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LINEAR MODEL GENEALOGICAL TREE APPLICATION TO AN ODONTOLOGY EXPERIMENT

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Abstract

Commutative Jordan algebras play a central part in orthogonal models. We apply the concepts of genealogical tree of an Jordan algebra associated to a linear mixed model in an experiment conducted to study optimal choosing of dentist materials. Apart from the conclusions of the experiment itself, we show how to proceed in order to take advantage of the great possibilities that Jordan algebras and mixed linear models give to practitioners.

Keywords: commutative Jordan algebra, binary operations, Kronecker matrix product, lattice, projectors.

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1. Introduction

Jordan algebras were first introduced by [7] as part of a new framework for quantum mechanics. The use of these algebras in statistical inference started with the seminal papers of Seely, [13, 14] and [15]. This work as been carried on by many authors, see for instance [6], [9, 10], [12] and [8]. We are mainly interested in commutative Jordan algebras which, see for instance [5], play a central part in the study of orthogonal models.

Consider an experiment carried out to study the differences of two different cements (C_1 and C_2) which were putted in the market for tooth treatments. These differences are measured in terms of an index S (which is the response variable) that measures the solidification of the cement. The idea is that the sooner the cement gets solidified, the better it is, so that the dentist can call in the next patient. This cement is applied conjointly with three different photopolymerizers (F_1, F_2 and F_3) intended to aid the solidification of the cement. The index of solidification, S was measured at two distinct times (t_1 and t_2) since depending on the treatment, some degree of solidification is enough (or, "I have only t_i minutes to spare with this patient... which cement with which polimeryzer should I use?") in 5 disks (d_1, d_2, d_3, d_4 and d_5) that gave 3 observations each (3 replicates).

The design and the analysis of the experiment is made according to the properties and the pertinent basis of the Jordan algebra that is associated to the linear mixed model used to interpret the experiment. These are well defined and explained in [2] and [3]. As part of these properties, we will make use of two different binary operations in Jordan algebras, the *Kronecker product* ($\otimes$) and the *restricted Kronecker product* ($\star$), which were first introduced in [5] and were developed in [2]. The most important theoretical results are resumed in the next section.

2. Theoretical results

2.1. Binary operations and the genealogical tree

We start by defining the Kronecker product between two families of matrices.

Definition 1. Given the families of matrices, $\mathscr{M}_1 = \{\mathbf{M}_{1i}, i = 1, ..., w_1\}$ and $\mathscr{M}_2 = \{\mathbf{M}_{2i}, i = 1, ..., w_2\}$, we take

$$\mathscr{M}_1 \otimes \mathscr{M}_2 = \{\mathbf{M}_{1i} \otimes \mathbf{M}_{2j} : i = 1, ..., w_1; j = 1, ..., w_2\}.$$

Suppose that $\mathscr{A}_i = sp(\mathscr{M}_i)$, $i = 1, 2$, are Commutative Jordan Algebras (CJA). In [13] we can see that it exists, for each CJA, a unique principal base, i.e., a base constituted by a family of mutual orthogonal orthogonal projection matrices ($FMOOPM$). Let $\mathscr{Q}_1 = \{\mathbf{Q}_{1i}, i = 1, ..., w_1\}$ and $\mathscr{Q}_2 = \{\mathbf{Q}_{2i}, i = 1, ..., w_2\}$ be the principal basis of $\mathscr{A}_1$ and $\mathscr{A}_2$. Also, with $\mathscr{Q} = \mathscr{Q}_1 \otimes \mathscr{Q}_2$ we put

$$\mathscr{A}_1 \otimes \mathscr{A}_2 = sp(\mathscr{Q}).$$

Proposition 1. *$\mathscr{A}_1 \otimes \mathscr{A}_2$ is a CJA and $\mathscr{Q}_1 \otimes \mathscr{Q}_2$ is it's principal basis.*

Let $\mathbf{M}_i \in \mathscr{M} = \mathscr{M}_1 \otimes \mathscr{M}_2$, then $\mathbf{M}_i = \mathbf{M}_{1i_1} \otimes \mathbf{M}_{2i_2}$ with $\mathbf{M}_{1i_1} \in \mathscr{M}_1$ and $\mathbf{M}_{2i_2} \in \mathscr{M}_2$. Supposing

$$\mathbf{M}_{1i_1} = \sum_{j_1=1}^{w_1} b_{1i_1j_1}\mathbf{Q}_{1j_1} \text{ and } \mathbf{M}_{2i_2} = \sum_{j_2=1}^{w_2} b_{2i_2j_2}\mathbf{Q}_{2j_2},$$

we have

$$\mathbf{M}_i = \sum_{j=1}^{w_1w_2} b_{ij}\mathbf{Q}_j,$$

where $b_{ij} = b_{1i_1j_1}b_{2i_2j_2}$ and $\mathbf{Q}_j = \mathbf{Q}_{1j_1} \otimes \mathbf{Q}_{2j_2}$, with $i = i_2 + (i_1 - 1)w_2$ and $j = j_2 + (j_1 - 1)w_2$. From here, it's straightforward to show that the transition matrix (this matrix is the matrix such that the i-th line are the coordinates of matrix $\mathbf{M}_i$ with respect to the matrices of family $\mathscr{Q}$, please see [3]) between $\mathscr{M}$ and $\mathscr{Q}$ is

$$\mathbf{B} = \mathbf{B}_1 \otimes \mathbf{B}_2, \tag{1}$$

where $\mathbf{B}_1$ is the transition matrix between $\mathscr{M}_1$ and $\mathscr{Q}_1$ and $\mathbf{B}_2$ is the transition matrix between $\mathscr{M}_2$ and $\mathscr{Q}_2$.

The identity element of $\mathscr{A}_i$ is

$$\mathbf{K}_i = \sum_{j=1}^{w_i} \mathbf{Q}_{ij}. \tag{2}$$

Proposition 2. *Given $k = 1, ..., w_2 - 1$, the family*

$$\mathscr{Q}_k = \{\mathbf{Q}_{1h} \otimes \mathbf{Q}_{2h'}, h = 1, ..., w_1,\ h' = 1, ..., k\} \cup \{\mathbf{K}_1 \otimes \mathbf{Q}_{2h}, h = k+1, ..., w_2\}$$

is a FMOOPM.

The CJA with principal basis $\mathscr{Q}_k$ will be the *restricted k Kronecker product* of $\mathscr{A}_1$ and $\mathscr{A}_2$. We represent this CJA by $\mathscr{A}_1 \star_k \mathscr{A}_2$. When $k = 1$, we write $\mathscr{A}_1 \star \mathscr{A}_2$.

