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SELECTIVE LACK-OF-MEMORY AND ITS APPLICATION

Czesław Stępniak

Institute of Mathematics, University of Rzeszów
Rejtana 16 A, PL–35–959 Rzeszów, Poland

e-mail: cees@univ.rzeszow.pl

Abstract

We say that a random variable X taking nonnegative integers has selective lack-of-memory (SLM) property with selector s if $P(X \geq n + s/X \geq n) = P(X \geq s)$ for $n = 0, 1, \ldots$ .This property is characterized in an elementary manner by probabilities $p_n = P(X = n)$. An application in car insurance is presented.

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1. Introduction

A nonnegative random variable X is said to have the lack-of-memory (or no memory) property, if

$$P(X \geq a + b/X \geq a) = P(X \geq b) \tag{1.1}$$

for any nonnegative a and b (cf., for instance, Feller [7], Galambos and Kotz [8], Brémaud [1], Stirzacker [15]). Most of known results in this area deals with the continuous case (see, e.g., Marsaglia and Tubilla [11], Schimizu [14]). It is well known that in this case all solutions of the functional equation (1.1) are represented by exponential distributions. More advanced methodology involving Cauchy integrated equation with generalizations to the bivariate case was suggested by Lin [10] , Rao *et al.* [12], Roy [13] and Kulkarni [9].

For random variables taking only nonnegative integers the condition (1.1) reduces to $P(X \geq n+m/X \geq n) = P(X \geq m)$ for all nonnegative integers m and n. It appears that any nontrivial distribution having this property is the geometric one (see [7] Ch. XIII, Sec. 9, [8], [1] p. 48).

In this paper we consider a weaker property, $P(X \geq n+s/X \geq n) = P(X \geq s)$ for a given positive integer s, called selector. Some results in this area are scattered under the name of the almost-lack-of-memory (ALM) property in a series of papers [2]–[6] by Chukova, Dimitrov, Green and Khalil. Instead of ALM we shall use the term *selective lack-of-memory* (*SLM*) with selector s which seems to be more informative (cf. Szala [16]).

All our results are derived in a simple and direct way and presented in a readable form. They are also supported by application in car insurance.

For convenience, let us begin from some classical results.

2. Discrete lack-of-memory distributions

Let X be a random variable taking nonnegative integer values.

Definition 1. The random variable X is said to have the lack-of-memory property if

$$P(X \geq n+m/X \geq n) = P(X \geq m) \tag{2.1}$$

for all nonnegative integers m and n.

The equation (2.1) may be presented in a simpler equivalent form.

Lemma 1. *The condition* (2.1) *holds if and only if*

$$P(X \geq n+1) = P(X \geq n)P(X \geq 1) \tag{2.2}$$

for any nonnegative integer n.

Proof. The implication (2.1) $\Longrightarrow$ (2.2) is evident. The converse one may be verified by induction with respect to m.

By (2.2) the condition (2.1) is satisfied for $m = 1$. Now suppose (2.1) is met for all $m \leq k$. Then, by definition of the conditional probability and by (2.2),

$$\begin{aligned} P(X \quad &\geq \quad n+m+1/X \geq n) = \frac{P(X \geq n+m+1)}{P(X \geq n)} \\ &= \quad \frac{P(X \geq n+m)P(X \geq 1)}{P(X \geq n)} = P(X \geq n+m/X \geq n)P(X \geq 1) \\ &= \quad P(X \geq m)P(X \geq 1) = P(X \geq m+1) \end{aligned}$$

yielding the desired result. ∎

For completeness let us recall the well known result by Feller ([7], Sec.XIII.9) on characterization of the geometric distribution.

Theorem 1. *Let* X *be a not degenerated random variable taking nonnegative integers with distribution* $P(X = n) = p_n$ *for* $n = 0, 1, \ldots$ *. Then* X *has the lack-of-memory property, if and only if,*

$$p_n = p^n(1-p) \quad \textit{for some } p \in (0,1).$$

As we see, the family of all distributions having the lack-of-memory property is very narrow. To extend it we shall replace the integer 1, appearing in the condition (2.2) by arbitrary positive integer s.

3. Discrete distributions with selective lack-of-memory

Let us start from the following definition.

Definition 2. We shall say that a random variable X, taking integer (not necessarily nonnegative) values $N, N+1, N+2, \ldots$, has the selective lack-of-memory (SLM) property with (positive) selector s, if

$$P(X \geq n+s) = P(X \geq n)P(X \geq s) \text{ for all integers } n \geq N. \tag{3.1}$$

As a direct consequence of this definition we get the following corollary.

Corollary 1. *A discrete random variable X has the selective lack-of-memory property with selector s, if and only if, $X+k$ has this property, where k is an arbitrary nonnegative integer.*

This corollary will be useful in the further consideration. Among others, we may and shall restrict our consideration to the case $N=0$.

