s\uparrow 1}\lim_{\rho\uparrow 1-a}\Big(\int_{(\rho,s]} \tilde{g}_{\tau-}d\tilde{Z}_\tau\Big).$$

Using properties of the indicator function and Itô-type stochastic integral, and Dominated Convergence Theorem we get

$$J = -\lim_{s\uparrow 1}\lim_{\rho\uparrow 1-a}\Big(\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{(\rho,s]}(\tau)d\tilde{Z}_\tau\Big) = -\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[1-a,1)}(\tau)d\tilde{Z}_\tau.$$

(ii) By Proposition 6(i) and above proved equality (i) we get

$$\begin{aligned}
J &= \int_{(a,b]} gd^+Z = \int_{(0,b]} gd^+Z - \int_{(0,a]} gd^+Z\\
&= -\int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[1-b,1)}(\tau)d\tilde{Z}_\tau + \int_{(0,1]} \tilde{g}_{\tau-}\cdot \mathbb{I}_{[1-a,1)}(\tau)d\tilde{Z}_\tau.
\end{aligned}$$

Using definition of Itô-type stochastic integral over $[a, b)$, (see e.g. [30]), and Dominated Convergence Theorem we get

$$J = -\int_{(0,1]} \tilde{g}_{\tau-} \cdot 1\!\!1_{[1-b,1-a)}(\tau) d\tilde{Z}_{\tau}.$$

(iii) By Proposition 6(v) and Lemma 7(i) we get

$$\int_{[a,1)} g d^{+} Z = -\int_{(0,1-a]} \tilde{g} d\tilde{Z} = -\int_{(0,1]} \tilde{g}_{\tau-} \cdot 1\!\!1_{[0,1-a]}(\tau) d\tilde{Z}_{\tau}.$$

■

Now we define a set-valued Stratonovich-type stochastic integral.

We recall now some basic definitions. Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be as before. A set-valued function $G : \Omega \to cc(I\!R^n)$ is $\mathcal{F}$-measurable if for every closed subset $C \in I\!R^n$ one has: $\{\omega \in \Omega \,:\, G(\omega) \cap C \neq \emptyset\} \in \mathcal{F}$.

By a set-valued stochastic process G with values in $cc(I\!R^n)$ we consider a family of $\mathcal{F}$-measurable set-valued mappings $G_t \,:\, \Omega \to cc(I\!R^n)$, each $t \in [0, 1]$. A stochastic set-valued process $G : [0, 1] \times \Omega \to cc(I\!R^n)$ is $\mathbf{F}$-adapted if for every $t \in [0, 1]$ a map $G_t : \Omega \to cc(I\!R^n)$ is $\mathcal{F}_t$-measurable.

For every fixed $\omega \in \Omega$ as a sample path of the set-valued process G we mean a set-valued function $G_\omega \,:\, [0, 1] \to cc(I\!R^n)$ such that $G_\omega(t) = G(t, \omega)$.

A set-valued process $G = (G_t)_{t\in[0,1]}$ is $\mathbf{F}$-predictable if it is $\mathcal{P}(\mathbf{F})$-measurable and the family of all such processes is also denoted by $\mathcal{P}(\mathbf{F})$. One has $\mathcal{P}(\mathbf{F}) \subset \beta \otimes \mathcal{F}$, where β denotes the Borel σ-algebra on $[0, 1]$, (see e.g. [29, 34]).

Definition 10 ([29])**.** A stochastic set-valued process G is càdlàg if it has right continuous sample paths with left limits with respect to the Hausdorff metric. Similarly we define a set-valued càglàd process. A stochastic set-valued process G is RV-càdlàg (RV-càglàd) if it is càdlàg (càglàd) and continuous for $t = 0$ and $t = 1$.

Definition 11 ([29])**.** For a stochastic set-valued càdlàg process G we set

$$\tilde{G}_t = (G_t)^{\sim} = G_{(1-t)-},$$

which is called a time-reversed process. The limit of the set-valued map is taken with respect to the Hausdorff metric.

Let $(\Omega, \mathcal{F}, [\mathbf{F}, \mathbf{H}], P)$ be as before.

Definition 12 ([29])**.** A set-valued càdlàg process G is $(\mathbf{F}, \mathbf{H})$-reversible if G is an $\mathbf{F}$-adapted process on $[0, 1]$ and $\tilde{G}$ is an $\mathbf{H}$-adapted process on $[0, 1]$.

Lemma 13 ([29])**.** *Let G be a set-valued $(\mathbf{F}, \mathbf{H})$-reversible process. Then there exists a selection g of G being an $(\mathbf{F}, \mathbf{H})$-reversible process.*

Remark 14 ([29])**.** Lemma 13 is also true if we additionally assume that paths of G are left continuous at $t = 1$, (then the obtained selection g is RV-càdlàg).

Definition 15 ([29])**.** Let G be a set-valued $(\mathbf{F}, \mathbf{H})$-reversible RV-càdlàg process and let Z be an $(\mathbf{F}, \mathbf{H})$-reversible semimartingale, $Z_0 = 0$. Let $S(G)$ denote the family of all $(\mathbf{F}, \mathbf{H})$-reversible RV-càdlàg selections of G. For every $0 \le a < b \le 1$ we define

$$\int_{(a,b]} G \circ dZ = \left\{ 1/2 \left(\int_{(a,b]} g d^- Z + \int_{(a,b]} g d^+ Z \right) \ : \ g \in S(G) \right\} \tag{1}$$

$$= \left\{ 1/2 \left(\int_{(a,b]} g_{\tau-} dZ_\tau - \int_{[1-b,1-a)} \tilde{g}_{\tau-} d\tilde{Z}_\tau \right) \ : \ g \in S(G) \right\}. \tag{2}$$

Remark 16 ([29])**.** It follows by Lemma 13 that the set $S(G)$ is nonempty and therefore, the set-valued integral defined above exists.