Remark that $\mathscr{A}_1 \star_{w_2} \mathscr{A}_2 = \mathscr{A}_1 \otimes \mathscr{A}_2$. The operation $\star_k$ can, in fact, be generalized to any two families of matrices. We are interested in the case of when, instead of only dealing with the principal basis of CJA's, we operate families $\mathscr{M}_1$ and $\mathscr{M}_2$ of commuting symmetric matrices such that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are the principal basis of $\mathscr{A}_1 = sp(\mathscr{M}_1)$ and $\mathscr{A}_2 = sp(\mathscr{M}_2)$. Putting

$$\mathscr{M}_1 \star_k \mathscr{M}_2 = \{\mathbf{M}_{ih} \otimes \mathbf{M}_{2h'}, h = 1, ..., w_1,\ h' = 1, ..., k\}$$

$$\cup\{\mathbf{K}_1 \otimes \mathbf{M}_{2h}, h = k+1, ..., w_2\},$$

any matrix, say $\mathbf{M}$, of $sp(\mathscr{M}_1 \star_k \mathscr{M}_2)$ will be of the form

$$\mathbf{M} = \sum_{i_1=1}^{w_1} \sum_{i_2=1}^{k} a_{1i_1i_2}\mathbf{M}_{1i_1} \otimes \mathbf{M}_{2i_2} + \sum_{i_3=k+1}^{w_2} a_{2i_3}\mathbf{K}_1 \otimes \mathbf{M}_{2i_3}. \tag{3}$$

We now have

Proposition 3. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two families of commuting symmetric matrices and $\mathscr{Q}_1$, $\mathscr{Q}_2$ the principal basis of $\mathscr{A}_1 = sp(\mathscr{M}_1)$ and $\mathscr{A}_2 = sp(\mathscr{M}_2)$, assume also that $\mathscr{A}_2$ is segregated with separation value k, i.e.,*

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{0} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix},$$

where $\mathbf{B}_{11}$ is of the size $k \times k$. Then,

$$\mathscr{A}_1 \star_k \mathscr{A}_2 = sp(\mathscr{M}_1 \star_k \mathscr{M}_2).$$

Besides this proposition, it's straightforward to see that, if $\mathscr{A}_2$ has segregation value k, given $\mathbf{B}_1$, the transition matrix of $\mathscr{A}_1$, the transition matrix of $\mathscr{A}_1 \star_k \mathscr{A}_2$ will be

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \otimes \mathbf{B}_{11} & \mathbf{0} \\ \mathbf{1}'_{w_1} \otimes \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}. \tag{4}$$

Moreover, it's trivial to see that, $\mathscr{A}_1 \star_k \mathscr{A}_2$ will be segregated with separation value $w_1 k$.

One case of singular importance, as we shall see later on, is the operation $\mathscr{A}_1 \star \mathscr{A}_2$, when $\mathscr{A}_2$ is complete and segregated with separation value 1. In this case, we have

$$\mathbf{B}_2 = \begin{bmatrix} n_2 & \mathbf{0} \\ \mathbf{b} & \mathbf{B}_{22} \end{bmatrix},$$

where $\mathbf{b}$ is of type $(w_2-1)\times 1$ and $\mathbf{B}_{22}$ is "almost" $\mathbf{B}_2$, since it's only missing the first line and the first column of $\mathbf{B}_2$. The matrix $\mathbf{B}$ is then given by

$$\mathbf{B} = \begin{bmatrix} n_2\mathbf{B}_1 & \mathbf{0} \\ \mathbf{1}'_{w_1} \otimes \mathbf{b} & \mathbf{B}_{22} \end{bmatrix}. \tag{5}$$

These concepts are closely connected to linear mixed models. In [5] and [2] we may see that all crossing, nesting and replicates in a mixed linear model can be explained trough the $\otimes$ and $\star$ products of CJA's. In fact it is possible to trace back the model building until we reach singular CJA's, drawing a *genealogical tree* for a model. This concept is deeply explained in [2] where a singular CJA is defined by the one of the simplest linear model, the random sample. This CJA has principal basis given by $\{\frac{1}{n}\mathbf{J}, \bar{\mathbf{J}}\}$, where $\mathbf{J} = \mathbf{1}\mathbf{1}'$ and $\bar{\mathbf{J}} = \mathbf{I} - \frac{1}{n}\mathbf{J}$, and is denoted by $\mathscr{A}(n)$.

This procedure is useful to obtain the principal basis of CJA's associated to models, starting from very simple input. We just write the factor by lexicographic order and, between them, we write $\otimes$ if the first crosses the following, or $\star$ if the second is nested in the first. We will illustrate this procedure later on, when writing the model to interpret the experiment referred in the introduction.

2.2. Optimal estimators

Let

$$\mathbf{Y} \sim \mathcal{N}\left(\mathbf{1}\mu + \sum_{i=2}^{m} \mathbf{X}_i\boldsymbol{\beta}_i, \sum_{j=m+1}^{w-1} \sigma_j^2\mathbf{M}_j + \sigma^2\mathbf{I}\right)$$

be an orthogonal linear model. Putting $\mathbf{M}_1 = \mathbf{1}\mathbf{1}', \mathbf{M}_i = \mathbf{X}_i\mathbf{X}_i', i = 2, ..., m$ and $\mathbf{M}_w = \mathbf{I}$, we have the $\mathscr{M}$ family, $\{\mathbf{M}_1, ..., \mathbf{M}_w\}$ and the principal basis $\mathscr{Q} = \{\mathbf{Q}_1, ..., \mathbf{Q}_w\}$ of $\mathscr{A} = sp(\mathscr{M}) = sp(\mathscr{Q})$. In [2] we have necessary and sufficient conditions for this last equality to hold. The transition matrix is given by $\mathbf{B} = [b_{ij}]$ which we suppose to be segregated with separation value m, so that

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{0} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \mathbf{B}' = \begin{bmatrix} \mathbf{B}'_{11} & \mathbf{B}'_{21} \\ \mathbf{0} & \mathbf{B}'_{22} \end{bmatrix} \tag{6}$$

and

$$(\mathbf{B}')^{-1} = \mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix}. \tag{7}$$

We point out that the variance covariance matrix can be rewritten as

$$\mathbf{V} = \sum_{j=1}^{w} \gamma_j \mathbf{Q}_j, \tag{8}$$

where, with $\sigma_w^2 = \sigma^2$, we have $\gamma_j = \sum_{i=m+1}^{w} b_{ij}\sigma_i^2$. The projection matrix on the range space of the mean vector is

$$\mathbf{Q} = \sum_{i=1}^{m} \mathbf{Q}_i. \tag{9}$$

We suppose that $\mathbf{V}$ and $\mathbf{Q}$ commute and therefore, please see [16], we have the following

Theorem 4. *If* $\mathbf{C}\boldsymbol{\beta}$ *is estimable,* $\widehat{\mathbf{C}\boldsymbol{\beta}} = \mathbf{C}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\mathbf{Y}$ *is it's BLUE.*

Putting $\mathbf{A}' = [\mathbf{A}'_1 \cdots \mathbf{A}'_m]$, we have $\mathbf{Q} = \mathbf{A}'\mathbf{A}$ such that we may write

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{A}'\boldsymbol{\eta}, \tag{10}$$

where $\boldsymbol{\eta} = \mathbf{A}\mathbf{X}\boldsymbol{\beta}$ and consider these, instead of the $\boldsymbol{\beta}$, as parameters of the model. Since $\mathbf{A}$ and $\mathbf{X}$ are known, we have $\widehat{\boldsymbol{\eta}} = \mathbf{A}\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{A}\mathbf{X}\mathbf{X}^{+}\mathbf{Q}\mathbf{Y}$, and, remembering that $\mathbf{X}\mathbf{X}^{+} = \mathbf{Q}$, we get

$$\widehat{\boldsymbol{\eta}} = \mathbf{A}\mathbf{Y} \tag{11}$$

and consequently

$$\widehat{\mathbf{C}\boldsymbol{\eta}} = \mathbf{C}\mathbf{A}\mathbf{Y}. \tag{12}$$

We can also write, for each $i \in \{1, ..., m\}$, $\boldsymbol{\eta}_i = \mathbf{A}\mathbf{X}_i\boldsymbol{\beta}_i$ and $\widehat{\boldsymbol{\eta}}_i = \mathbf{A}_i\mathbf{Y}$. Using this parameterization has some advantages, as we shall see later on.

