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TESTING HYPOTHESES IN UNIVERSAL MODELS*

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Abstract

A linear regression model, when a design matrix has not full column rank and a covariance matrix is singular, is considered. The problem of testing hypotheses on mean value parameters is studied. Conditions when a hypothesis can be tested or when need not be tested are given. Explicit forms of test statistics based on residual sums of squares are presented.

Key words: universal linear model, unbiased estimator, tests hypotheses.

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1. Introduction

Let a linear regression model be under consideration. Generally, no assumptions on the rank of design and covariance matrices are given. When testing linear hypotheses on mean value parameters in universal (singular) models, three typical situations can occur; either a hypothesis cannot be tested, or a hypothesis need not be tested, since it is automatically true, or a hypothesis can be tested.

The aim of the paper is to investigate possible situations which can occur when testing hypothesis in universal models and to find proper test statistics based on residual sums of squares.

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2. Notations and auxiliary statements

Let $\mathbf{A}$ be an $m \times n$ matrix. Let $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$ and $Ker(\mathbf{A}) = \{\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^n, \mathbf{A}\boldsymbol{u} = \mathbf{0}\} \subset \mathbb{R}^n$ denote the column space and the null space of the matrix $\mathbf{A}$, respectively. Let $\mathbf{W}$ be an $m \times m$ symmetric positive semidefinite matrix such that $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$. Then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}} = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{-}\mathbf{A}'\mathbf{W}$ denotes a projector on $\mathcal{M}(\mathbf{A})$ in the $\mathbf{W}$-seminorm. The symbol $\mathbf{M}_{\mathbf{A}}^{\mathbf{W}}$ means $\mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\mathbf{W}}$. If $\mathbf{W} = \mathbf{I}$ (identity matrix), symbols $\mathbf{P}_{\mathbf{A}}$ and $\mathbf{M}_{\mathbf{A}}$ are used. The $\mathbf{W}$-seminorm of $\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^m$, is given by $\|\boldsymbol{x}\|_{\mathbf{W}} = \sqrt{\boldsymbol{x}'\mathbf{W}\boldsymbol{x}}$. Symbols $\mathbf{A}^{-}$ and $\mathbf{A}^{+}$ mean the g-inverse and the Moore-Penrose inverse of the matrix $\mathbf{A}$, respectively.

Let $\mathbf{N}$ be an $n \times n$ symmetric positive semidefinite matrix. The symbol $\mathbf{A}_{m(\mathbf{N})}^{-}$ denotes the minimum $\mathbf{N}$-seminorm g-inverse of the matrix $\mathbf{A}$, i.e., the matrix $\mathbf{A}_{m(\mathbf{N})}^{-}$ satisfies equations

$$(1) \qquad \mathbf{A}\mathbf{A}_{m(\mathbf{N})}^{-}\mathbf{A} = \mathbf{A}, \quad \mathbf{N}\mathbf{A}_{m(\mathbf{N})}^{-}\mathbf{A} = \mathbf{A}'\left(\mathbf{A}_{m(\mathbf{N})}^{-}\right)'\mathbf{N}.$$

One of representation of the matrix $\mathbf{A}_{m(\mathbf{N})}^{-}$ is

$$\mathbf{A}_{m(\mathbf{N})}^{-} = \begin{cases} \mathbf{N}^{-}\mathbf{A}'(\mathbf{A}\mathbf{N}^{-}\mathbf{A}')^{-} & \text{if } \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N}), \\ (\mathbf{N} + \mathbf{A}'\mathbf{A})^{-}\mathbf{A}'[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^{-}\mathbf{A}']^{-} & \text{otherwise.} \end{cases}$$

In more detail cf. [4].