Remark 17. Note that the family $S(G)$ of all $(\mathbf{F}, \mathbf{H})$-reversible RV-càdlàg selections of G is, in some sense, a set of the form: $S(G) = S_-(G) \cup S_+(\tilde{G})$, where $S_-(G)$ is a set of $\mathbf{F}$-adapted RV-càdlàg selections of G and $S_+(\tilde{G})$ is a set of $\mathbf{H}$-adapted RV-càdlàg selections of $\tilde{G}$.

Definition 18. A stochastic process x is $(\mathbf{F}, \mathbf{H})$-adapted if there exist stochastic processes u, v such that u is an $\mathbf{F}$-adapted process on $[0, 1]$, v is an $\mathbf{H}$-adapted process on $[0, 1]$ and $x = u + v$.

Definition 19 ([29])**.** A set-valued process G is integrably bounded if there exists an $(\mathbf{F}, \mathbf{H})$-reversible RV-càdlàg process m, $\|m\|_{S^\infty} < \infty$ and such that $H(G_t, \{0\}) \le m_t$, each $t \in [0, 1]$.

Remark 20. Note that integrably boundedness of a process G means that G is μ_Z-squareintegrably bounded and $\tilde{G}$ is $\mu_{\tilde{Z}}$-squareintegrably bounded.

5. Stratonovich-type stochastic differential inclusion

In this section we prove non-emptiness and closedness of the solution set of Stratonovich-type stochastic differential inclusion of the form (SSI), (in Definition 23).

Let $(\Omega, \mathcal{F}, [\mathbf{F}, \mathbf{H}], P)$ be as before.

Let $F : [0, 1] \times I\!R^n \to cc(I\!R^n)$ be a $(\beta \otimes \mathcal{F})$-measurable multifunction.

Let $S^2([0, 1])$ denote the space of $I\!R^n$-valued $(\mathbf{F}, \mathbf{H})$-adapted cádlág processes with a norm $\|x\|_{S^2} = \| \sup_{t \in [0,1]} |x_t| \|_{L^2(\Omega)}$.

For any $x \in S^2([0, 1])$ and a multifunction F, by $(F \circ x)_-$, where $x_{t-} = \lim_{s \uparrow t} x_s$, we denote a set-valued process $(F(t, x_{t-}(\omega)))_{t \in [0,1]}$.

Let $x \in S^2([0,1])$, $Z \in \mathcal{H}^\infty$ and $F(t,\cdot)$ be continuous for any $t \in [0,1]$. If a process $F \circ x$ is integrably bounded, then the sets $\mathcal{S}_{\mu_Z}((F \circ x)_-)$ and $\mathcal{S}_{\mu_{\tilde{Z}}}((F \circ \tilde{x})_-)$ are nonempty in $L^2_{\mu_Z}$ and $L^2_{\mu_{\tilde{Z}}}$, respectively. It follows by Lemma 3 and Remark 20.

By Definition 15 we have two equivalent possibilities to define the Stratonovich-type stochastic integral: (1) and (2). In the following we use (2), but we expand the choice of selections.

Definition 21. Let G be a set-valued $(\mathbf{F},\mathbf{H})$-reversible RV-càdlàg process and let Z be an $(\mathbf{F},\mathbf{H})$-reversible semimartingale, $Z_0 = 0$. For every $0 \le a < b \le 1$ we define

$$\int_{(a,b]} G \circ dZ = \Bigg\{ 1/2 \left(\int_{(a,b]} g_\tau dZ_\tau - \int_{[1-b,1-a)} \tilde{h}_\tau d\tilde{Z}_\tau \right) : \\ g \in \mathcal{S}_{\mu_Z}(G_-) \text{ and } \tilde{h} \in \mathcal{S}_{\mu_{\tilde{Z}}}(\tilde{G}_-) \Bigg\}.$$

For convenience we introduce the following notation

$$\int_{(a,b]} (g,\tilde{h}) \circ dZ := 1/2 \left(\int_{(a,b]} g_{\tau-} dZ_\tau - \int_{[1-b,1-a)} \tilde{h}_{\tau-} d\tilde{Z}_\tau \right).$$

Remark 22. Let us note that for $g \in S(G)$, (in Definition 15), we get that $g_- \in \mathcal{S}_{\mu_Z}(G_-)$ and $\tilde{g}_- \in \mathcal{S}_{\mu_{\tilde{Z}}}(\tilde{G}_-)$, (in Definition 21). Thus, the set-valued Stratonovich-type stochastic integral defined by Definition 15 is, in some sense, a subset of this defined by Definition 21.