We will now focus on equation (8). Putting $\sigma_1^2 = \cdots = \sigma_m^2 = 0$, $\boldsymbol{\sigma^2} = [\sigma_1^2 \cdots \sigma_w^2]'$ and $\boldsymbol{\gamma} = [\gamma_1 \cdots \gamma_w]'$ we can write

$$\boldsymbol{\gamma} = \mathbf{B}'\boldsymbol{\sigma^2}, \tag{13}$$

and, with

- $\boldsymbol{\sigma^2}_{[1]} = [\sigma_1^2 \cdots \sigma_m^2]'$,
- $\boldsymbol{\sigma^2}_{[2]} = [\sigma_{m+1}^2 \cdots \sigma_w^2]'$,
- $\boldsymbol{\gamma}_{[1]} = [\gamma_1 \cdots \gamma_m]'$
- $\boldsymbol{\gamma}_{[2]} = [\gamma_{m+1} \cdots \gamma_w]'$,

we have

$$\boldsymbol{\gamma}_{[1]} = \mathbf{B}_{21}'\boldsymbol{\sigma^2}_{[2]} \tag{14}$$

as well

$$\boldsymbol{\sigma^2}_{[2]} = \mathbf{U}_{22}\boldsymbol{\gamma}_{[2]}. \tag{15}$$

These two last expressions are of extreme importance, since they show that once we have an unbiased estimator for $\boldsymbol{\gamma}_{[2]}$ we also have for $\boldsymbol{\sigma^2}_{[2]}$ and $\boldsymbol{\gamma}_{[1]}$.

Since $\mathbb{E}[\mathbf{Y}] = \mathbf{1}\mu + \sum_{i=2}^{m} \mathbf{X}_i\boldsymbol{\beta}_i$, we have that $\mathbb{E}[\mathbf{Y}] \in R\left(\bigoplus_{i=1}^{m} \mathbf{M}_i\right)$, which, due to the segregation of the transition matrix, belongs to the sub-space $R\left(\bigoplus_{i=1}^{m} \mathbf{Q}_i\right)$ that is orthogonal to $R\left(\bigoplus_{i=m+1}^{w} \mathbf{Q}_i\right)$. Thus

$$\mathbb{E}[\mathbf{A}_i\mathbf{Y}] = \mathbf{0}, i = m+1, ..., w, \tag{16}$$

where $\mathbf{A}_i'\mathbf{A}_i = \mathbf{Q}_i$. The variance covariance matrix of $\mathbf{A}_i\mathbf{Y}$, $i = 1, ..., w$, is

$$\begin{aligned}
\mathbf{\Sigma}(\mathbf{A}_i\mathbf{Y}) &= \mathbf{A}_i\mathbf{V}\mathbf{A}_i' && (17)\\
&= \mathbf{A}_i\sum_{j=1}^{w}\gamma_j\mathbf{Q}_j\mathbf{A}_i' && (18)\\
&= \sum_{j=1}^{w}\gamma_j\mathbf{A}_i\mathbf{A}_j'\mathbf{A}_j\mathbf{A}_i' && (19)\\
&= \gamma_i\mathbf{I}_{g_i}. && (20)
\end{aligned}$$

Since $\mathbf{\Sigma}(\mathbf{A}_i\mathbf{Y}) = \mathbb{E}[(\mathbf{A}_i\mathbf{Y} - \mathbb{E}[\mathbf{A}_i\mathbf{Y}])(\mathbf{A}_i\mathbf{Y} - \mathbb{E}[\mathbf{A}_i\mathbf{Y}])']$, for $i = m+1, ..., w$, we have

$$\gamma_i\mathbf{I}_{g_i} = \mathbb{E}\big[(\mathbf{A}_i\mathbf{Y})(\mathbf{A}_i\mathbf{Y})'\big]\,. \tag{21}$$

From (16) and (21), we get

$$\mathbb{E}\big[(\mathbf{A}_i\mathbf{Y})'(\mathbf{A}_i\mathbf{Y})\big] = tr(\gamma_i\mathbf{I}_{g_i}) = \gamma_i g_i. \tag{22}$$

Putting

$$S_i = \|\mathbf{A}_i\mathbf{Y}\|^2 = (\mathbf{A}_i\mathbf{Y})'(\mathbf{A}_iY) = \mathbf{Y}'\mathbf{Q}_i\mathbf{Y} = tr(\mathbf{Q}_i\mathbf{Y}\mathbf{Y}') = < \mathbf{Q}_i, \mathbf{Y}\mathbf{Y}' >, \tag{23}$$

we have

$$\mathbb{E}[S_i] = \gamma_i g_i, \tag{24}$$

which immediately leads us to take

$$\widetilde{\gamma_i} = \frac{S_i}{g_i} \tag{25}$$

as an unbiased estimator of γ_i, and therefore, $\widetilde{\boldsymbol{\gamma}_{[2]}} = [\widetilde{\gamma}_{m+1} \cdots \widetilde{\gamma}_w]'$ is an unbiased estimator of $\boldsymbol{\gamma}_{[2]}$, from which we obtain for $\boldsymbol{\sigma^2}_{[2]}$ and $\boldsymbol{\gamma}_{[1]}$ the unbiased estimators

$$\widetilde{\boldsymbol{\sigma^2}_{[2]}} = \mathbf{U}_{22}\widetilde{\boldsymbol{\gamma}_{[2]}} \tag{26}$$

and

$$\widetilde{\boldsymbol{\gamma}_{[1]}} = \mathbf{B}'_{21}\widetilde{\boldsymbol{\sigma^2}_{[2]}}. \tag{27}$$

Having

1. $det(\mathbf{V}) = \prod_{j=1}^{w} \gamma_j^{g_j}$

2. $\mathbf{V}^{-1} = \sum_{j=1}^{w} \gamma_j^{-1}\mathbf{Q}_j$

the density of $\mathbf{Y}$ will be

$$\begin{aligned} n(\mathbf{y}|\boldsymbol{\mu}, \mathbf{V}) &= \frac{exp\left(-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})'\mathbf{V}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right)}{(2\pi)^{\frac{n}{2}} \prod_{j=1}^{w} \gamma_j^{\frac{g_j}{2}}} \\ &= \frac{exp\left(-\frac{1}{2}\sum_{j=1}^{w}(\mathbf{y}-\boldsymbol{\mu})'\mathbf{Q}_j(\mathbf{y}-\boldsymbol{\mu})\right)}{(2\pi)^{\frac{n}{2}} \prod_{j=1}^{w} \gamma_j^{\frac{g_j}{2}}}. \end{aligned} \tag{28}$$

Since $\mathbf{Q}_j = \mathbf{A}_j'\mathbf{A}_j$ and, for $j > m$, $\mathbf{Q}_j\boldsymbol{\mu} = \mathbf{0}$, we have that

$$(\mathbf{y}-\boldsymbol{\mu})'\mathbf{Q}_j(\mathbf{y}-\boldsymbol{\mu}) = \begin{cases} \|\mathbf{A}_j(\mathbf{y}-\boldsymbol{\mu})\|^2 = \|\boldsymbol{\eta}_j - \widehat{\boldsymbol{\eta}_j}\|^2 & j \le m \\ \|\mathbf{A}_j\mathbf{Y}\|^2 = S_j & j > m \end{cases}, \tag{29}$$

and therefore,

$$n(\mathbf{y}|\boldsymbol{\mu},\mathbf{V}) = \frac{e^{-\frac{1}{2}\left(\sum_{j=1}^{m}\frac{1}{\gamma_j}\|\boldsymbol{\eta}_j-\widehat{\boldsymbol{\eta}_j}\|^2+\sum_{j=m+1}^{w}\frac{S_j}{\gamma_j}\right)}}{(2\pi)^{\frac{n}{2}}\prod_{j=1}^{m}\gamma_j^{\frac{g_j}{2}}}. \tag{30}$$