Lemma 2.1. *Let* $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{A})$ *and* $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C})$. *Then*

$$(2) \qquad (\mathbf{A} - \mathbf{B}\mathbf{C}^{-}\mathbf{B}')^{-} = \mathbf{A}^{-} + \mathbf{A}^{-}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-}\mathbf{B})^{-}\mathbf{B}'\mathbf{A}^{-}.$$

Proof. It is an obvious consequence of Rohde theorem (cf., e.g., [1], p. 446, Lemma 10.1.40). ∎

3. Universal model

The universal linear model is considered in the form

$$\boldsymbol{Y} \sim N_n\Big(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}\Big), \quad \boldsymbol{\beta} \in \mathbb{R}^k, \tag{3}$$

where $\boldsymbol{Y}$ is an n-dimensional normally distributed random vector, $\mathbf{X}\boldsymbol{\beta}$ is the mean value of $\boldsymbol{Y}$ and $\boldsymbol{\Sigma}$ its covariance matrix. $\mathbf{X}$ is a given matrix of the type $n \times k$ and $\boldsymbol{\Sigma}$ is a given $n \times n$ symmetric positive semidefinite matrix.

The best linear unbiased estimator (BLUE) of the function $\mathbf{X}\boldsymbol{\beta}$ in the universal model (3) is (cf. [4], p. 148)

$$\widehat{\mathbf{X}\boldsymbol{\beta}} = \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\boldsymbol{Y} = \mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\boldsymbol{Y} \tag{4}$$

with the covariance matrix

$$\mathrm{Var}\left(\widehat{\mathbf{X}\boldsymbol{\beta}}\right) = \mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\boldsymbol{\Sigma}\mathbf{T}^-\mathbf{X}\mathbf{D}^-\mathbf{X}' = \mathbf{X}(\mathbf{D}^- - \mathbf{I})\mathbf{X}',$$

where

$$\mathbf{T} = \boldsymbol{\Sigma} + \mathbf{X}\mathbf{X}', \quad \mathbf{D} = \mathbf{X}'\mathbf{T}^-\mathbf{X}.$$

Let a null hypothesis

$$H_0 : \boldsymbol{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}, \quad \boldsymbol{h} \in \mathcal{M}(\mathbf{H}),$$

where $\mathbf{H}$ is a given $h \times k$ matrix and $\boldsymbol{h}$ is a given h-dimensional vector, be tested in the universal model (3) against an alternative hypothesis

$$H_a : \boldsymbol{h} + \mathbf{H}\boldsymbol{\beta} = \boldsymbol{\xi} \neq \mathbf{0}.$$

If the hypothesis is taken into account as constraints on the parameter $\boldsymbol{\beta}$,the estimator of $\mathbf{X}\boldsymbol{\beta}$ can be determined in the following way. Let $\boldsymbol{\beta}_0$ be any solution of the equation $\boldsymbol{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}$. Then the parameter $\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \{\boldsymbol{u} : \boldsymbol{h} + \mathbf{H}\boldsymbol{u} = \mathbf{0}\}$, can be expressed by the help of a new parameter $\boldsymbol{\gamma}$

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{K}_{\mathbf{H}}\gamma, \quad \gamma \in \mathbb{R}^{k-\mathrm{rank}(\mathbf{H})},$$

where $\mathbf{K}_{\mathbf{H}}$ is a $k \times [k - \mathrm{rank}(\mathbf{H})]$ matrix such that $\mathcal{M}(\mathbf{K}_{\mathbf{H}}) = Ker(\mathbf{H})$.

The new model without constraints is in the form

$$\boldsymbol{Y} \sim N_n\left(\mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}\mathbf{K}_{\mathbf{H}}\boldsymbol{\gamma}, \boldsymbol{\Sigma}\right),\ \boldsymbol{\gamma} \in \mathbb{R}^{k-\mathrm{rank}(\mathbf{H})}.$$

Hence the BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the universal model (3) respecting the null hypothesis is

$$\begin{aligned}\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} &= \mathbf{X}\boldsymbol{\beta}_0 + \widehat{\mathbf{X}\mathbf{K}_{\mathbf{H}}\boldsymbol{\gamma}}\\ &= \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}\mathbf{K}_{\mathbf{H}}\left[(\mathbf{K}_{\mathbf{H}}'\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0).\end{aligned}$$

