A NOTE ON THE STRONG CONSISTENCY OF LEAST SQUARES ESTIMATES

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Abstract
The strong consistency of least squares estimates in multiples regression models with i.i.d. errors is obtained under assumptions on the design matrix and moment restrictions on the errors.

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1. Introduction

Many statisticians have considered the problem of strong consistency of the least squares estimates in multiple regression models. In the seventies, this problem was completely solved under weak moment conditions on the errors, namely, assuming their finite variance (see [5, 6] and [9]). More recently, some other authors had studied this same problem for the case where the variance of the errors is infinite. Nevertheless, these works reveals very restrictive conditions on the design matrix (see [8]) or particular scenarios for the errors (see [11, 12] or [13]).

In this paper, we establish the strong consistency of the least squares estimates for the parameters $\beta_j$ of the multiple regression model

$$y_i = \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \varepsilon_i \quad (i = 1, 2, \ldots)$$
under suitable assumptions on the design matrix \( x_{ij} \) when the error variance is infinite. Specially, we shall assume that

\[
\varepsilon_i \text{ are i.i.d. with } \mathbb{E}|\varepsilon_i|^r < \infty \text{ for some } r \in (0, 2) \\
\text{and } \mathbb{E}\varepsilon_i = 0 \text{ whenever } r \in (1, 2)
\tag{1.2}
\]

admitting cases where the errors don’t have mean value. Let us stress that in [6] or [7] only the errors \( \varepsilon_i \) with \( \sup_i \mathbb{E}|\varepsilon_i|^r < \infty \) for some \( 1 \leq r < 2 \) are considered leaving the cases where \( 0 < r < 1 \) unsolved. In particular, on the paper [6], the authors establish the strong consistency of the least squares estimates for the case \( 1 \leq r < 2 \) using the Hölder inequality (see Corollary 3 and Lemma 4), which is no more useful when \( 0 < r < 1 \).

Throughout this work, we shall let \( X_n \) denote the design matrix

\[
\left( x_{ij} \right)_{1 \leq i \leq n, 1 \leq j \leq p}
\]

and let \( y_n = (y_1, \ldots, y_n)' \) and \( \beta = (\beta_1, \ldots, \beta_p)' \), where prime denotes transpose. For \( n \geq p \), the least squares estimate \( b_n = (b_{n1}, \ldots, b_{np})' \) of the vector \( \beta \) based on the design matrix \( X_n \) and the response vector \( y_n \) is given by

\[
b_n = (X_n'X_n)^{-1}X_n'y_n
\tag{1.3}
\]

provided that

\[
X_n'X_n = \begin{pmatrix}
\sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1}x_{i2} & \ldots & \sum_{i=1}^{n} x_{i1}x_{ip} \\
\sum_{i=1}^{n} x_{i1}x_{i2} & \sum_{i=1}^{n} x_{i2}^2 & \ldots & \sum_{i=1}^{n} x_{i2}x_{ip} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{n} x_{i1}x_{ip} & \sum_{i=1}^{n} x_{i2}x_{ip} & \ldots & \sum_{i=1}^{n} x_{ip}^2
\end{pmatrix}
\]
is nonsingular for all \( n \geq n_0 \). From the expression of \( b_n \) it follows that the strong consistency of the least squares estimates is equivalent to

\[
(X'_n X_n)^{-1} \sum_{i=1}^{n} x_i \xrightarrow{a.s.} 0
\]

where \( x_i = (x_{i1}, \ldots, x_{ip})' \).

2. Auxiliary tools

It is well-known that every positive definite matrix \( A = (a_{ij})_{1 \leq i,j \leq p} \) satisfies

\[
\det(A) \leq a_{11} \cdots a_{pp}
\]

and the equality holds if and only if \( A \) is diagonal (see [3], page 477). This classical result due to Hadamard leads us to the following definition.

**Definition.** A sequence of \( p \times p \) positive definite matrices \( A_n = (a_{ij}^{(n)})_{1 \leq i,j \leq p} \) is said asymptotically diagonal dominant if \( \det(A_n) \asymp a_{11}^{(n)} \cdots a_{pp}^{(n)} \), \( n \to \infty \).

An important tool in proving the strong consistency of \( b_n \) for error structures satisfying (1.2) is the next lemma which is an extension of Marcinkiewicz-Zygmund theorem presented in [1] (page 118).