Theorem 5. *In a linear mixed normal model, the statistics* $\widehat{\boldsymbol{\eta}_j}$ *and* S_j*, defined above, are sufficient and complete.*

Given the Blackwell-Lehmann-Scheffé theorem we then have

Corollary 6. *The estimators* $\widetilde{\boldsymbol{\gamma}_{[2]}}$, $\widetilde{\boldsymbol{\sigma^2}}_{[2]}$, $\widetilde{\boldsymbol{\gamma}_{[1]}}$ *and* $\widehat{\boldsymbol{\eta}}_j$*, defined above, are UMVUE.*

From equations (17) to (20), we have that

$$\widehat{\mathbf{C}\boldsymbol{\eta}_j} \sim \mathcal{N}\left(\mathbf{C}\boldsymbol{\eta}_j, \gamma_j\mathbf{C}\mathbf{C}'\right), j = 1, ..., m, \tag{31}$$

$$S_j \sim \gamma_j\chi^2_{(g_j)}, j = m+1, ..., w \tag{32}$$

are mutually independent.

2.3. Pivot variables

According to the preceding section, we get the pivot variables

$$(33)\ \frac{1}{\gamma_j}\left(\widehat{\mathbf{C}\boldsymbol{\eta}_j}-\mathbf{C}\boldsymbol{\eta}_j\right)'(\mathbf{C}\mathbf{C}')^+\left(\widehat{\mathbf{C}\boldsymbol{\eta}_j}-\mathbf{C}\boldsymbol{\eta}_j\right)\sim\chi^2_{(c)}, j=1,...,m, c=r(\mathbf{C})$$

$$\frac{S_j}{\gamma_j}\sim\chi^2_{(g_j)}, j=m+1,...,w. \tag{34}$$

Clearly, all the γ_j, $j=1,...,m$, are (would be) nuisance parameters.
From equations (14) and (15), we may write

$$\boldsymbol{\gamma}_{[1]}=\mathbf{B}'_{21}\mathbf{U}_{22}\boldsymbol{\gamma}_{[2]}. \tag{35}$$

This last equation enables us to write (33) in such a way that it only depends on $\boldsymbol{\gamma}_{[\mathbf{2}]}$. If $\mathbf{c}_j$ is such that, for any given $j\in 1,...,m$, $\gamma_j=\mathbf{c}'_j\boldsymbol{\gamma}_{[2]}$, we have

$$\frac{1}{\mathbf{c}'_j\boldsymbol{\gamma}_{[2]}}\left(\widehat{\mathbf{C}\boldsymbol{\eta}_j}-\mathbf{C}\boldsymbol{\eta}_j\right)'(\mathbf{C}\mathbf{C}')^+\left(\widehat{\mathbf{C}\boldsymbol{\eta}_j}-\mathbf{C}\boldsymbol{\eta}_j\right)\sim\chi^2_{(g_j)}, j=1,...,m. \tag{36}$$

Writing this equation in such fashion entails an enormous advantage, since we may induce a density function for any $\gamma_j, j=m+1,...,w$, say $f(\gamma_j)$. This is possible since $\frac{S_j}{\gamma_j}$ is an inducing pivot variable, in fact it is an invertible (with respect to γ_j) function and, moreover, given the observed value s_j of S_j, it's invertible function is $m(z)=\frac{s_j}{z}$, which is measurable since it is continuous. We may read about this subject with much more detail in [1], where we have the induced density of $\gamma_j,, j=m+1,...,w$,

$$f(\gamma_j|s_j) = \frac{1}{\Gamma\left(\frac{g_j}{2}\right)\gamma_j}\left(\frac{s_j}{2\gamma_j}\right)^{\frac{g_j}{2}}e^{-\frac{s_j}{2\gamma_j}};\ \gamma_j>0. \tag{37}$$

The statistics S_j, $j = m+1, ..., w$, are independent, thus the joint density is

$$f(\boldsymbol{\gamma}_{[2]}|s_{m+1}, ..., s_w) = \prod_{j=m+1}^{w} f(\gamma_j|s_j), \tag{38}$$

with marginals

$$f(\gamma_j|s_{m+1}, ..., s_w), j = 1, ..., m. \tag{39}$$

If $\zeta_j(x|s_{m+1}, ..., s_w, \gamma_j)$ is the density of the product of two independent random variables one with density $f(\gamma_j|s_{m+1}, ..., s_w)$ and the other a $\chi^2_{(g_j)}$, since the $\widetilde{\boldsymbol{\eta}_1}, ..., \widetilde{\boldsymbol{\eta}_m}$ are independent between themselves as well as from the $S_{m+1}, ..., S_w$, we may rewrite equations (33) and (34) as

$$\left(\widetilde{\mathbf{C}\boldsymbol{\eta}}_j - \mathbf{C}\boldsymbol{\eta}_j\right)'(\mathbf{CC}')^+\left(\widetilde{\mathbf{C}\boldsymbol{\eta}}_j - \mathbf{C}\boldsymbol{\eta}_j\right) \sim \zeta_j(x|s_{m+1}, ..., s_w, \gamma_j), j = 1, ..., m, \tag{40}$$

$$\gamma_j \sim f(\gamma_j|s_j), j = m+1, ..., w. \tag{41}$$

The density function ζ_j has nuisance parameters, so we may apply Monte-Carlo methods.

It seems easy to obtain confidence intervals or to test hypothesis for γ_j, but for $\boldsymbol{\eta}_j$ it is not that evident. The work of obtaining confidence ellipsoids for $\boldsymbol{\eta}_j$ has already been pursued by [4]. Taking $c = r(\mathbf{C})$ we have the $1-q$ level confidence ellipsoid

$$(\widetilde{\mathbf{C}\boldsymbol{\eta}}_j - \mathbf{C}\boldsymbol{\eta}_j)'(\mathbf{CC}')^+(\widetilde{\mathbf{C}\boldsymbol{\eta}}_j - \mathbf{C}\boldsymbol{\eta}_j) \leq \zeta_{1-q,j}$$

with $\zeta_{1-q,j}$ the $1-q$ quantile probability of ζ_j. By the Scheffé Theorem, $\mathbf{C}\boldsymbol{\eta}_j$ lies inside the previous ellipsoid if and only if

$$\bigcap_{\mathbf{z}} \left(|\mathbf{z}'\widehat{\boldsymbol{\eta}} - \mathbf{z}'\boldsymbol{\eta}| \leq \sqrt{c\zeta_{1-q,j}\mathbf{z}'\mathbf{C}\mathbf{C}'\mathbf{z}} \right), \tag{42}$$

so we obtain simultaneous confidence intervals for the $\mathbf{z}'\boldsymbol{\eta}_j$. Whenever

$$|\mathbf{z}'\boldsymbol{\eta}_{j0} - \mathbf{z}'\boldsymbol{\eta}_j| > \sqrt{c\zeta_{1-q,j}\mathbf{z}'\mathbf{C}\mathbf{C}'\mathbf{z}}$$

we may reject

$$H_0 : \mathbf{z}'\boldsymbol{\eta}_j = \mathbf{z}'\boldsymbol{\eta}_{j0}$$

with a risk less or equal than q.

3. The experiment

For better understanding of the experiment referred in the introduction, we now describe it in more detail.