Definition 23. Let Z be an $(\mathbf{F},\mathbf{H})$-reversible semimartingale from $\mathcal{H}^\infty$, $Z_0 = 0$, $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ and $s,t \in [0,1]$, $s < t$. We consider the Stratonovich-type stochastic inclusion

$$\text{(SSI)} \qquad x_t - x_s \in cl_{L^2(\Omega)}\left(\int_{(s,t]} F(\tau, x_\tau) \circ dZ \right)$$

with $x_0 = \xi \in L^2(\Omega, [\mathcal{F}_0, \mathcal{H}_1], P; I\!R^n)$, i.e., $\xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$ and $\tilde{\xi} \in L^2(\Omega, \mathcal{H}_1, P; I\!R^n)$. A process $x \in S^2([0,1])$ is a solution of the stochastic inclusion (SSI), if $x_0 = \xi$ and for any $s,t \in [0,1]$, $s < t$ a random variable $x_t - x_s$ belongs to the set

$$cl_{L^2(\Omega)}\left(\int_{(s,t]} F(\tau, x_\tau) \circ dZ \right).$$

A set of all solutions of the stochastic inclusion (SSI) is denoted by

$$\mathcal{T}(\xi, Z, F) = \{x \in S^2([0,1]) : x \text{ is a solution of (SSI)}\}.$$

We say that $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ is a Lipschitz multifunction if there exists a constant D such that for all $t \in [0,1]$ and $x, y \in I\!R^n$

$$H(F(t,x), F(t,y)) \leq D|x-y|.$$

Assumption 1
Let $F : [0,T] \times I\!R^n \to cc(I\!R^n)$ be a multifunction satisfying

(i) $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ is $(\beta \otimes \mathcal{F})$-measurable,

(ii) $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ is a Lipschitz multifunction,

(iii) for any $x \in S^2([0,1])$ a set-valued process $F \circ x$ is integrably bounded.

Now we prove non-emptiness and closedness of the set of solutions $\mathcal{T}(\xi, Z, F)$.

Theorem 24. *Let Z be an $(\mathbf{F}, \mathbf{H})$-reversible semimartingale from $\mathcal{H}^\infty$, $Z_0 = 0$. Let $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ be a multifunction satisfying Assumption 1. Then for any $\xi \in L^2(\Omega, [\mathcal{F}_0, \mathcal{H}_1], P; I\!R^n)$ the set $\mathcal{T}(\xi, Z, F)$ is nonempty.*

Proof. In the proof we use Covitz-Nadler Theorem, (see e.g.: [19] Th.II.4.4). Let $N^1 + A^1$ be a decomposition of the semimartingale Z, i.e., $Z = N^1 + A^1$ and $\tilde{N}^2 + \tilde{A}^2$ be a decomposition of the semimartingale $\tilde{Z}$, i.e., $\tilde{Z} = \tilde{N}^2 + \tilde{A}^2$. Let us divide the interval $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$. Let $c^i_Z = (\int_{(t_{i-1},t_i]} d[N^1, N^1]_\tau)^{1/2} + \int_{(t_{i-1},t_i]} |dA^1_\tau|$ and $c^i_{\tilde{Z}} = (\int_{[1-t_i,1-t_{i-1})} d[\tilde{N}^2, \tilde{N}^2]_\tau)^{1/2} + \int_{[1-t_i,1-t_{i-1})} |d\tilde{A}^2_\tau|$, for $i = 1, \ldots, k$. We choose the points t_i such that

$$\max_{t_i} \{Dc_2 \|c^i_Z\|_{L^\infty(\Omega)}, Dc_2 \|c^i_{\tilde{Z}}\|_{L^\infty(\Omega)}\} < 1,$$

where a constant c_2 comes from [30] Th.V.2.2.

First, we construct a solution of the stochastic inclusion (SSI) on $[0, t_1]$. For any $\xi \in L^2(\Omega, [\mathcal{F}_0, \mathcal{H}_1], P; I\!R^n)$ and $x \in S^2([0, t_1])$ we define a map Γ by

$$\Gamma(x) = \Bigg\{ y \in S^2([0,t_1]) : \ y_t = \xi + 1/2 \left(\int_{(0,t]} f^1_\tau dZ_\tau - \int_{[1-t,1)} \tilde{f}^2_\tau d\tilde{Z}_\tau \right),$$
$$\text{where } f^1 \in \mathcal{S}_{\mu_Z}(F \circ x)_- \text{ and } \tilde{f}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F \circ x)^{\sim}_-), \text{ for } (t,\omega) \in (0,t_1] \times \Omega \Bigg\}.$$

Let x be an arbitrary element of $S^2((0, t_1])$.

Thanks to Assumption 1(iii), Remark 20 and Lemma 3 it follows that $\Gamma(x)$ is nonempty.

$\Gamma(x)$ is not necessarily a closed set in $S^2((0, t_1])$ (in a sense of $\|\cdot\|_{S^2}$-norm).

Let us consider a set $cl_{S^2}(\Gamma(x))$. It is a closure of the set $\Gamma(x)$ in $S^2((0, t_1])$. This set is a nonempty, bounded and closed subset of $S^2((0, t_1])$.

We show that a map $x \to cl_{S^2}(\Gamma(x))$ is a set-valued contraction in $S^2((0,t_1])$.

