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MOMENTS OF ORDER STATISTICS OF THE GENERALIZED T DISTRIBUTION

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Abstract

We derive an explicit expression for the single moments of order statistics from the generalized t (GT) distribution. We also derive an expression for the product moment of any two order statistics from the same distribution. Then the location-scale estimating problem of a real data set is solved alternatively by the best linear unbiased estimates which are based on the moments of order statistics.

Keywords: best linear unbiased estimates, generalized Kampé de Fériet function, generalized t (GT) distribution, moments of order statistics.

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1. Introduction

The generalized t (GT) distribution has the following probability density function (pdf)

$$f(x;p,q) = \frac{p}{2q^{1/p}B(1/p,q)}\left(1+\frac{|x|^p}{q}\right)^{-q-1/p}, \tag{1}$$

where $p>0$ and $q>0$ are shape parameters and $B(\cdot,\cdot)$ is the beta function. The GT distribution was defined by McDonald and Newey [11] to develop a partially adaptive M-regression procedure. The procedure includes many other estimation methods such as least squares, least absolute deviation and L_p.

The shape parameters p and q control the tails of the distribution. Larger values of p and q are associated with thinner tails of the distribution. Similarly, smaller values of the shape parameters correspond to thicker tails. Thus, the GT distribution is useful in accommodation both leptokurtic an platykurtic symmetric unimodal distributions.

The GT distribution includes some subdistributions such as, for $p = 2$ we get the usual t distribution with degrees of freedom $2q$, and for $p \to \infty$ and $q \to \infty$ we get the uniform and power exponential distributions, respectively. When $p \leq 1$, we have the cuspidate distributions.

The cdf of the GT distribution is

$$F(x; p, q) = \frac{1}{2}\left[1 + \operatorname{sgn}(x) I_{g(x)}\left(\frac{1}{p}, q\right)\right],$$

where $g(x) = |x|^p/(q + |x|^p)$ and $I_x(a, b)$ is the incomplete beta function ratio defined by the following integral

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x w^{a-1}(1 - w)^{b-1}\, dw.$$

This function has the following series expansion

$$I_x(a, b) = \frac{x^a}{B(a, b)} \sum_{k=0}^{\infty} \frac{(1 - b)_k x^k}{(a + k)k!}, \tag{2}$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ denotes the ascending factorial with $(z)_0 = 1$.

The GT distribution is known as an alternative heavy-tailed distribution in robust statistical procedures. Arslan and Genç [2] considered the distribution in location-scale estimating problem. Arslan [3] introduced the multivariate extension of the distribution and studied its properties in a more general class of distributions within the family of elliptically contoured distributions. Wang and Romagnoli [18] proposed to use the GT distribution to characterize the process data in case of the violation of the normality assumption. Nadarajah [13] studied the cumulative distribution function of the GT distribution and derived several explicit formulas for it. Choy and Chan [4] developed a scale mixtures of uniform representation of the GT distribution and used this representation in Gibbs sampling algorithm efficiently. Finally, Fung and Seneta [8] defined another generalized version of the multivariate version of the distribution by using extended generalized inverse gamma distribution in a mixture. Genç [9] used the GT distribution to obtain an extended Birnbaum-Saunders distribution. Wang, Choy and Chan [19] modeled financial return time series and time-varying volatility data with the GT distribution. Recently, Vu [17] considered the GT distribution in data reconciliation estimation.

On the other hand, moments of order statistics play an important role in various fields especially in statistical inference. Nadarajah [12, 14] obtained exact expressions for the moments of order statistics from several well known distributions by using the generalized Kampé de Fériet function which is a kind of special function. This function of n variables ([10, 6]) is defined as

$$\begin{aligned}
&F^{A:B}_{C:D}\left[\begin{matrix} a_1,\dots,a_A : b_{1,1},\dots,b_{1,B};\ \dots\ ;b_{n,1},\dots,b_{n,B}; \\ c_1,\dots,c_C : d_{1,1},\dots,d_{1,D};\dots;d_{n,1},\dots,d_{n,D}; \end{matrix} x_1,\dots,x_n\right] \\
&= F^{A:B}_{C:D}\left[\begin{matrix} (a):(b_1);\dots;(b_n); \\ (c):(d_1);\dots;(d_n); \end{matrix} x_1,\dots,x_n\right] \\
&= \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} \frac{\left\{\prod_{j=1}^{A}\left(a_j,\sum_{i=1}^{n} m_i\right)\right\}\left\{\prod_{j=1}^{B}(b_{1,j},m_1)\cdots(b_{n,j},m_n)\right\}}{\left\{\prod_{j=1}^{C}\left(c_j,\sum_{i=1}^{n} m_i\right)\right\}\left\{\prod_{j=1}^{D}(d_{1,j},m_1)\cdots(d_{n,j},m_n)\right\}} \\
&\quad\times \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \\
&= \sum \frac{\left((a),\sum m\right)\left((b_1),m_1\right)\cdots\left((b_n),m_n\right)}{\left((c),\sum m\right)\left((d_1),m_1\right)\cdots\left((d_n),m_n\right)} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!},
\end{aligned}$$