Theorem 2. *A not degenerated random variable X taking nonnegative integers has the selective lack-of-memory property with selector s, if and only if,*

$$P(X = n) = q^k p_m \tag{3.2}$$

for some nonnegative $p_0, p_1, \ldots, p_{s-1}$, such that

$$0 < \sum_{i=0}^{s-1} p_i < 1,$$

where m and k are uniquely determined by the conditions $m = n(\operatorname{mod} s)$ and

$$k = \frac{n-m}{s}, \quad \text{while} \quad q = 1 - \sum_{i=0}^{s-1} p_i.$$

This theorem follows directly from the following proposition.

Proposition 1. *For arbitrary convergent sequence $\{p_n\}_{n\geq 0}$ of nonnegative numbers, given positive q and positive integer s, the following are equivalent:*

(a) $$p_{n+s} = qp_n, \quad \text{for} \quad n = 0, 1, 2, ...$$

(b) $$\sum_{i=n+s}^{\infty} p_i = q \sum_{i=n}^{\infty} p_i, \quad \text{for} \quad n = 0, 1, 2, ...$$

Moreover, the number q, if such exists, satisfies the condition

$$q = \frac{\sum_{i=0}^{\infty} p_i - \sum_{i=0}^{s-1} p_i}{\sum_{i=0}^{\infty} p_i} = 1 - \frac{\sum_{i=0}^{s-1} p_i}{\sum_{i=0}^{\infty} p_i}. \tag{3.3}$$

Proof. (of the proposition)

$(a) \Longrightarrow (b)$

If (a) holds then the sequence $\{p_n\}$ may be decomposed on s geometric subsequences $\{p_{m+ns}\}_{n\geq 0}$, $m = 0, 1, ..., s-1$, with the common ratio q.

This implies directly (b) for all n of the form $n = ks$. Now, by Corollary 1, the problem with arbitrary n reduces to the case $n = ks$. In this way the implication $(a) \Longrightarrow (b)$ is proved.

$(b) \Longrightarrow (a)$
We observe that

$$p_n = \sum_{i=n}^{\infty} p_i - \sum_{i=n+1}^{\infty} p_i.$$

Thus, if (b) holds, then

$$p_{n+s} = \sum_{i=n+s}^{\infty} p_i - \sum_{i=n+s+1}^{\infty} p_i = q\left(\sum_{i=n}^{\infty} p_i - \sum_{i=n+1}^{\infty} p_i\right) = qp_n$$

completing the proof of the implication.

In order to verify (3.3), let us rewrite

$$\begin{aligned}\sum_{i=0}^{\infty} p_i &= \sum_{m=0}^{s-1} \sum_{n=0}^{\infty} p_{m+ns} = \sum_{m=0}^{s-1} \sum_{n=0}^{\infty} q^n p_m \\ &= \sum_{m=0}^{s-1} p_m \sum_{n=0}^{\infty} q^n = \frac{\sum_{m=0}^{s-1} p_m}{1-q}.\end{aligned}$$

This implies the desired result. ■

Let us note that if the condition (a) in the Proposition 1 holds for some s and q then it also holds for $s' = ms$ and $q' = q^m$, where m is any positive integer.

Definition 3. The minimal s, such that X has the selective lack-of-memory with selector s is said to be the principal selector of X.

We state the following elementary lemma.

Lemma 2. *Assume X has the selective lack-of-memory with selector s*

(a) *If s is a prime number then it is the principal selector of X.*

(b) *If s is not a prime number, say $s = rt$, then s is the principal selector of X, if and only if, the finite sequence $p_0, p_1, ..., p_{s-1}$, appearing in Theorem 2 can not be decomposed on r geometric subsequences of the form $\{p_{ti}\}, \{p_{ti+1}\}, ..., \{p_{ti+r-1}\}$ with the same rate, for $i = 0, ..., r-1$.*

4. Application in car insurance

An insurance company applies 3 levels of insurance rate, depending on the number of the accidences caused by driver in the last two years: basic - if one, reduced - if none, and raised - if more than one. Suppose, for simplicity, that a driver may cause not more than one accident per year with probability p and the numbers of accidents in different years are independent.

Let T be the first passing time (in years) from the basic one to an other rate of the insurance. It is easy to verify that

$$P(T = n) = \begin{cases} 0, & \text{if} \quad n < 2 \\ (1 - 2p + 2p^2)(p - p^2)^{\frac{n-2}{2}}, & \text{if } n \geq 2 \quad \text{and even} \\ (p - p^2)^{\frac{n-1}{2}}, & \text{if } n \geq 2 \quad \text{and odd.} \end{cases}$$

Thus, by Theorem 2, the random variable $T - 2$ has the selective lack-of-memory with selector $s = 2$ and, moreover, $p_0 = 1 - 2p + 2p^2$and $p_1 = q = p - p^2$. In consequence, by Corollary 1, the random variable T also has the selective lack-of-memory. It is evident that $s = 2$ is its principal selector.

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