Let u and v be arbitrary elements of $S^2((0,t_1])$. We show that there exists a constant $K \in [0,1)$ such that

$$H_{S^2}(cl_{S^2}(\Gamma(u)), cl_{S^2}(\Gamma(v))) \leq K\|u-v\|_{S^2}.$$

Let y be an arbitrary element of $cl_{S^2}(\Gamma(u))$. For any $\epsilon > 0$ there exists a stochastic process $y^1 \in \Gamma(u)$ such that $\|y - y^1\|_{S^2} < \epsilon$. It can be represented as $y^1_t = \xi + 1/2(\int_{(0,t]} f^1_\tau dZ_\tau - \int_{[1-t,1)} \tilde{f}^2_\tau d\tilde{Z}_\tau)$ for some $f^1 \in \mathcal{S}_{\mu_Z}((F \circ u)_-)$ and $\tilde{f}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F \circ u)^{\sim}_-)$ on $(0,t_1] \times \Omega$ and on $[1-t_1,1) \times \Omega$, respectively. It follows by the definition of the set $\Gamma(u)$. From Filippov Theorem (see e.g.: [19] Th.II.3.12) there exist $g^1 \in \mathcal{S}_{\mu_Z}((F \circ v)_-)$ such that

$$|f^1(t,\omega) - g^1(t,\omega)| \leq \operatorname{dist}(f^1(t,\omega), F(t, v(t-,\omega))) + \epsilon, \tag{3}$$

for any $t \in (0,t_1]$ and a.a. $\omega \in \Omega$. Similarly, there exists $\tilde{g}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F \circ v)^{\sim}_-)$ such that

$$|\tilde{f}^2(t,\omega) - \tilde{g}^2(t,\omega)| \leq \operatorname{dist}(\tilde{f}^2(t,\omega), (F(t, v(\cdot,\omega)))^{\sim}_{t-}) + \epsilon, \tag{4}$$

for any $t \in [1-t_1,1)$ and a.a. $\omega \in \Omega$.

Let $y^2_t = \xi + 1/2(\int_{(0,t]} g^1_\tau dZ_\tau - \int_{[1-t,1)} \tilde{g}^2_\tau d\tilde{Z}_\tau)$ for $t \in [0,t_1]$. From the definition of the set $\Gamma(v)$ we get $y^2 \in \Gamma(v)$. Let us estimate the distance between y and y^2 in $S^2([0,t_1])$. We get

$$\begin{aligned} J \;=\; & \|y - y^2\|_{S^2} \leq \|y - y^1\|_{S^2} + \|y^1 - y^2\|_{S^2} \\ \leq\; & \epsilon + 1/2\|\int (f^1_\tau - g^1_\tau) dZ_\tau\|_{S^2} + 1/2\|\int (\tilde{f}^2_\tau - \tilde{g}^2_\tau) d\tilde{Z}_\tau\|_{S^2}. \end{aligned}$$

For arbitrary $f^1 \in (F \circ u)_-$ and $\tilde{f}^2 \in ((F \circ u)_-)^{\sim}$ we have

$$\operatorname{dist}(f^1(t,\omega), F(t, v(t-,\omega))) \leq H(F(t, u_{t-}), F(t, v_{t-})),$$

for any $t \in (0,t_1]$ and a.a. $\omega \in \Omega$, and

$$\operatorname{dist}(\tilde{f}^2(t,\omega), (F(t, v(t-,\omega)))^{\sim}) \leq H((F(t,u_{t-})^{\sim}), (F(t, v_{t-}))^{\sim}),$$

for any $t \in [1-t_1,1)$ and a.a. $\omega \in \Omega$.

So by (3) and (4) we get

$$\begin{aligned} J \;\leq\; & \epsilon + 1/2c_2\|c^1_Z\|_{L^\infty(\Omega)}\|\sup_{t\in(0,t_1]}(H(F(t,u_{t-}), F(t,v_{t-})) + \epsilon)\|_{L^2(\Omega)} \\ & + 1/2c_2\|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\|\sup_{t\in[1-t_1,1)}(H((F(t,u_{t-}))^{\sim}, (F(t,v_{t-}))^{\sim}) + \epsilon)\|_{L^2(\Omega)}. \end{aligned}$$

By Lipschitz condition for the multifunction F we get

$$\begin{aligned} J &\le \epsilon + 1/2c_2\|c^1_Z\|_{L^\infty(\Omega)}\|\sup_{t\in(0,t_1]}(D|u_t-v_t|+\epsilon)\|_{L^2(\Omega)} \\ &+ 1/2c_2\|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\|\sup_{t\in[1-t_1,1)}(D|u_t-v_t|+\epsilon)\|_{L^2(\Omega)} \\ &\le 1/2Dc_2(\|c^1_Z\|_{L^\infty(\Omega)}\|\sup_{t\in(0,t_1]}|u_t-v_t|\|_{L^2(\Omega)} \\ &+ \|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\|\sup_{t\in[1-t_1,1)}|u_t-v_t|\|_{L^2(\Omega)}) + \epsilon_1 \\ &\le Dc_2\max\{\|c^1_Z\|_{L^\infty(\Omega)}, \|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\}\|u-v\|_{S^2} + \epsilon_1, \end{aligned}$$

where $\epsilon_1 = (1/2c_2(\|c^1_Z\|_{L^\infty(\Omega)} + \|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}) + 1)\epsilon$. Thus there exists a constant $K = Dc_2\max\{\|c^1_Z\|_{L^\infty(\Omega)}, \|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\}$ which does not depend on the choice of the element y from the set $cl_{S^2}(\Gamma(u))$. Therefore,