where $(a) = (a_1,\dots,a_A)$ and $(b_j) = (b_{j,1},\dots,b_{j,B})$ are sequences of numbers. $(a,n) = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$, and $(a,0) = 1$ for convenience. We will also obtain our results in terms of this function in the paper. We derive an exact expression for the moments of order statistics from the *GT* distribution and further, we also search for an exact expression for the product moment of any two order statistics from the same distribution.

2. Single moments

The order statistics are one of the important topics in statistics and related fields (see e.g. [1]). For a random sample $X_1, X_2, \dots, X_n$ of size n from the distribution with cdf $F(x)$ and pdf $f(x)$, the pdf of the rth order statistic $X_{r:n}$, denoted by $f_{r:n}(x)$, for $1 \le r \le n$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\,f(x). \tag{3}$$

For the *GT* distribution we have

$$f_{r:n} = r\binom{n}{r}\frac{p}{2^n q^{1/p}B(1/p,q)}\left(1+\frac{|x|^p}{q}\right)^{-q-1/p}\left[1+\operatorname{sgn}(x)I_{g(x)}\left(\frac{1}{p},q\right)\right]^{r-1}$$
$$\times\left[1-\operatorname{sgn}(x)I_{g(x)}\left(\frac{1}{p},q\right)\right]^{n-r},$$

where $-\infty < x < \infty$. For brevity we shall hereafter use I_x to denote $I_x(1/p,q)$.

For the kth moment of $X_{r:n}$ from a symmetric distribution we have

$$\begin{aligned}
E(X_{r:n}^k) &= r\binom{n}{r}\int_{-\infty}^{\infty} x^k[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx \\
&= r\binom{n}{r}\left[\int_0^{\infty} x^k[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx\right. \\
&\quad\left. +(-1)^k\int_0^{\infty} x^k[F(x)]^{n-r}[1-F(x)]^{r-1}f(x)\,dx\right] \\
&= A(k,n,r)+(-1)^k A(k,n,n-r+1)
\end{aligned}$$

so that it is sufficient to find $A(k,n,r)$ in searching k th moment of an order statistic from a symmetric distribution defined on the whole real line.

For the GT distribution we have

$$A(k,n,r) = C\int_0^{\infty} x^k\left(1+\frac{x^p}{q}\right)^{-q-1/p}\left[1+I_{g(x)}\right]^{r-1}\left[1-I_{g(x)}\right]^{n-r},$$

where

$$C = r\binom{n}{r}\frac{p}{2^n q^{1/p}B(1/p,q)}.$$

By change of variable $y = x^p/(q+x^p)$, we have

$$A(k,n,r) = \frac{Cq^{(k+1)/p}}{p}\int_0^1 y^{(k+1)/p-1}(1-y)^{-k/p+q-1}(1+I_y)^{r-1}(1-I_y)^{n-r}\,dy.$$

By binomial expansions of the two incomplete beta function ratio factors in the integrand, we have

$$\begin{aligned}
A(k,n,r) = \frac{Cq^{(k+1)/p}}{p}&\sum_{i=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{i}\binom{n-r}{j}(-1)^j\int_0^1 y^{(k+1)/p-1} \\
&\times(1-y)^{-k/p+q-1}I_y^{i+j}\,dy. \qquad (4)
\end{aligned}$$

By (2), the integral, say I, in (4) becomes

$$I = \int_0^1 y^{(k+1)/p-1}(1-y)^{-k/p+q-1}\left(\frac{y^{1/p}}{B(1/p,q)}\sum_{m=0}^{\infty}\frac{(1-q)_m y^m}{m!(1/p+m)}\right)^{i+j} dy.$$