The experimenter intends to evaluate the differences of two different cements (C_1 and C_2) which are just now in market. These cements are intended for tooth treatments. The differences between the cements are measured in terms of an index that measures the solidification of the cement and that we take as the response variable, i.e., $\mathbf{Y}$. The cements are ranked inversely to the time needed to solidification (in practice the sooner the cement is solidified, the sooner the treatment is complete and the sooner the dentist can call in the next patient, maximizing he's profit).

The process of solidification is made under the effect of intensive light (the same for both cements), aided by the presence of a photopolymerizer. There are a few photopolymerizers in the market, from which the three most common were taken into the experiment (F_1, F_2 and F_3).

Depending on the tooth treatment made, some degree of solidification can be enough, so the experimenter was interested in seeing if there were differences in solidification with time. For example, if only some small grade of solidification is needed, (meaning more time is spared), it is interesting to ask which cement with which photopolymerizer should one use. For this reason, the experiment was repeated at two given times (t_1 and t_2).

The experiment was conducted in 5 different disks (d_1, d_2, d_3, d_4 and d_5), which constitute the cells, that were big enough to give three uncorrelated observations (r_1, r_2 and r_3).

The results of the experiment are resumed in Table 1, in which we present the averages of the observations in each disk.

Table 1. Averages of the disks

	t_1					t_2				
	d_1	d_2	d_3	d_4	d_5	d_1	d_2	d_3	d_4	d_5
C_1F_1	26.03	28.43	27.40	26.10	26.77	29.37	30.53	30.27	29.80	29.63
F_2	26.10	26.47	29.90	25.60	24.17	27.13	25.97	29.20	30.77	28.60
F_3	9.83	10.00	10.17	10.67	11.37	9.83	10.00	10.17	10.67	11.37
C_2F_1	26.00	26.93	25.17	26.50	25.17	29.97	29.53	29.00	29.03	25.47
F_2	27.37	26.43	26.23	26.37	27.77	30.67	27.83	28.07	27.40	22.07
F_3	6.07	6.40	6.63	6.60	6.43	6.07	6.40	6.63	6.60	6.43

3.1. The genealogical tree and the resulting algebraic structure

In this, three times replicated, experiment we have three crossed factors, "cement" (C) which is fixed with two levels, "photopolymerizer" (F) which is fixed with three levels and "time" (T) that which is random with 2 levels and nests the factor "disk" (D) which is random with 5 levels.

Therefore, as referred in the second section, the genealogical tree is

$$[C_1, C_2]' \otimes [F_1, F_2, F_3]' \otimes [t_1, t_2]' \star [d_1, d_2, d_3, d_4, d_5]' \star [r_1, r_2, r_3]'$$

and the CJA is

$$(\mathscr{A}(2) \otimes \mathscr{A}(3) \otimes \mathscr{A}(2)) \star \mathscr{A}(5) \star \mathscr{A}(3).$$

This Genealogical Tree is, in fact, very practical since it allows us to get not only the $\mathscr{M}$ family and the principal basis of the associated CJA, but also the incidence matrices of the model. From the definitions of $\otimes$ and $\star$, easily we get

1. $\mathbf{M}_1 = \mathbf{J}_2 \otimes \mathbf{J}_3 \otimes \mathbf{J}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
2. $\mathbf{M}_2 = \mathbf{J}_2 \otimes \mathbf{J}_3 \otimes \mathbf{I}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
3. $\mathbf{M}_3 = \mathbf{J}_2 \otimes \mathbf{I}_3 \otimes \mathbf{J}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
4. $\mathbf{M}_4 = \mathbf{J}_2 \otimes \mathbf{I}_3 \otimes \mathbf{I}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
5. $\mathbf{M}_5 = \mathbf{I}_2 \otimes \mathbf{J}_3 \otimes \mathbf{J}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
6. $\mathbf{M}_6 = \mathbf{I}_2 \otimes \mathbf{J}_3 \otimes \mathbf{I}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
7. $\mathbf{M}_7 = \mathbf{I}_2 \otimes \mathbf{I}_3 \otimes \mathbf{J}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
8. $\mathbf{M}_8 = \mathbf{I}_2 \otimes \mathbf{I}_3 \otimes \mathbf{I}_2 \otimes \mathbf{J}_5 \otimes \mathbf{J}_3$
9. $\mathbf{M}_9 = \mathbf{I}_{12} \otimes \mathbf{I}_5 \otimes \mathbf{J}_3$
10. $\mathbf{M}_{10} = \mathbf{I}_{60} \otimes \mathbf{I}_3$

and

1. $\mathbf{Q}_1 = \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{3}\mathbf{J}_3 \otimes \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
2. $\mathbf{Q}_2 = \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{3}\mathbf{J}_3 \otimes \bar{\mathbf{J}}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
3. $\mathbf{Q}_3 = \frac{1}{2}\mathbf{J}_2 \otimes \bar{\mathbf{J}}_3 \otimes \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
4. $\mathbf{Q}_4 = \frac{1}{2}\mathbf{J}_2 \otimes \bar{\mathbf{J}}_3 \otimes \bar{\mathbf{J}}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
5. $\mathbf{Q}_5 = \bar{\mathbf{J}}_2 \otimes \frac{1}{3}\mathbf{J}_3 \otimes \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
6. $\mathbf{Q}_6 = \bar{\mathbf{J}}_2 \otimes \frac{1}{3}\mathbf{J}_3 \otimes \bar{\mathbf{J}}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
7. $\mathbf{Q}_7 = \bar{\mathbf{J}}_2 \otimes \bar{\mathbf{J}}_3 \otimes \frac{1}{2}\mathbf{J}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
8. $\mathbf{Q}_8 = \bar{\mathbf{J}}_2 \otimes \bar{\mathbf{J}}_3 \otimes \bar{\mathbf{J}}_2 \otimes \frac{1}{5}\mathbf{J}_5 \otimes \frac{1}{3}\mathbf{J}_3$
9. $\mathbf{Q}_9 = \mathbf{I}_{12} \otimes \bar{\mathbf{J}}_5 \otimes \frac{1}{3}\mathbf{J}_3$
10. $\mathbf{Q}_{10} = \mathbf{I}_{60} \otimes \bar{\mathbf{J}}_3$

depending if we start with the basis of $\mathscr{A}(p)$ constituted by $\{\mathbf{J}_p, \mathbf{I}_p\}$ or $\{\frac{1}{p}\mathbf{J}_p, \bar{\mathbf{J}_p}\}$. To get the incidence matrices of the model, it's not difficult to see that we only have to correspond the set of the usual incidence matrices for the random sample, $\{\mathbf{1}_p, \mathbf{I}_p\}$, and proceed in the same way. Thus,

1. $\mathbf{X}_1 = \mathbf{1}_2 \otimes \mathbf{1}_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
2. $\mathbf{X}_2 = \mathbf{1}_2 \otimes \mathbf{1}_3 \otimes \mathbf{I}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
3. $\mathbf{X}_3 = \mathbf{1}_2 \otimes \mathbf{I}_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
4. $\mathbf{X}_4 = \mathbf{1}_2 \otimes \mathbf{I}_3 \otimes \mathbf{I}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
5. $\mathbf{X}_5 = \mathbf{I}_2 \otimes \mathbf{1}_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
6. $\mathbf{X}_6 = \mathbf{I}_2 \otimes \mathbf{1}_3 \otimes \mathbf{I}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
7. $\mathbf{X}_7 = \mathbf{I}_2 \otimes \mathbf{I}_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
8. $\mathbf{X}_8 = \mathbf{I}_2 \otimes \mathbf{I}_3 \otimes \mathbf{I}_2 \otimes \mathbf{1}_5 \otimes \mathbf{1}_3$
9. $\mathbf{X}_9 = \mathbf{I}_{12} \otimes \mathbf{I}_5 \otimes \mathbf{1}_3$
10. $\mathbf{X}_{10} = \mathbf{I}_{60} \otimes \mathbf{I}_3$
11. $\mathbf{X} = [\mathbf{X}_1\ \mathbf{X}_2\ \mathbf{X}_3\ \mathbf{X}_4]$.