$$\|y_t - y^2_t\|_{S^2} \le K\|u-v\|_{S^2} + \epsilon_1.$$

Since $\epsilon > 0$ was arbitrarily chosen, the distance from an element $y \in cl_{S^2}(\Gamma(u))$ to the set $cl_{S^2}(\Gamma(v))$ can be estimated by

$$\mathrm{dist}_{S^2}(y, cl_{S^2}(\Gamma(v))) \le K\|u-v\|_{S^2}.$$

Thus

$$H_{S^2}(cl_{S^2}(\Gamma(u)), cl_{S^2}(\Gamma(v))) \le K\|u-v\|_{S^2}.$$

The constant $K = Dc_2\max\{\|c^1_Z\|_{L^\infty(\Omega)}, \|c^1_{\tilde{Z}}\|_{L^\infty(\Omega)}\}$ is a nonnegative number less than 1, and therefore, the map $cl_{S^2}(\Gamma(\cdot))$ is a set-valued contraction in $S^2((0,t_1])$.

From Covitz-Nadler Theorem we conclude there exists a process $y \in S^2((0,t_1])$ such that $y \in cl_{S^2}(\Gamma(x))$. For any $\epsilon > 0$ we can choose $y^\epsilon \in \Gamma(x)$ satisfying

$$\|y - y^\epsilon\|_{S^2} < \epsilon.$$

By the definition of the set $\Gamma(x)$ there exist $g^{1,\epsilon} \in \mathcal{S}_{\mu_Z}((F\circ x)_-)$ and $\tilde{g}^{2,\epsilon} \in \mathcal{S}_{\mu_{\tilde{Z}}}((F\circ x)^{\sim}_-)$ such that $y^\epsilon_t = \xi + 1/2\int_{(0,t]} g^{1,\epsilon}_\tau dZ_\tau - 1/2\int_{[1-t,1)}\tilde{g}^{2,\epsilon}_\tau d\tilde{Z}_\tau$ for any $t \in (0,t_1]$. Therefore,

$$\left\|\sup_{t\in(0,t_1]}\left|y_t - \left(\xi + 1/2\int_{(0,t]} g^{1,\epsilon}_\tau dZ_\tau - 1/2\int_{[1-t,1)}\tilde{g}^{2,\epsilon}_\tau d\tilde{Z}_\tau\right)\right|\right\|_{L^2(\Omega)} < \epsilon.$$

Thus for any $t \in (0, t_1]$

$$\left\| y_t - \left(\xi + 1/2 \int_{(0,t]} g_\tau^{1,\epsilon} dZ_\tau - 1/2 \int_{[1-t,1)} \tilde{g}_\tau^{2,\epsilon} d\tilde{Z}_\tau\right)\right\|_{L^2(\Omega)} < \epsilon. \tag{5}$$

Now we show that the above process y is a solution of the stochastic inclusion (SSI) on $(0, t_1]$. We do this by checking that $y_t - y_s \in cl_{L^2(\Omega)}(\int_{(s,t]} F(\tau, x_\tau) \circ dZ)$ for any $s, t \in (0, t_1]$, $s < t$.

By Definition 21 and (5) we get that for every $s, t \in (0, t_1]$, $s < t$

$$\left\| y_t - y_s - 1/2 \int_{(s,t]} g_\tau^{1,\epsilon} dZ_\tau + 1/2 \int_{[1-t,1-s)} \tilde{g}_\tau^{2,\epsilon} d\tilde{Z}_\tau \right\|_{L^2(\Omega)} < \epsilon.$$

Since $\epsilon > 0$ was arbitrarily chosen, we obtain

$$y_t - y_s \in cl_{L^2(\Omega)}\left(\int_{(s,t]} F(\tau, x_\tau) \circ dZ\right), \text{ for any } s, t \in (0, t_1], \quad s < t.$$

Let $i = 2$. The proof is similar to the $i = 1$ case. We should only change the interval $(0, t_1]$ into $(t_1, t_2]$ and take the starting point of the constructed solution equal to y_{t_1}. In a similar way we obtain a process $y \in S^2((t_1, t_2])$ and for an arbitrary $\epsilon > 0$ processes $g^{1,\epsilon} \in \mathcal{S}_{\mu_Z}((F \circ y)_-)$ and $\tilde{g}^{2,\epsilon} \in \mathcal{S}_{\mu_{\tilde{Z}}}((F \circ y)_-^{\sim})$ such that

$$\left\| y_t - \left(y_{t_1} + 1/2 \int_{(t_1,t]} g_\tau^{1,\epsilon} dZ_\tau - 1/2 \int_{[1-t,1-t_1)} \tilde{g}_\tau^{2,\epsilon} d\tilde{Z}_\tau\right)\right\|_{L^2(\Omega)} < \epsilon,$$

for any $t \in (t_1, t_2]$.

The above inequality means that for every $s, t \in (t_1, t_2]$, $s < t$, the stochastic process y is an element of the closure in a sense of an $L^2(\Omega)$-norm of the set

$$\int_{(s,t]} F(\tau, x_\tau) \circ dZ,$$

Therefore, y is a solution of the inclusion (SSI) on the interval $(t_1, t_2]$.

When we repeat the above construction for $i = 2, 3, \ldots, k-1$, taking starting points of the constructed solutions equal to y_{t_i}, we get solutions of the inclusion (SSI) on the intervals $(t_i, t_{i+1}]$.