Now by the generalized multinomial theorem we have

$$\begin{aligned}
I &= \frac{1}{[B(1/p,q)]^{i+j}}\sum_{m_1,\ldots,m_{i+j}=0}^{\infty}\prod_{t=1}^{i+j}\frac{(1-q)_{m_t}}{m_t!(1/p+m_t)}\int_0^1 y^{(k+1+i+j)/p+\sum_{t=1}^{i+j} m_t-1} \\
&\quad\times (1-y)^{-k/p+q-1}\,dy \\
&= \sum_{m_1,\ldots,m_{i+j}=0}^{\infty}\prod_{t=1}^{i+j}\frac{(1-q)_{m_t}}{m_t!(1/p+m_t)}\frac{B((k+i+j+1)/p+\sum_{t=1}^{i+j} m_t, q-k/p)}{[B(1/p,q)]^{i+j}},
\end{aligned}$$

where $pq-k>0$. Since $(z)_k = \Gamma(z+k)/\Gamma(z)$, we have

$$\begin{aligned}
I &= \sum_{m_1,\ldots,m_{i+j}=0}^{\infty}\frac{\left(\frac{k+i+j+1}{p}\right)_{\sum_{t=1}^{i+j} m_t} B\left(q-\frac{k}{p},\frac{k+i+j+1}{p}\right)p^{i+j}\prod_{t=1}^{i+j}(1-q)_{m_t}\left(\frac{1}{p}\right)_{m_t}}{\left(q+\frac{i+j+1}{p}\right)_{\sum_{t=1}^{i+j} m_t}\left[B\left(\frac{1}{p},q\right)\right]^{i+j}\prod_{t=1}^{i+j} m_t!\left(\frac{1}{p}+1\right)_{m_t}} \\
&= \frac{B\left(q-\frac{k}{p},\frac{k+i+j+1}{p}\right)p^{i+j}}{\left[B\left(\frac{1}{p},q\right)\right]^{i+j}} F^{1:1}_{1:2}\left[\begin{matrix}\frac{k+i+j+1}{p}: & (1-q,\frac{1}{p});\ \ldots\ ;(1-q,\frac{1}{p}); \\ q+\frac{i+j+1}{p}: & \frac{1}{p}+1\ \ ;\cdots;\ \ \frac{1}{p}+1\ \ ;\end{matrix}\ 1,\ldots,1\right].
\end{aligned}$$

By putting this last result in (4) we obtain

$$\begin{aligned}
A(k,n,r) &= r\binom{n}{r}\frac{q^{k/p}}{2^n}\sum_{i=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{i}\binom{n-r}{j}(-1)^j\frac{B\left(q-\frac{k}{p},\frac{k+i+j+1}{p}\right)p^{i+j}}{\left[B\left(\frac{1}{p},q\right)\right]^{i+j+1}} \\
&\quad\times F^{1:2}_{1:1}\left[\begin{matrix}\frac{k+i+j+1}{p}: & (1-q,\frac{1}{p});\ \ldots\ ;(1-q,\frac{1}{p}); \\ q+\frac{i+j+1}{p}: & \frac{1}{p}+1\ \ ;\cdots;\ \ \frac{1}{p}+1\ \ ;\end{matrix}\ 1,\ldots,1\right].
\end{aligned}$$

where $pq-k>0$. This result contains only two finite sums and a special function, that is the generalized Kampé de Fériet function. This special function is not available in most mathematical softwares. However, the relation

$$F_{1:1}^{1:2}\left[\begin{matrix} a: & (c_1,d_1);\ldots;(c_n,d_n); \\ a+b: & f_1 \quad ;\ldots; \quad f_n \quad ; \end{matrix} s_1,\ldots,s_n\right] = \frac{1}{B(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}$$
$$\times\, {}_2F_1(c_1,d_1;f_1;s_1x)\cdots {}_2F_1(c_n,d_n;f_n;s_nx)\,dx$$

which can be found in Exton [7] can be used in computations since generalized hypergeometric functions are implemented in many mathematical programs for example in Mathematica Software [20].

3. Product moment

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = C_{s,s,n}\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s}\,f(x)f(y),$$

where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

and $-\infty < x < y < \infty$.