Since

$$\mathcal{M}\left(\mathbf{M}_{\mathbf{H}'}\right) = Ker(\mathbf{H}), \quad \mathcal{M}\left(\mathbf{X}\mathbf{K}_{\mathbf{H}}\right) = \mathcal{M}\left(\mathbf{X}\mathbf{M}_{\mathbf{H}'}\right),$$

and

$$\mathcal{P}\left\{\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0 \in \mathcal{M}(\boldsymbol{\Sigma})\right\} = 1,$$

it holds that

$$\mathbf{X}\mathbf{K}_{\mathbf{H}}\left[(\mathbf{K}_{\mathbf{H}}'\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) = \mathbf{X}\mathbf{M}_{\mathbf{H}'}\left[(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)$$

and thus

$$\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\left[(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0). \tag{5}$$

The symbol $\widehat{\ }$ means the estimator in the universal model (3) and $\widehat{\widehat{\ }}$ means the estimator in the universal model (3) respecting the null hypothesis.

4. Testing linear hypotheses

Here approach of χ^2-tests based on residual sums of squares

$$R_0^2 = \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)'\left[\mathrm{Var}\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)\right]^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right),$$

$$R_1^2 = \left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)' \left[\mathrm{Var}\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)\right]^- \left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)$$

is used (cf. [3], p. 153-157, the first and the second theorems of the least squares theory).

Lemma 4.1. *Let in the universal model* (3) *the null hypothesis be considered.*

(i) *Matrices* $\boldsymbol{\Sigma}^-$, $(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-$ *and* $(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{X}')^-$ *can be chosen as a g-inverses of the matrix* $\mathrm{Var}(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}})$.

(ii) *Matrices* $\boldsymbol{\Sigma}^-$ *and* $(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-$ *can be chosen as a g-inverses of the matrix* $\mathrm{Var}(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}})$.

Proof. Obviously covariance matrices of residual vectors are

$$\mathrm{Var}\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) = \boldsymbol{\Sigma} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma},$$

$$\mathrm{Var}\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right) = \boldsymbol{\Sigma} - \mathbf{X}\mathbf{M}_{\mathbf{H}'}\left[(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma}.$$

Let the matrix $(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-$ be chosen. Then

$$\left\{\boldsymbol{\Sigma} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma}\right\}(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left\{\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\right\}$$
$$= \left\{\mathbf{I} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{\Sigma}(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}' - \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)$$
$$\times\left\{\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\right\}$$
$$= \left\{\mathbf{I} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{\Sigma}\left\{\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\right\} = \mathrm{Var}\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right).$$

The other statements can be proved in an analogous way. ∎

Theorem 4.2. *Let in the universal model* (3) *the null hypothesis be considered. Let* $\mathcal{M}(\mathbf{H}') \cap \mathcal{M}(\mathbf{X}') = \{\mathbf{0}\}$*. Then* $R_1^2 - R_0^2 = 0$*, i.e, the hypothesis cannot be tested by the help of the statistic* $R_1^2 - R_0^2$*.*

Proof. If $\mathcal{M}(\mathbf{H}') \cap \mathcal{M}(\mathbf{X}') = \{\mathbf{0}\}$, then $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{H}'}) = \mathcal{M}(\mathbf{X})$, since

$$\operatorname{rank}(\mathbf{X}) + \operatorname{rank}(\mathbf{H}) = \operatorname{rank}\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix} = \operatorname{rank}(\mathbf{X}\mathbf{M}_{\mathbf{H}'}) + \operatorname{rank}(\mathbf{H})$$