The transition matrix can also be taken from the genealogical tree. For $\mathscr{A}(p)$ the transition matrix is given by $\mathbf{B} = \begin{bmatrix} p & 0 \\ 1 & 1 \end{bmatrix}$, thus from both equations (1) and (5), we get

$$\mathbf{B} = \left[\begin{array}{cccc|cccccc}
180 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
90 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 30 & 30 & 30 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
90 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 \\
45 & 45 & 0 & 0 & 45 & 45 & 0 & 0 & 0 & 0 \\
30 & 0 & 30 & 0 & 30 & 0 & 30 & 0 & 0 & 0 \\
15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 0 & 0 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]. \tag{43}$$

We have identified matrices $\mathbf{B}_{11}$, $\mathbf{B}_{21}$ and $\mathbf{B}_{22}$ accordingly to equation (6). Matrix $\mathbf{U}$ defined in equation (7) is given by

$$\mathbf{U} = \left[\begin{array}{cccc|cccccc}
\frac{1}{180} & -\frac{1}{180} & -\frac{1}{180} & \frac{1}{180} & -\frac{1}{180} & \frac{1}{180} & \frac{1}{180} & -\frac{1}{180} & 0 & 0 \\
0 & \frac{1}{90} & 0 & -\frac{1}{90} & 0 & -\frac{1}{90} & 0 & \frac{1}{90} & 0 & 0 \\
0 & 0 & \frac{1}{60} & -\frac{1}{60} & 0 & 0 & -\frac{1}{60} & \frac{1}{60} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{30} & 0 & 0 & 0 & -\frac{1}{30} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \frac{1}{90} & -\frac{1}{90} & -\frac{1}{90} & \frac{1}{90} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{90} & 0 & -\frac{2}{90} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & -\frac{1}{30} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{30} & -\frac{2}{30} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & -\frac{1}{30} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right], \tag{44}$$

where we also identified $\mathbf{U}_{11}$, $\mathbf{U}_{12}$ and $\mathbf{U}_{22}$.

It is trivial to write a procedure to obtain each matrix $\mathbf{M}_i$, $\mathbf{Q}_i$, $\mathbf{X}_i$, $i = 1, ..., 10$ and matrix $\mathbf{B}$ which illustrates the enormous advantage of the genealogical tree.

In order to calculate one matrix $\mathbf{P}$, i.e., one common diagonalizer of for all matrices $\mathbf{M}_i$, $i = 1, ..., w$, it is easier to calculate the roots of matrices $\mathbf{Q}_i$, $i = 1, ..., w$, which are the matrices $\mathbf{A}_i$, $i = 1, ..., w$, referred in the previous section. In fact, it is as easy as easy as calculating a singular value decomposition of each matrix $\mathbf{Q}_i$. This can be made in most matrix manipulation software packages where we can obtain matrices $\mathbf{U}_i$, $\mathbf{S}_i$ and $\mathbf{T}_i$ such that $\mathbf{U}_i\mathbf{S}_i\mathbf{T}_i' = \mathbf{Q}_i$ where $\mathbf{S}_i$ is a diagonal matrix of the same dimension as $\mathbf{Q}_i$ with nonnegative diagonal elements in decreasing order, and $\mathbf{U}$ and $\mathbf{T}$ are unitary matrices. Once the singular value decomposition is obtained, we have that $\mathbf{A}_i$ is constituted by the first g_i lines of the transpose of $\mathbf{Q}_i\mathbf{T}_i$, where g_i is the trace of $\mathbf{Q}_i$. Observe that, in our case, these calculus are even easier to carry out since $\mathbf{Q}_i$ is symmetric, meaning that $\mathbf{U}_i = \mathbf{T}_i$ and, since $\mathbf{Q}_i$ has eigenvalues 1 or 0, $\mathbf{S}$ has either 0 or 1 in the diagonal.

3.2. Estimation

3.2.1. Fixed effects

The fixed effects considered in the experiment were the cement, the photopolymerizer and, therefore, the interactions between these.

Accordingly to the objectives explained before, we are interested in estimating differences between the different levels of cement ($[1 \;\; (-1)]\boldsymbol{\beta}_2$), photopolymerizer (we chose $[1 \;\; (-1) \;\; 0]\boldsymbol{\beta}_3$ and $[0 \;\; 1 \;\; (-1)]\boldsymbol{\beta}_3$) and interactions (we chose $[1 \;\; (-1) \;\; 0 \;\; 0 \;\; 0 \;\; 0]\boldsymbol{\beta}_4$ and $[0 \;\; 0 \;\; 0 \;\; 1 \;\; (-1) \;\; 0]\boldsymbol{\beta}_4$). For this purposes, choosing

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

from Theorem 4, we get that

$$\widehat{\mathbf{C}\boldsymbol{\beta}} = \begin{bmatrix} 10.5794 \\ 0.7650 \\ 3.9811 \\ 5.8700 \\ -1.3683 \\ 1.8683 \end{bmatrix} \text{ is the estimate of } \begin{bmatrix} \mu \\ C_1 - C_2 \\ F_1 - F_2 \\ F_2 - F_3 \\ C_1F_1 - C_1F_2 \\ C_2F_1 - C_2F_2 \end{bmatrix}.$$

To any other estimates, we just need to choose any other matrix $\mathbf{C}$. We note that only contrasts are estimable.

3.2.2. Random effects

The random effects and interactions considered in the experiment are, in the design order, time (for which we want to test σ_5^2), the interaction time×cement (for which we want to test σ_6^2), the interaction time × photopolymerizer (for which we want to test σ_7^2), the interaction time × cement× photopolymerizer (for which we want to test σ_8^2), and disk (for which we want to test σ_9^2). Observe that there are no interactions between nested factors and that we will also estimate $\sigma_{10}^2 = \sigma^2$ which correspond to the technical error.