For the product moment of $X_{r:n}$ and $X_{s:n}$ from a symmetric distribution, we have

$$\begin{aligned}
\frac{1}{C_{r,s,n}}E(X_{r:n}X_{s:n}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} xy[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1}[1-F(x)]^{n-s} \\
&\quad \times f(x)f(y)\,dy\,dx \\
&= \int_{-\infty}^{0}\int_{y}^{-y} xy[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1}[1-F(x)]^{n-s} \\
&\quad \times f(x)f(y)\,dx\,dy \\
&\quad + \int_{0}^{\infty}\int_{-x}^{x} xy[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1}[1-F(x)]^{n-s} \\
&\quad \times f(x)f(y)\,dy\,dx.
\end{aligned} \tag{5}$$

Now let

$$D(n,r,s) = \int_0^{\infty}\int_{-x}^{x} xy[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1}[1-F(x)]^{n-s}f(x)f(y)\,dy\,dx.$$

Then using appropriate change of variable and the symmetry of the distribution, the first double integral in (5) becomes

$$\int_0^\infty \int_{-t}^{t} (-1)^{s-r-1}(-x)t[1-F(t)]^{r-1}[F(-x)-F(t)]^{s-r-1}[F(-x)]^{n-s} \times f(-x)f(t)\,dx\,dt.$$

Continuing this process once again, this last integral equals

$$\int_0^\infty \int_{-x}^{x} xy[1-F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[F(y)]^{n-s} f(x)f(y)\,dy\,dx,$$

which is $D(n, n-s+1, n-r+1)$. Then the product moment is given by

$$E(X_{r:n}X_{s:n}) = C_{r,s,n}\left[D(n,r,s) + D(n, n-s+1, n-r+1)\right].$$

As in the single moments case, it is sufficient to find $D(n,r,s)$ in searching the product moment of two order statistics from a symmetric distribution defined on the whole real line.

We also have

$$D(n,r,s) = \int_0^\infty x[1-F(x)]^{n-s} f(x)[\xi_1(x) + \xi_2(x)]\,dx, \tag{6}$$

where

$$\xi_1(x) = \int_{-x}^{0} y[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1} f(y)\,dy$$

and

$$\xi_2(x) = \int_0^x y[F(y)]^{r-1}[F(x)-F(y)]^{s-r-1} f(y)\,dy.$$

Let us proceed with $\xi_2(x)$. We have

$$\xi_2(x) = \frac{C^*}{2^{s-2}} \int_0^x y\left[1 + I_{g(y)}\right]^{r-1}\left[I_{g(x)} - I_{g(y)}\right]^{s-r-1}\left(\frac{q+y^p}{q}\right)^{-q-1/p} dy,$$

where C^* is the normalizing constant of the *GT* distribution. By change of variable $u = y^p/(q+y^p)$, we have

$$\xi_2(x) = \frac{C^* q^{2/p}}{2^{s-2}p} \int_0^{g(x)} u^{2/p-1}(1-u)^{q-1/p-1}(1+I_u)^{r-1}\left[I_{g(x)} - I_u\right]^{s-r-1} du.$$

By binomial expansions of the two factors involving incomplete beta function ratios in the integrand, we have

$$\xi_2(x) = \frac{C^* q^{2/p}}{2^{s-2}p} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \binom{r-1}{i} \binom{s-r-1}{j} (-1)^j I_{g(x)}^{s-r-j-1} \int_0^{g(x)} u^{2/p-1} \times (1-u)^{q-1/p-1} I_u^{i+j}\, du. \tag{7}$$

Now, first using the series representation (2) of the incomplete beta function ratio and then using the generalized multinomial theorem, the integral, say II, in (7) becomes

$$\begin{aligned} II &= \int_0^{g(x)} u^{2/p-1}(1-u)^{q-1/p-1} \left(\frac{u^{1/p}}{B(1/p,q)} \sum_{m=0}^{\infty} \frac{(1-q)_m u^m}{m!(1/p+m)} \right)^{i+j} du \\ &= \sum_{m_1,\ldots,m_{i+j}=0}^{\infty} \prod_{t=1}^{i+j} \frac{(1-q)_{m_t}}{m_t!(1/p+m_t)} \frac{B((2+i+j)/p + \sum_{t=1}^{i+j} m_t, q-1/p)}{[B(1/p,q)]^{i+j}} \\ &\quad \times I_{g(x)}\left((2+i+j)/p + \sum_{t=1}^{i+j} m_t, q-1/p \right). \end{aligned}$$