(cf. [4], p. 137). Thus

$$\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}} = \left\{\mathbf{I} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{Y} = \left\{\mathbf{I} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\right\}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)$$

$$= \left\{\mathbf{I} - \mathbf{X}\mathbf{M}_{\mathbf{H}'}\left[(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\right\}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) = \boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}$$

and with respect to Lemma 4.1 we obtain $R_1^2 - R_0^2 = 0$. ■

The last theorem implies that those rows of the matrix $\mathbf{H}$, which cannot be obtained from rows of the matrix $\mathbf{X}$ by a linear combination, cannot be used in the hypothesis. Therefore in the following text $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ is assumed. Moreover, the assumption $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ implies that the vector function $\mathbf{H}\boldsymbol{\beta}$ is unbiasedly estimable in the universal model (3) as

$$\widehat{\mathbf{H}\boldsymbol{\beta}} = \mathbf{H}\left[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{Y} = \mathbf{H}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\boldsymbol{Y},$$

$$\operatorname{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right) = \mathbf{H}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\boldsymbol{\Sigma}\mathbf{T}^-\mathbf{X}\mathbf{D}^-\mathbf{H}' = \mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}'.$$

Before a theorem on a test of a linear hypothesis some auxiliary statements must be proved.

Lemma 4.3. *Let in the universal model* (3) *the null hypothesis be considered.*

(i) *As a minimum* $\boldsymbol{\Sigma}$*-seminorm g-inverse* $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}$ *of the matrix* $\mathbf{X}'$ *also matrices* $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')}$ *and* $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{X}')}$ *can be chosen.*

(ii) *As a minimum* $\mathbf{\Sigma}$*-seminorm g-inverse* $(\mathbf{M_{H'}X'})^-_{m(\mathbf{\Sigma})}$ *of the matrix* $\mathbf{M_{H'}X'}$ *also the matrix* $(\mathbf{M_{H'}X'})^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}$ *can be chosen.*

Proof. (i) Both matrices $(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}$ and $(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XX'})}$ are g-inverses of the matrix $\mathbf{X}'$. Thus it suffices to prove the symmetry of matrices

$$\mathbf{\Sigma}(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}\mathbf{X}' \quad \text{and} \quad \mathbf{\Sigma}(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XX'})}\mathbf{X}'.$$

The matrix

$$(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}\mathbf{X}'$$

is symmetric with respect to definition of the matrix $(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}$. Since

$$(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}\mathbf{X}' = \mathbf{\Sigma}(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}\mathbf{X}' + \mathbf{XM_{H'}X'},$$

the matrix $\mathbf{\Sigma}(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})}\mathbf{X}'$ is symmetric too.

Analogously for the matrix $\mathbf{\Sigma}(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{XX'})}\mathbf{X}'$.

(ii) It can be proved in the same way as (i). ■

Lemma 4.4. *Let in the universal model (3) the null hypothesis be under consideration. Let* $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})$ *and* $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$*. Then one choice of the g-inverse of the matrix* $\mathrm{Var}(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h})$ *is*

$$\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})^-\mathbf{X}\right]^-\mathbf{H}'\right\}^-.$$

Proof. The covariance matrix of the random vector $\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}$ is

$$\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right) = \mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})^-\mathbf{X}\right]^-\mathbf{X}'(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})^-\mathbf{\Sigma}$$

$$\times(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})^-\mathbf{X}\left[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{XM_{H'}X'})^-\mathbf{X}\right]^-\mathbf{H}',$$

since the assumption $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$ implies that one version of the minimum $\mathbf{\Sigma}$-seminorm g-inverse of the matrix $\mathbf{X}'$ is

$$\left(\mathbf{X}'\right)^{-}_{m(\mathbf{\Sigma})} = (\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}.$$

The last term of the expression for $\mathrm{Var}(\widehat{\mathbf{H}\boldsymbol{\beta}})$ is the matrix $\mathbf{H}'$ and thus it is sufficient to prove the equality

$$\mathbf{H}'\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-}\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right) = \mathbf{H}'.$$

It holds that

$$\begin{aligned}
&\mathbf{H}'\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-}\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right)\\
&\quad= \mathbf{H}'\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-}\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\\
&\qquad\times \mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}' - \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)\\
&\qquad\times (\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\\
&\quad= \mathbf{H}'\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-}\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\\
&\qquad\times \left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]\left[\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\\
&\quad= \mathbf{H}'.
\end{aligned}$$

■

The following theorem deals with a test of a linear hypothesis under the special condition $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$.