The solution of the inclusion (SSI) for $s, t \in [0, 1]$, $s < t$ is a composition of the solutions constructed on the intervals $(0, t_1]$ and $(t_i, t_{i+1}]$, $i = 1, \ldots, k-1$. ■

Assumption 2

Let Z be an $(\mathbf{F}, \mathbf{H})$-reversible semimartingale from $\mathcal{H}^\infty$, $Z_0 = 0$ such that

(i) $Z = N^1 + A^1$, where N^1 – local $\mathbf{F}$-martingale and A^1 – deterministic FV-process,

(ii) $\tilde{Z} = \tilde{N}^2 + \tilde{A}^2$, where $\tilde{N}^2$ – local $\mathbf{H}$-martingale and $\tilde{A}^2$ – deterministic FV-process,

Theorem 25. *Let Z be an $(\mathbf{F}, \mathbf{H})$-reversible semimartingale from $\mathcal{H}^\infty$, $Z_0 = 0$ and satisfies Assumption 2. Let $F : [0,1] \times I\!R^n \to cc(I\!R^n)$ be a multifunction satisfying Assumption 1. Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$ the set $\mathcal{T}(\xi, Z, F)$ is closed in $S^2([0,1])$.*

Proof. Let $\{x^k\}_{k\geq 1}$ be a sequence of elements of the set of all solutions of the stochastic inclusion (SSI), which converges to the limit x in $S^2([0,1])$.

We have to show that the limit x belongs to the set $\mathcal{T}(\xi, Z, F)$, i.e., for any $s, t \in [0,1]$, $s < t$

$$\mathrm{dist}_{L^2(\Omega)}\bigg(x_t - x_s, \int_{(s,t]} F(\tau, x_\tau) \circ dZ\bigg) = 0.$$

Observe that

$$\begin{aligned} I \;=\; & \mathrm{dist}_{L^2(\Omega)}\bigg(x_t - x_s, \int_{(s,t]} F(\tau, x_\tau) \circ dZ\bigg) \\ \leq\; & \|x_t - x_s - (x_t^k - x_s^k)\|_{L^2(\Omega)} \\ +\; & H_{L^2(\Omega)}\bigg(\int_{(s,t]} F(\tau, x_\tau^k) \circ dZ, \int_{(s,t]} F(\tau, x_\tau) \circ dZ\bigg) = I_1 + I_2, \end{aligned}$$

and

$$I_1 = \|x_t - x_s - (x_t^k - x_s^k)\|_{L^2(\Omega)} \leq 2\|x - x^k\|_{S^2} \to 0,$$

while $k \to \infty$.

In order to analyze I_2, let $g^1 \in \mathcal{S}_{\mu_Z}(F \circ x^k)_-$ and $\tilde{g}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F \circ x^k)^{\sim}_-)$ be arbitrary elements. We have

$$\begin{aligned} & \inf_{f^1 \in \mathcal{S}_{\mu_Z}(F\circ x)_-} \inf_{\tilde{f}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F\circ x)^{\sim}_-)} \bigg\| \int_{(s,t]} (g^1, \tilde{g}^2) \circ dZ - \int_{(s,t]} (f^1, \tilde{f}^2) \circ dZ \bigg\|^2_{L^2(\Omega)} \\ =\; & 1/4 \inf_{f^1 \in \mathcal{S}_{\mu_Z}(F\circ x)_-} \inf_{\tilde{f}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F\circ x)^{\sim}_-)} \bigg\| \int_{(s,t]} g^1_\tau dZ_\tau - \int_{[1-t,1-s)} \tilde{g}^2_\tau d\tilde{Z}_\tau \\ -\; & \int_{(s,t]} f^1_\tau dZ_\tau + \int_{[1-t,1-s)} \tilde{f}^2_\tau d\tilde{Z}_\tau \bigg\|^2_{L^2(\Omega)} \\ \leq\; & 1/2 \bigg(\inf_{f^1 \in \mathcal{S}_{\mu_Z}(F\circ x)_-} \bigg\| \int_{(s,t]} (g^1_\tau - f^1_\tau) dZ_\tau \bigg\|^2_{L^2(\Omega)} \\ +\; & \inf_{\tilde{f}^2 \in \mathcal{S}_{\mu_{\tilde{Z}}}((F\circ x)^{\sim}_-)} \bigg\| \int_{[1-t,1-s)} (\tilde{g}^2_\tau - \tilde{f}^2_\tau) d\tilde{Z}_\tau \bigg\|^2_{L^2(\Omega)} \bigg) = 1/2(J_1 + J_2). \end{aligned}$$

Using Lemma 1 and [15] Theorem 2.2 for J_1, we get

$$J_1 \leq 2\int_{(s,t]\times\Omega} \inf_{y^1\in F(\tau,x_{\tau-})} |g^1_\tau - y^1|^2 d\mu_Z = 2\int_{(s,t]\times\Omega} \operatorname{dist}^2(g^1_\tau, F(\tau,x_{\tau-}))d\mu_Z.$$