Now $\xi_2(x)$ becomes

$$\begin{aligned} \xi_2(x) &= \frac{C^* q^{2/p}}{2^{s-2}p} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \binom{r-1}{i} \binom{s-r-1}{j} (-1)^j I_{g(x)}^{s-r-j-1} \sum_{m_1,\ldots,m_{i+j}=0}^{\infty} \\ &\quad \times \prod_{t=1}^{i+j} \frac{(1-q)_{m_t}}{m_t!(1/p+m_t)} \frac{[g(x)]^{(2+i+j)/p+\sum_{t=1}^{i+j} m_t}}{[B(1/p,q)]^{i+j}} \\ &\quad \times \sum_{u=0}^{\infty} \frac{(1-q+1/p)_u [g(x)]^u}{u!\left((2+i+j)/p + \sum_{t=1}^{i+j} m_t + u\right)}. \end{aligned} \tag{8}$$

After putting (8) in (6), the integral

$$\int_0^{\infty} x[1-F(x)]^{n-s} f(x) \xi_2(x)\, dx \tag{9}$$

becomes

$$
\begin{aligned}
&\frac{p}{2^n[B(1/p,q)]^2}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{k=0}^{n-s}\binom{r-1}{i}\binom{s-r-1}{j}\binom{n-s}{k}(-1)^{k+j}\\
&\times\frac{1}{[B(1/p,q)]^{i+j}}\sum_{m_1,\dots,m_{i+j}=0}^{\infty}\prod_{t=1}^{i+j}\frac{(1-q)_{m_t}}{m_t!(1/p+m_t)}\\
&\times\sum_{u=0}^{\infty}\frac{(1-q+1/p)_u}{u!\left((2+i+j)/p+\sum_{t=1}^{i+j}m_t+u\right)}\\
&\times\int_0^\infty xI_{g(x)}^{s-r-j-1+k}(1+x^p/q)^{-q-1/p}[g(x)]^{(2+i+j)/p+\sum_{t=1}^{i+j}m_t+u}\,dx.
\end{aligned}
\tag{10}
$$

The integral in (10) is similar to the one studied in the previous section, and after using the same steps, for $pq > 1$ it becomes

$$
\begin{aligned}
&\frac{q^{2/p}B\left(q-\frac{1}{p},\frac{3+i+s-r+k}{p}+u+\sum_{t=1}^{i+j}m_t\right)}{p^{2-s+r+j-k}\left[B\left(\frac{1}{p},q\right)\right]^{s-r-j-1+k}}\\
&\times F_{1:1}^{1:2}\left[\begin{matrix}\frac{3+i+s-r+k}{p}+u+\sum_{t=1}^{i+j}m_t: & (1-q,\frac{1}{p});\ \dots\ ;(1-q,\frac{1}{p}); \\ q+\frac{2+i+s-r+k}{p}+u+\sum_{t=1}^{i+j}m_t: & \frac{1}{p}+1\quad;\cdots;\quad\frac{1}{p}+1\quad;\end{matrix}1,\dots,1\right].
\end{aligned}
\tag{11}
$$

After putting (11) in (10) and then doing some arrangements, (9) becomes

$$
\begin{aligned}
&\frac{1}{2^n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\sum_{k=0}^{n-s}C_{n,r,s}(i,j,k)\frac{B(q-1/p,(3+i+s-r+k)/p)p^{i+s-r+k}}{(2+i+j)[B(1/p,q)]^{i+s-r+k+1}}\sum_{m_1,\dots,m_{i+j}=0}^{\infty}\\
&\times\prod_{t=1}^{i+j}\frac{(1-q)_{m_t}(1/p)_{m_t}}{m_t!(1/p+1)_{m_t}}\frac{((2+i+j)/p)_{\sum_{t=1}^{i+j}m_t}((3+i+s-r+k)/p)_{\sum_{t=1}^{i+j}m_t}}{(1+(2+i+j)/p)_{\sum_{t=1}^{i+j}m_t}(q+(2+i+s-r+k)/p)_{\sum_{t=1}^{i+j}m_t}}\\
&\times\sum_{u=0}^{\infty}\frac{(1-q+1/p)_u((3+i+s-r+k)/p+\sum_{t=1}^{i+j}m_t)_u}{u!((2+i+j)/p+\sum_{t=1}^{i+j}m_t+1)_u(q+(2+i+s-r+k)/p+\sum_{t=1}^{i+j}m_t)_u}\\
&\times F_{1:1}^{1:2}\left[\begin{matrix}\frac{3+i+s-r+k}{p}+u+\sum_{t=1}^{i+j}m_t: & (1-q,\frac{1}{p});\ \dots\ ;(1-q,\frac{1}{p}); \\ q+\frac{2+i+s-r+k}{p}+u+\sum_{t=1}^{i+j}m_t: & \frac{1}{p}+1\quad;\cdots;\quad\frac{1}{p}+1\quad;\end{matrix}1,\dots,1\right].
\end{aligned}
\tag{12}
$$