Theorem 4.5. *Let in the universal model* (3) *the null hypothesis be considered. If* $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$ *and* $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$, *the test statistic is*

$$R_1^2 - R_0^2 = \left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right)' \left\{\mathbf{H}\left[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-} \left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right).$$

The statistic $R_1^2 - R_0^2$ *has the central chi-squared distribution when the null hypothesis is true and the noncentral chi-squared distribution when the null hypothesis is not true; the parameter of noncentrality is*

$$\delta = (\mathbf{H}\boldsymbol{\beta}^* + \boldsymbol{h})' \left\{\mathbf{H}\left[\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-} (\mathbf{H}\boldsymbol{\beta}^* + \boldsymbol{h})$$

where $\boldsymbol{\beta}^*$ *is an actual value of the parameter* $\boldsymbol{\beta}$*. Degrees of freedom are*

$$f = \operatorname{rank}(\mathbf{H}).$$

Proof. According to Lemma 4.3 the matrix $(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')}$ is one choice of the minimum $\boldsymbol{\Sigma}$-seminorm g-inverse of the matrix $\mathbf{M}_{\mathbf{H}'}\mathbf{X}'$. Thus using relations

$$\left[(\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')}\right]'$$

$$= \left[\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\mathbf{M}_{\mathbf{H}'}\right]^{-}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-},$$

$$\left[\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\mathbf{M}_{\mathbf{H}'}\right]^{-}$$

$$= \left[\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\right]^{-} - \left[\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\right]^{-}\mathbf{H}'$$

$$\times \left\{\mathbf{H}\left[\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\right]^{-}\mathbf{H}'\right\}^{-}\mathbf{H}\left[\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{-}\mathbf{X}\right]^{-},$$

$$(\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma})} = (\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')}$$

and

$$\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}') \quad \Rightarrow \quad \mathbf{H}\left[(\mathbf{X}')^-_{m(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')}\right]'\mathbf{X}\boldsymbol{\beta}_0 = \mathbf{H}\boldsymbol{\beta}_0 = -\boldsymbol{h}$$

the estimator $\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}$ given by (5) can be rewritten as

$$\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = \widehat{\mathbf{X}\boldsymbol{\beta}} + \boldsymbol{k},$$

where

$$\boldsymbol{k} = -\mathbf{X}\left[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\mathbf{X}\right]^-\mathbf{H}'\left\{\mathbf{H}\left[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\mathbf{X}\right]^-\mathbf{H}'\right\}^- \times\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right).$$

Then

$$\begin{aligned} R_1^2 - R_0^2 &= \left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right) \\ &\quad - \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) \\ &= \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}} - \boldsymbol{k}\right)'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}} - \boldsymbol{k}\right) \\ &\quad - \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) \\ &= -2\boldsymbol{k}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) + \boldsymbol{k}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\boldsymbol{k}. \end{aligned}$$

It is easy to show that

$$\boldsymbol{k}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) = 0$$

and

$$\begin{aligned} &\boldsymbol{k}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\boldsymbol{k} \\ &\quad = \left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)'\left\{\mathbf{H}[\mathbf{X}'(\mathbf{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\mathbf{X}]^-\mathbf{H}'\right\}^-\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right). \end{aligned}$$