Considering the component J_2 we get

$$\begin{aligned} J_2 &= \inf_{\tilde f^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x)\tilde{}_-)} \left\| \int_{[1-t,1-s)} (\tilde g^2_\tau - \tilde f^2_\tau) d\tilde Z_\tau \right\|^2_{L^2(\Omega)} \\ &= \lim_{\alpha\uparrow 1-t}\lim_{\beta\uparrow 1-s} \inf_{\tilde f^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x)\tilde{}_-)} \left\| \int_{(\alpha,\beta]} (\tilde g^2_\tau - \tilde f^2_\tau) d\tilde Z_\tau \right\|^2_{L^2(\Omega)} \\ &\leq 2\lim_{\alpha\uparrow 1-t}\lim_{\beta\uparrow 1-s} \int_{(\alpha,\beta]\times\Omega} \operatorname{dist}^2(\tilde g^2_\tau, \tilde F(\tau,x_{\tau-}))d\mu_{\tilde Z}. \end{aligned}$$

Using Lebesgue Dominated Convergence Theorem we obtain

$$J_2 \leq 2\int_{[1-t,1-s)\times\Omega} \operatorname{dist}^2(\tilde g^2_\tau, \tilde F(\tau,x_{\tau-}))d\mu_{\tilde Z},$$

and finally

$$\begin{aligned} &1/2(J_1+J_2) \\ &\leq \int_{(s,t]\times\Omega} \operatorname{dist}^2(g^1_\tau, F(\tau,x_{\tau-}))d\mu_Z + \int_{[1-t,1-s)\times\Omega} \operatorname{dist}^2(\tilde g^2_\tau, \tilde F(\tau,x_{\tau-}))d\mu_{\tilde Z}. \end{aligned}$$

Observe that we get similar result for $\inf_{g^1\in\mathcal{S}_{\mu_Z}(F\circ x^k)_-} \inf_{\tilde g^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x^k)\tilde{}_-)} \| \int_{(s,t]}(g^1,\tilde g^2)\circ dZ - \int_{(s,t]}(f^1,\tilde f^2)\circ dZ\|^2_{L^2(\Omega)}$, when $f^1\in\mathcal{S}_{\mu_Z}(F\circ x)_-$ and $\tilde f^2\in\mathcal{S}_{\mu_{\tilde Z}}((F\circ x)\tilde{}_-)$ be arbitrary elements. Moreover,

$$\begin{aligned} &\sup_{g^1\in\mathcal{S}_{\mu_Z}(F\circ x^k)_-} \sup_{\tilde g^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x^k)\tilde{}_-)} \inf_{f^1\in\mathcal{S}_{\mu_Z}(F\circ x)_-} \inf_{\tilde f^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x)\tilde{}_-)} \\ &\left\| \int_{(s,t]} (g^1,\tilde g^2)\circ dZ - \int_{(s,t]} (f^1,\tilde f^2)\circ dZ \right\|^2_{L^2(\Omega)} \\ &\leq \int_{(s,t]\times\Omega} \overline{h}^2(F(\tau,x^k_{\tau-}),F(\tau,x_{\tau-}))d\mu_Z + \int_{[1-t,1-s)\times\Omega} \overline{h}^2(\tilde F(\tau,x^k_{\tau-}),\tilde F(\tau,x_{\tau-}))d\mu_{\tilde Z} \end{aligned}$$

and similar for $\sup_{f^1\in\mathcal{S}_{\mu_Z}(F\circ x)_-} \sup_{\tilde f^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x)\tilde{}_-)} \inf_{g^1\in\mathcal{S}_{\mu_Z}(F\circ x^k)_-} \inf_{\tilde g^2\in \mathcal{S}_{\mu_{\tilde Z}}((F\circ x^k)\tilde{}_-)}$ $\| \int_{(s,t]}(g^1,\tilde g^2)\circ dZ - \int_{(s,t]}(f^1,\tilde f^2)\circ dZ\|^2_{L^2(\Omega)}$.

Finally, we get

$$(I_2)^2 \leq \int_{(s,t]\times\Omega} H^2(F(\tau,x^k_{\tau-}),F(\tau,x_{\tau-}))d\mu_Z + \int_{[1-t,1-s)\times\Omega} H^2(\tilde{F}(\tau,x^k_{\tau-}),\tilde{F}(\tau,x_{\tau-}))d\mu_{\tilde{Z}}.$$

Using Lipschitz condition for the set-valued process F, we get

$$\begin{aligned}
&(I_2)^2 \leq D^2\int_{(s,t]\times\Omega}|x^k_{\tau-}-x_{\tau-}|^2 d\mu_Z + \int_{[1-t,1-s)\times\Omega}|x^k_{\tau-}-x_{\tau-}|^2 d\mu_{\tilde{Z}}\\
&= D^2\cdot\Bigg(E\int_{(s,t]}|x^k_{\tau-}-x_{\tau-}|^2 d[N^1,N^1]_\tau + E\Bigg(c_{A^1}\cdot\int_{(s,t]}|x^k_{\tau-}-x_{\tau-}|^2|dA^1_\tau|\Bigg)\\
&+ E\int_{[1-t,1-s)}|x^k_{\tau-}-x_{\tau-}|^2 d[\tilde{N}^2,\tilde{N}^2]_\tau + E\Bigg(c_{\tilde{A}^2}\cdot\int_{[1-t,1-s)}|x^k_{\tau-}-x_{\tau-}|^2|d\tilde{A}^2_\tau|\Bigg)\Bigg),
\end{aligned}$$

where: c_{A^1} and $c_{\tilde{A}^2}$ are defined in Section 3. Since A^1 and $\tilde{A}^2$ are deterministic FV-processes, we get