**Lemma 1.** If \( \{X_n\} \) are i.i.d. r.v.'s with \( \mathbb{E}|X_1|^r < \infty \) and \( \{a_n\} \) are real numbers such that \( a_n = O\left(n^{-1/r}\right) \) for some \( 0 < r < 2 \) then

\[
\sum_{n=1}^{\infty} (a_n X_n - \mathbb{E}Y_n)
\]

converges a.s., where \( Y_n = a_n X_{nI_{\{|X_n| \leq n^{1/r}\}}} \). Furthermore, if either (i) \( 0 < r < 1 \) or (ii) \( 1 < r < 2 \) and \( \mathbb{E}X_1 = 0 \), then \( \sum_{n=1}^{\infty} a_n X_n \) converges a.s.

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\(*_{a_n \asymp b_n, n \to \infty} \) means that \( a_n = O(b_n) \) and \( b_n = O(a_n) \) as \( n \to \infty \).
**Proof.** Set \( A_j = \{(j - 1)^{1/r} \leq |X_1| \leq j^{1/r}\}, \ j \geq 1. \) Then for \( \alpha > r > 0 \)

\[
\sum_{n=1}^{\infty} \mathbb{E} |Y_n|^\alpha \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} |a_n|^\alpha \int_{A_j} |X_1|^\alpha
\]

\[
\leq C \sum_{n=1}^{\infty} \sum_{j=1}^{n} n^{-\alpha/r} \int_{A_j} |X_1|^\alpha
\]

\[
(2.1)
\]

\[
\leq C \sum_{n=1}^{\infty} \left( j^{-\alpha/r} + \frac{r}{\alpha - r} j^{(r-\alpha)/r} \right) \int_{A_j} |X_1|^\alpha
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\alpha}{\alpha - r} \int_{A_j} |X_1|^r
\]

\[
\leq \frac{\alpha}{\alpha - r} \mathbb{E} |X_1|^r < \infty,
\]

whence \((\alpha = 2) \sum_{n=1}^{\infty} (Y_n - \mathbb{E} Y_n)\) converges a.s. by Khintchine’s theorem (see [1], page 113). Since

\[
\sum_{n=1}^{\infty} \mathbb{P} \{a_n X_n \neq Y_n\} = \sum_{n=1}^{\infty} \mathbb{P} \{|X_1| > n^{1/r}\} \leq \mathbb{E} |X_1|^r < \infty
\]

the sequences \(\{a_n X_n\}, \{Y_n\}\) are equivalent and so \(\sum_{n=1}^{\infty} (a_n X_n - \mathbb{E} Y_n)\) converges a.s.

In case (i), where \(0 < r < 1, \sum_{n=1}^{\infty} |\mathbb{E} Y_n| < \infty\) via (1.1) with \(\alpha = 1\). In case (ii), where \(1 < r < 2\) and \(\mathbb{E} X_1 = 0\), we have
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\[
\sum_{n=1}^{\infty} |E Y_n| \leq \sum_{n=1}^{\infty} |a_n| \int_{\{|X_n|>n^{1/r}\}} |X_n|
\]

\[
\leq C \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n^{-1/r} \int_{A_j} |X_1|
\]

\[
= C \sum_{j=2}^{\infty} \sum_{n=1}^{j-1} n^{-1/r} \int_{A_j} |X_1|
\]

\[
\leq C \frac{r}{r-1} \sum_{j=1}^{\infty} (j-1)^{(r-1)/r} \int_{A_j} |X_1|^r
\]

\[
\leq C \frac{r}{r-1} \sum_{j=1}^{\infty} \int_{A_j} |X_1|^r
\]

\[
= C \frac{r}{r-1} E |X_1|^r < \infty.
\]

Thus, the second part of Lemma 1 follows from the first.

\[\square\]

**Remark.** If \( \{X_n, n \geq 1\} \) is a i.i.d. sequence of r.v.’s with \( E |X_1| < \infty \) and \( \{a_n, n \geq 1\} \) are real numbers such that \( a_n = O(n^{-1}) \) and \( \sum_{n=1}^{\infty} a_n \) converges then \( \sum_{n=1}^{\infty} a_n X_n \) converges a.s.

3. Strong consistency

In this section we shall prove the main result of this paper.

**Theorem 1.** Suppose that in model (1.1), \( \varepsilon_1, \varepsilon_2, \ldots \) are random variables satisfying (1.2) and \( \{x_{ij}\} \) \( (i = 1, 2, \ldots; j = 1, \ldots, p) \) is an arbitrary double
array of constants. If $X_n'X_n$ is nonsingular for all $n \geq n_0$ and asymptotically diagonal dominant with constants $x_{ij}$ satisfying $\sum_{k=1}^{n} x_{kj}^2 \to \infty$ for all $j$ and

$$\frac{x_{nj}}{\left( \sum_{k=1}^{n} x_{ki}^2 \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2}} = O\left( n^{-1/r} \right) \text{ for all } i, j = 1, \ldots, p \text{ when } r \neq 1$$

or

$$\frac{x_{nj}}{\left( \sum_{k=1}^{n} x_{ki}^2 \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2}} = O\left( n^{-1} \right) \text{ and } \sum_{n=1}^{\infty} \frac{x_{nj}}{\left( \sum_{k=1}^{n} x_{ki}^2 \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2}}$$

converges for all $i, j = 1, \ldots, p$ whenever $r = 1$.

then $b_n \overset{a.s.}{\to} \beta$.