where

$$C_{n,r,s}(i,j,k) = \binom{r-1}{i}\binom{s-r-1}{j}\binom{n-s}{k}(-1)^{k+j}.$$

This last result contains finite and infinite sums. Unfortunately, we do not know whether the Kampé de Fériet function in (12) is reduced so that we can obtain a simpler expression for (9). On the other hand, the evaluation of the integral

$$\int_0^\infty x[1-F(x)]^{n-s} f(x)\xi_1(x)\,dx$$

follows the same steps as we did for (9), and is equal to (12), in which $(-1)^{k+j}$ is replaced by $(-1)^{i+k+1}$ only.

As a submodel of the GT distribution, Vaughan [16] also derived an exact expression for the product moment of any two order statistics from the Cauchy distribution in terms of an infinite series. He also analyzed the convergence of the series and tried to find some bound on truncation error of the series. For the GT distribution, we note that the complexity of the expressions derived here makes it very difficult.

4. An application

We consider the Rosner data set [15]. This data set consists of 10 monthly diastolic blood pressure measurements and as follows: 90, 93, 86, 92, 95, 83, 75, 40, 88, 80. We note that the observation 40 is far from the other observations. Thus, it is a (possible) outlier. The sample mean of the data is 82.2, and the standard deviation is 19.1. They seem to be influenced by the outlier badly. In order to summarize the location and scale of the data more accurately, one should use a robust method. Since outliers in data produce thick-tailed distributions, one may use a thick-tailed distribution like the GT for modeling and then estimate the location and scale parameters. We follow the second way and model the data with the $GT(\mu,\sigma)$ distribution where μ is the location parameter and σ is the scale parameter. We give the shape parameters the role of robustness tuning constants like the ordinary Student's t distribution, and fix them at $p = 1.5$ and $q = 2$. This specific member of the GT distributions family is relatively a heavy-tailed one so that it is suitable for modeling such a data set. To apply the moments of order statistics from the GT distribution, we search for the best linear unbiased estimates (BLUE's) of the location-scale parameters. For a reference, see David and Nagaraja [5] p. 185.

Now let $y_{1:10} \le \cdots \le y_{10:10}$ be the ordered Rosner data. Then the BLUE vector of μ and σ is given by

$$\begin{pmatrix}\hat\mu\\ \hat\sigma\end{pmatrix} = \mathbf{Cy},$$

where

$$\mathbf{C} = (\mathbf{A}^T\mathbf{V}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{V}^{-1};$$

$\mathbf{A} = (\mathbf{1}, \boldsymbol{\alpha})$, $\mathbf{1}^T = (1, \dots, 1)$, $\mathbf{y}^T = (y_{1:10}, \dots, y_{10:10})$, $\boldsymbol{\alpha}^T = (\alpha_{1:10}, \dots, \alpha_{10:10})$, $\alpha_{r:10} = E(Y_{r:10})$ and $\mathbf{V}$ is the variance-covariance matrix of the ordered observations. All the vectors are 10×1. Then the coefficients for the BLUE of μ, that is the first row of $\mathbf{C}$, are computed as

$$\begin{gathered} -0.008733, -0.012176, 0.051563, 0.172370, 0.299810, \\ 0.299524, 0.171067, 0.047562, -0.014850, -0.006137 \end{gathered}$$

and the coefficients for the BLUE of σ, that is the second row of $\mathbf{C}$, are computed as

$$\begin{gathered} -0.029986, -0.184240, -0.232130, -0.214326, -0.091642, \\ 0.092750, 0.214739, 0.251252, 0.149359, 0.044223, \end{gathered}$$

by Mathematica software [20]. The BLUE's of μ and σ are then computed as $\hat{\mu} = 87.11867$ and $\hat{\sigma} = 9.437156$. We note that $\hat{\mu}$ is very close to 87 which is the sample median, a robust estimate of location. Further, there exist variances of the estimates on the diagonal of $(\mathbf{A}^T\mathbf{V}^{-1}\mathbf{A})^{-1}\sigma^2$. They are computed as $Var(\hat{\mu}) = 0.102006\sigma^2$, $Var(\hat{\sigma}) = 0.145115\sigma^2$ and $Cov(\hat{\mu}, \hat{\sigma}) = 0$.

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