$$\begin{aligned}
&(I_2)^2 \leq D^2\cdot\max\{1,c_{A^1},c_{\tilde{A}^2}\}\\
&\cdot\Bigg(E\int_{(s,t]}|x^k_{\tau-}-x_{\tau-}|^2 d[N^1,N^1]_\tau + E\Bigg(\int_{(s,t]}|x^k_{\tau-}-x_{\tau-}|^2|dA^1_\tau|\Bigg)\\
&+ E\int_{[1-t,1-s)}|x^k_{\tau-}-x_{\tau-}|^2 d[\tilde{N}^2,\tilde{N}^2]_\tau + E\Bigg(\int_{[1-t,1-s)}|x^k_{\tau-}-x_{\tau-}|^2|d\tilde{A}^2_\tau|\Bigg)\Bigg).
\end{aligned}$$

Using Emery's inequality, we get

$$\begin{aligned}
&(I_2)^2 \leq D^2\cdot\max\{1,c_{A^1},c_{\tilde{A}^2}\}\cdot\||x^k-x|^2\|_{S^1}\\
&\cdot(\|[N^1,N^1]\|_{\mathcal{H}^\infty} + \||dA^1|\|_{\mathcal{H}^\infty} + \|[\tilde{N}^2,\tilde{N}^2]\|_{\mathcal{H}^\infty} + \||d\tilde{A}^2|\|_{\mathcal{H}^\infty})\\
&= D^2\cdot\max\{1,c_{A^1},c_{\tilde{A}^2}\}\cdot\|x^k-x\|_{S^2}\\
&\cdot(\|[N^1,N^1]\|_{\mathcal{H}^\infty} + \||dA^1|\|_{\mathcal{H}^\infty} + \|[\tilde{N}^2,\tilde{N}^2]\|_{\mathcal{H}^\infty} + \||d\tilde{A}^2|\|_{\mathcal{H}^\infty})
\end{aligned}$$

which tends to 0 while $k\to\infty$, so we have the result. ■

Remark 26. The above theorem can be applied only to stochastic inclusions driven by a semimartingale with a deterministic FV-part. From a mathematical point of view this is a serious problem. However, real problems are often described by stochastic equations or inclusions of this type, see e.g. [1, 11].

As an example we recall a financial model presented in [28, 34]. Suppose we have a model of a free-arbitrage market defined on a filtered probability space. The capital of an investor (a writer of a contingent claim) is defined under a self-financing assumption by a relation

$$\xi_t(\omega,u)=\xi_0(\omega,u)+\int_0^t\theta_\tau(\omega,u)dB_\tau(\omega,u)+\int_0^t\gamma_\tau(\omega,u)dS_\tau(\omega,u),\ \ t\in[0,T],$$

where (θ,γ) is an investor's strategy (hedge) process, while B and S are price processes of a bond (an asset with a predictable price) and stock, respectively (see e.g.: [11] for details), u denotes a control parameter taken from a given set U of attainable controls.

If the model is based on daily returns of a stock, statistical tests reject hypotheses about normality distribution made in the model of the Black and Scholes type, (one of the most commonly used Gaussian model in financial mathematics). It follows that real prices are usually better characterized by the so-called heavy tailed distributions, skewness property, effects of clusters and so on. Moreover, an empirical study of the German stock price data shows that paths should be modeled by a discontinuous process instead of a continuous one.

Generalizations of the Gaussian model were proposed in many different manners. It was allowed in [1], that the price process has jumps and the resulting equation has the form (in a one dimensional case)

$$\xi_t(\omega,u)=\mu_t\xi_{t-}(\omega,u)dt+\sigma_t\xi_{t-}(\omega,u)dW_t(\omega,u)+\beta\xi_{t-}(\omega,u)dN_t(\omega,u),$$

where N is a point process counting the number of jumps of size β which the relative price $\xi_t(\omega,u)/\xi_{t-}(\omega,u)$ had before time t and W is a standard Wiener process.

Since $(N_t)_{t\geq 0}$ can be treated as a Poisson process with some intensity λ (see e.g.: [30]), then the above problem can be again rewritten equivalently as

$$\xi_t=\xi_0+\int_0^t f_\tau dZ_\tau;\ \ t\in[0,T],$$

or

$$\xi_t\in\xi_0+\int_0^t F_\tau dZ_\tau;\ \ t\in[0,T], \tag{6}$$

with $f_\tau(\omega,u)=(\mu_\tau\xi_{\tau-},\sigma_\tau\xi_{\tau-},\beta,\beta),\ Z_\tau=(0,W_\tau,N_\tau-\lambda\tau,0)+(\tau,0,0,\lambda\tau)=M_\tau+A_\tau$ and

$$F(\tau,\omega)=\bigcup_{u\in U}f(\tau,\omega,u).$$

The set-valued integral $\int F_\tau dZ_\tau$ driven by a semimartingale appears, instead of a single-valued $\int f_\tau dZ_\tau$, in a natural way, if we consider the control of financial problems connected with the models as those presented above. We obtain the stochastic inclusion (6), which describes the discussed financial problem. We can analyze it with respect to the whole set U of attainable controls.

In above example we obtain an Itô-type stochastic differential inclusion driven by a semimartingale with a deterministic FV-part. In our paper we give a tool to describe in a compact form some advanced models (financial, economical, technical) using Stratonovich-type stochastic integrals and inclusions.

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