Since matrices $\mathbf{Q}_i$, $i = 5, ..., 10$, and matrices $\mathbf{A}_i$, $i = 5, ..., 10$, are already obtained, according to equations (23) and (25), we have

$$\widetilde{\boldsymbol{\gamma}_{[2]}} = \begin{bmatrix} \frac{S_5}{g_5} = \frac{tr(\mathbf{Q}_5\mathbf{y}\mathbf{y}')}{tr(\mathbf{Q}_5)} \\ \frac{S_6}{g_6} = \frac{tr(\mathbf{Q}_6\mathbf{y}\mathbf{y}')}{tr(\mathbf{Q}_6)} \\ \frac{S_7}{g_7} = \frac{tr(\mathbf{Q}_7\mathbf{y}\mathbf{y}')}{tr(\mathbf{Q}_7)} \\ \frac{S_8}{g_8} = \frac{tr(\mathbf{Q}_8\mathbf{y}\mathbf{y}')}{tr(\mathbf{Q}_8)} \\ \frac{S_9}{g_9} = \frac{tr(\mathbf{Q}_9\mathbf{y}\mathbf{y}')}{tr(\mathbf{Q}_9)} \\ \frac{S_{10}}{g_{10}} = \frac{tr(\mathbf{Q}_{10}\mathbf{y}\mathbf{y}')}{tr\mathbf{Q}_{10}} \end{bmatrix} = \begin{bmatrix} \frac{2.2162 \times 10^3}{1} = 2.2162 \times 10^3 \\ \frac{2.7534 \times 10^3}{1} = 2.7534 \times 10^3 \\ \frac{898.2608}{2} = 449.1304 \\ \frac{687.3381}{2} = 343.6691 \\ \frac{173.1027}{48} = 3.6063 \\ \frac{739.3067}{120} = 6.1609 \end{bmatrix}$$

which, according to expression (26) enables us to use matrix $\mathbf{U}_{22}$ to calculate

$$(45) \qquad \mathbf{U}_{22}\widetilde{\gamma_{[2]}} = \begin{bmatrix} -7.1408 \\ 53.5501 \\ 3.5154 \\ 22.6709 \\ -0.8515 \\ 6.1609 \end{bmatrix} \text{ which is the estimate of } \begin{bmatrix} \sigma_5^2 \\ \sigma_6^2 \\ \sigma_7^2 \\ \sigma_8^2 \\ \sigma_9^2 \\ \sigma_{10}^2 \end{bmatrix}.$$

3.3. Testing

3.3.1. Fixed factors

The hypothesis of interest, at this point, are clear. Concerning

1. cement,

   H_0^C: There is no difference between C_1 and C_2

   vs.

   H_1^C: There is a difference between C_1 and C_2,

2. photopolymerizer,

   H_0^F: There are no differences between F_1, F_2 and F_3

   vs.

   H_1^F: There is at least a difference between F_1, F_2 or F_3,

3. interactions cement×photopolymerizer

   H_0^{CF}: There are no differences between any interaction C_1F_1, C_1F_2, C_1F_3, C_2F_1, C_2F_2 and C_3F_3

   vs.

   H_1^{CF}: There is at least a difference between interactions C_1F_1, C_1F_2, C_1F_3, C_2F_1, C_2F_2 or C_3F_3.

Accordingly to equation (10), these hypothesis are equivalent to

1. (for cement)

$$H_0^C\colon\ \eta_2 = 0$$

vs.

$$H_1^C\colon\ \eta_2 \neq 0,$$

2. (for photopolymerizer)

$$H_0^F\colon\ \boldsymbol{\eta}_3 = \mathbf{0}$$

vs.

$$H_1^F\colon\ \boldsymbol{\eta}_3 \neq \mathbf{0},$$

3. (for interactions cement×photopolymerizer)

$$H_0^{CF}\colon\ \boldsymbol{\eta}_4 = \mathbf{0}$$

vs.

$$H_1^{CF}\colon\ \boldsymbol{\eta}_4 \neq \mathbf{0}.$$

A remark is due at this point. η_2 is a scalar and $\boldsymbol{\eta}_i, i = 3, 4$, has two components. This is, off course, linked to the rank of the correspondent matrix $\mathbf{A}_i, i = 2, 3, 4$, and is something that can be found in any introductory book of analysis of variance, see for example [11]. According to the definition of effects and interactions, their sums has to be null, i.e., $C_1 + C_2 = 0$, $F_1 + F_2 + F_3 = 0$, and also

$$\begin{array}{c|ccc|c} & F_1 & F_2 & F_3 & sum \\ \hline C_1 & C_1F_1 & C_1F_2 & C_1F_3 & 0 \\ C_2 & C_2F_1 & C_2F_2 & C_2F_3 & 0 \\ \hline sum & 0 & 0 & 0 & 0 \end{array}.$$

This means that, for cement, there is only $(2-1) = 1$ effects "free" (or there is 1 degree of freedom), for photopolymerizer there are $(3 - 1) = 2$ degrees of freedom and for the interaction there are $(2 - 1)(3 - 1) = 2$ degrees of freedom. This is the reason why, for the cements to be equal, we only need to test if one contrast is null and for photopolymerizer and interactions we need to test if two (any two linearly independent) contrasts are simultaneous null.

The estimates of η_1 (that concerns the mean value, and therefore of no interest), η_2, $\boldsymbol{\eta}_3$ and $\boldsymbol{\eta}_4$ which, geometrically, are estimates of contrasts that belong to $R(\mathbf{Q}_2)$, $R(\mathbf{Q}_3)$ and $R(\mathbf{Q}_4)$, can be obtained using equation (12),

$$\eta_1 = -283.8763, \eta_2 = -6.8424, \boldsymbol{\eta}_3 = \begin{bmatrix} 48.2270 \\ -65.6120 \end{bmatrix}$$

$$\text{and } \boldsymbol{\eta}_4 = \begin{bmatrix} 41.7091 \\ 7.7001 \end{bmatrix}.$$

According to equations (33) and (34) and choosing

(46) $C = 1$ for cement,

(47) $\mathbf{C} = \mathbf{I}_2$ for photopolymerizer and

(48) $\mathbf{C} = \mathbf{I}_2$ for the interaction between them,

we have that, for a significance level α,

1. under H_0^C, $\frac{1}{\gamma_2} \times 46.8180$ should be smaller than the $(1-\alpha)$ quantile of the chi-square distribution with 1 degree of freedom,
2. under H_0^F, $\frac{1}{\gamma_3} \times 6.6308 \times 10^3$ should be smaller than the $(1-\alpha)$ quantile of the chi-square distribution with 2 degrees of freedom,
3. under H_0^{CF}, $\frac{1}{\gamma_4} \times 1.7989 \times 10^3$ should be smaller than the $(1-\alpha)$ quantile of the chi-square distribution with 2 degrees of freedom.

For practical reasons, in order to apply the theory of Section 2.3, we will estimate the parameters $\gamma_i, i = 2, 3, 4$, or, in fact, use equation (35) and (39) to generate samples for each $\gamma_i, i = 2, 3, 4$. Matrix $\mathbf{B}'_{21}\mathbf{U}_{22}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = [\mathbf{c}'_1 \ \ \mathbf{c}'_2 \ \ \mathbf{c}'_3 \ \ \mathbf{c}'_4]',$$

such that $\gamma_j = \mathbf{c}'_j \chi^2_{(g_j)}, j = 2, 3, 4,$ where $g_2 = 1, g_3 = 2, g_4 = 2$. Having chosen to generate ten thousand $\gamma_j, j = 2, 3, 4$, and $\alpha = 5\%$, we calculated the percentage of times that we rejected H_0^C, H_0^F and H_0^{CF}. We expect that, if H_0 is true, then this percentage is near 5%. The results obtained are the following.

	C	F	CF
H_0 rejections (%)	0%	67%	31%

from which we conclude that, at a significance level of 5%, there does not exist statistical evidence to say that both cements difer from one another, while there exists for photopolymerizers and interactions.

We give an illustration on Figure 1 of the interaction CF from which we see the significance of this interaction since both lines are not parallel.