**Proof.** Setting $C_n = (c_{ij}^{(n)})_{1 \leq i,j \leq p} = (X_n'X_n)^{-1}$ we have

$$\det(X_n'X_n) c_{ij}^{(n)} = (-1)^{i+j} \sum_{\sigma \in S_{ij}} \text{sgn}(\sigma) \prod_{m=1 \atop m \neq i}^{p} \sum_{k=1}^{n} x_{km} x_{\sigma(m)}$$

where the sum is computed over all bijections $\sigma$ of $\{1, \ldots, i - 1, i + 1, \ldots, p\}$ into $\{1, \ldots, j - 1, j + 1, \ldots, p\}$. Thus, using Cauchy-Schwarz inequality we get

$$\det(X_n'X_n) \left| c_{ii}^{(n)} \right| \leq (p - 1)! \prod_{m=1 \atop m \neq i}^{p} \sum_{k=1}^{n} x_{km}^2$$

and

$$\det(X_n'X_n) \left| c_{ij}^{(n)} \right| \leq$$

$$\leq (p - 1)! \left( \sum_{k=1}^{n} x_{ki}^2 \right)^{1/2} \left( \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2} \prod_{m=1 \atop m \neq i,j}^{p} \sum_{k=1}^{n} x_{km}^2, \quad i \neq j.$$
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Since $\det(\mathbf{X}_n' \mathbf{X}_n) \sim \sum_{k=1}^{n} x_{k1}^2 \cdots \sum_{k=1}^{n} x_{kp}^2$ as $n \to \infty$ it is sufficient to prove that

(3.1) $\frac{1}{\left( \sum_{k=1}^{n} x_{k1}^2 \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2}} \sum_{m=1}^{n} x_{mj} \varepsilon_m \overset{a.s.}{\longrightarrow} 0 \quad (i, j = 1, \ldots, p)$

By Lemma 1

$$\frac{1}{\left( \sum_{k=1}^{m} x_{k1}^2 \sum_{k=1}^{m} x_{kj}^2 \right)^{1/2}} \varepsilon_m \text{ converges a.s.} \quad (i, j = 1, \ldots, p)$$

and Kronecker’s lemma permit us to conclude (3.1) which establish the thesis.

$$n \sum_{k=1}^{n} x_{ki} x_{kj} \sim \frac{n^{\alpha_i + \alpha_j + 1}}{\alpha_i + \alpha_j + 1}, \quad i \neq j$$

which implies that $\mathbf{X}_n' \mathbf{X}_n$ is asymptotically diagonal dominant. Indeed, using induction on $p$ we have

$$\det(\mathbf{X}_n' \mathbf{X}_n) = \sum_{j=1}^{p} \sum_{k=1}^{n} x_{kp} x_{kj} \cdot (-1)^{p+j} \det(\mathbf{X}_n' \mathbf{X}_n(p|j))$$

where the modulus of each term of $(-1)^{p+j} \det(\mathbf{X}_n' \mathbf{X}_n(p|j))$, $j = 1, \ldots, p-1$ is bounded by

$$\left( \sum_{k=1}^{n} x_{kp}^2 \right)^{1/2} \left( \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2} \prod_{m=1}^{p} \sum_{m \neq j, p}^{n} x_{km}^2.$$
The conclusion now follows since

\[
\sum_{k=1}^{n} x_{kp} x_{kj} - \left( \sum_{k=1}^{n} x_{kp}^2 \right)^{1/2} \left( \sum_{k=1}^{n} x_{kj}^2 \right)^{1/2} \sim \frac{\sqrt{2\alpha_p + 1} \sqrt{2\alpha_j + 1}}{\alpha_p + \alpha_j + 1} < 1, \quad j = 1, \ldots, p - 1
\]

and \( \sum_{k=1}^{n} x_{kp}^2 \cdot \det (X_n'X_n(p|p)) \sim \sum_{k=1}^{n} x_{k1}^2 \cdots \sum_{k=1}^{n} x_{kp}^2 \) as \( n \to \infty \). The assumptions (i) or (ii) of Theorem 1 are also satisfied.

References


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