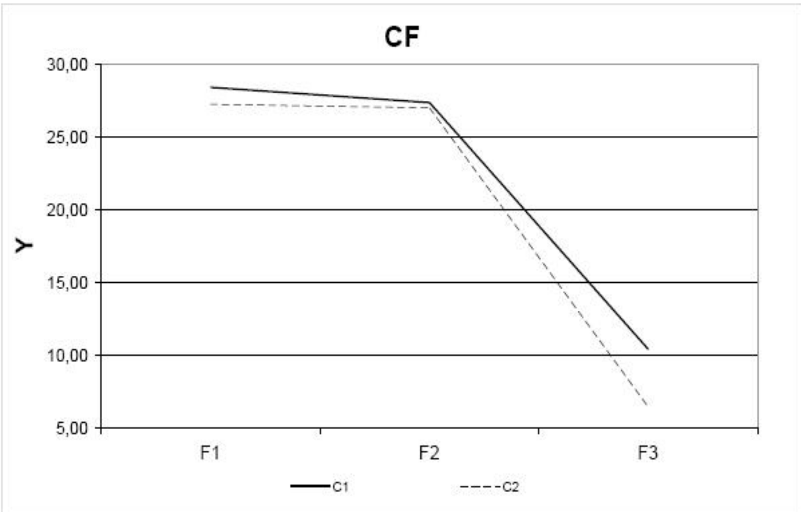

Figure 1. Interaction CF

3.3.2. Random factors

The hypothesis of interest in random factors are the following

1. for time

$$H_0^t\colon\ \sigma_5^2 = 0$$
vs.
$$H_1^t\colon\ \sigma_5^2 > 0$$

2. for the interaction cement×time

$$H_0^{Ct}\colon\ \sigma_6^2 = 0$$
vs.
$$H_1^{Ct}\colon\ \sigma_6^2 > 0$$

3. for the interaction photopolymerizer×time

$$H_0^{Ft}\colon\ \sigma_7^2 = 0$$
vs.
$$H_1^{Ft}\colon\ \sigma_7^2 > 0$$

4. for the interaction cement×photopolymerizer×time

$$H_0^{CFt}\colon\ \sigma_8^2 = 0$$
vs.
$$H_1^{CFt}\colon\ \sigma_8^2 > 0$$

5. for disks

$$H_0^d\colon\ \sigma_9^2 = 0$$

vs.

$$H_1^d\colon\ \sigma_9^2 > 0.$$

To test these hypothesis, we use again the results in Section 2.3. In practical sense, instead of calculating estimates of each $\sigma_i^2, i = 5, 6, 7, 8, 9$, like we did in equation (45), we intend to generate $1 - \alpha$ confidence intervals for them by generating ten thousand of each $\gamma_i, i = 5, 6, 7, 8, 9$ by the same procedure used for fixed factors.

We reject each H_0 at the significance level α if the $(1 - \alpha)$ confidence interval for each σ_i^2 does not covers 0.

With $\alpha = 5\%$ we obtained the confidence intervals

	t	Ct	Ft	CFt	d
lower bound	-35178.08	-145.67	-390.33	6.03	-1.58
upper bound	24642.79	73877.66	548.44	886.74	-0.06

from which we decide that, at a significance level of 5%, there only exists statistical evidence to say that there exists significant differences between different levels of interaction CFt. In Figure 2 we can see the illustration of this interaction.

3.4. Further analysis

In this section we intend to go through two decisions made in the previous section. We will use this to show how to make a possible aggregation and how to choose different contrasts to test.

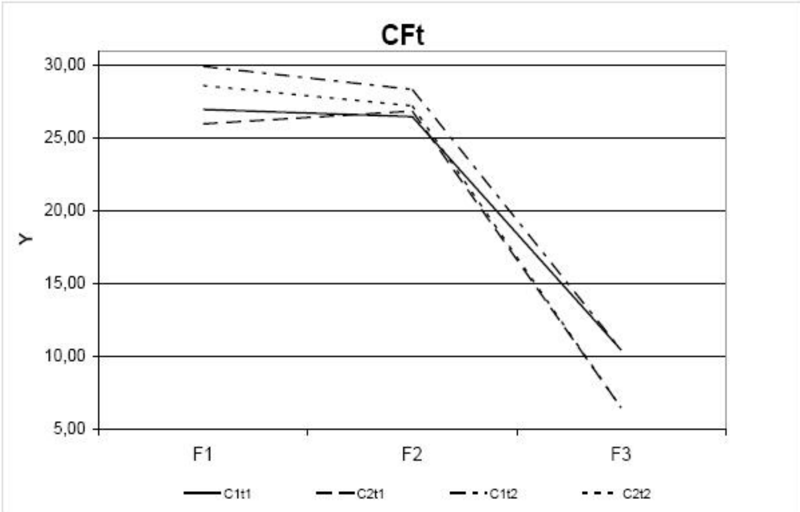

Figure 2. Interaction CFt

3.4.1. Aggregation

The first decision we would like to discuss concerns the disks used in the experience. Since there is no difference between disks we will aggregate this factor into the random error, i.e., we will, both,

1. consider the genealogical tree $(\mathscr{A}(2) \otimes \mathscr{A}(3) \otimes \mathscr{A}(2)) \star \mathscr{A}(15)$,
2. aggregate A_9 and A_{10} into the same matrix.

Following the entire procedure made in the previous sections, with this new model we have the following results which conduct to the same decisions.

3.4.2. Estimates

$$\widehat{\mathbf{C}\boldsymbol{\beta}} \text{ yields, as expected, the same value}$$

and

$$\mathbf{U}_{22}\widetilde{\gamma_{[2]}} = \begin{bmatrix} -7.1408 \\ 53.5501 \\ 3.5154 \\ 22.5492 \\ 5.4312 \end{bmatrix} \text{ which is the estimate of } \begin{bmatrix} \sigma_5^2 \\ \sigma_6^2 \\ \sigma_7^2 \\ \sigma_8^2 \\ \sigma_9^2 \end{bmatrix}$$

3.4.3. Testing

	C	F	CF
H_0 rejections %	0%	67%	31%

and

	t	Ct	Ft	CFt
lower bound	−36340.37	−136.20	−371.70	5.94
upper bound	25804.10	74750.87	581.90	832.29

3.4.4. Contrasts

The second decision concerns the effects of photopolymerizers. From the observation of data, it seems clear the the differences between photopolymerizers is due to the third level, while the first and second only differ slightly. We would like to test this hypothesis.

Taking a look at matrix $\mathbf{A}_4$ we see that

$$\mathbf{A}_4 = \begin{bmatrix} 0 & a_1 & -a_1 & 0 & a_1 & -a_1 \\ -2a_1 & a_1 & a_1 & -2a_1 & a_1 & a_1 \end{bmatrix} \otimes \mathbf{1}'_{30}$$

by which $\boldsymbol{\eta}_4 = [\eta_{41} \ \ \eta_{42}]'$ estimates the contrasts $F_2 - F_3$ and $-2F_1 + F_2 + F_3$. We are now interested in testing the contrasts

1. $F_1 + F_2 - 2F_3$ and

2. $F_1 - F_2$,

it being necessary to choose matrix $\mathbf{C}$ in equation (47) so that these new contrasts can be written in terms of the new one. Since this a trivial exercise of solving linear system, it's easy to find out $\mathbf{C} = [\frac{3}{2} \ \ -\frac{5}{6}]$ for the first new contrast and $\mathbf{C} = [-\frac{1}{2} \ \ -\frac{1}{2}]$ for the second new contrasts.

Computing the same statistics with the same decision rule we have

	$F_1 + F_2 - 2F_3$	$F_1 - F_2$
H_0 rejections %	73%	0%

which is in line with what we suspected.

4. Conclusions

Binary operations between algebras are extremely useful in defining models. The concept of genealogical tree illustrates how the associated algebra is constructed and enables us to easily conduct posterior analysis, as the, for example, the aggregation made in the experiment.

As far as the experiment results, all estimation and hypothesis testing was very easy to apply and the conclusions were in order to the experimenters intuition.

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