Further

$$
\begin{aligned}
f &= \operatorname{rank}\left[\operatorname{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right)\right] \\
&= \operatorname{rank}\Big\{\mathbf{H}\left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-} \\
&\quad \times(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'-\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-} \\
&\quad \times\mathbf{X}\left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'\Big\} \\
&= \operatorname{rank}\left\{\mathbf{H}\left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^{-}\mathbf{X}\right]^{-}\mathbf{H}'-\mathbf{H}\mathbf{M}_{\mathbf{H}'}\mathbf{H}'\right\} = \operatorname{rank}(\mathbf{H}).
\end{aligned}
$$

The rest of the proof is obvious. ■

The condition $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$, which enable us to utter the statement on testing linear hypotheses in the classical form, is an obstacle in a general solution of the problem. Therefore we shall investigate a situation in which we assume $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$, however $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$, is not assumed.

Theorem 4.6. *Let in the universal model* (3) *the null hypothesis be considered. Let* $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$. *Then the BLUE of the vector* $\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\boldsymbol{\beta}$ *is*

$$
\widehat{\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\boldsymbol{\beta}}} = \begin{pmatrix}\widehat{\mathbf{X}\boldsymbol{\beta}}\\ \widehat{\mathbf{H}\boldsymbol{\beta}}\end{pmatrix} - \begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}'\left[\mathbf{H}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}'\right]^{-}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right).
$$

The expression $\widehat{\widehat{\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\boldsymbol{\beta}}}$ *is invariant with respect to the choice of the g-inverse.*

Proof. The universal model (3) with the null hypothesis can be written in the form

$$
\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{h}\end{pmatrix} \sim N_{n+q}\left[\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\boldsymbol{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right].
$$

Then the sought estimator is

$$\widehat{\begin{pmatrix} \mathbf{X} \\ \mathbf{H} \end{pmatrix} \boldsymbol{\beta}} = \begin{pmatrix} \mathbf{X} \\ \mathbf{H} \end{pmatrix} \left[(\mathbf{X}', \mathbf{H}')^{-}_{m\begin{pmatrix} \Sigma, 0 \\ 0,\ 0 \end{pmatrix}} \right]' \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{h} \end{pmatrix}.$$

Further

$$\left[(\mathbf{X}', \mathbf{H}')^{-}_{m\begin{pmatrix} \Sigma, 0 \\ 0,\ 0 \end{pmatrix}} \right]'$$

$$= \left[(\mathbf{X}', \mathbf{H}') \begin{pmatrix} \mathbf{T}, & \mathbf{X}\mathbf{H}' \\ \mathbf{H}\mathbf{X}', & \mathbf{H}\mathbf{H}' \end{pmatrix}^{-} \begin{pmatrix} \mathbf{X} \\ \mathbf{H} \end{pmatrix} \right]^{-} (\mathbf{X}', \mathbf{H}') \begin{pmatrix} \mathbf{T}, & \mathbf{X}\mathbf{H}' \\ \mathbf{H}\mathbf{X}', & \mathbf{H}\mathbf{H}' \end{pmatrix}^{-}$$

$$= \left[(\mathbf{X}', \mathbf{H}') \begin{pmatrix} \mathbf{Q}, & \mathbf{R} \\ \mathbf{S}, & \mathbf{U} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{H} \end{pmatrix} \right]^{-} (\mathbf{X}', \mathbf{H}') \begin{pmatrix} \mathbf{Q}, & \mathbf{R} \\ \mathbf{S}, & \mathbf{U} \end{pmatrix},$$

where (cf. Rohde theorem, e.g., in [1], p. 446, Lemma 10.1.40)

$$\mathbf{Q} = \mathbf{T}^{-} + \mathbf{T}^{-}\mathbf{X}\mathbf{H}'[\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-}\mathbf{H}\mathbf{X}'\mathbf{T}^{-},$$

$$\mathbf{R} = -\mathbf{T}^{-}\mathbf{X}\mathbf{H}'[\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-},$$

$$\mathbf{S} = -[\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-}\mathbf{H}\mathbf{X}'\mathbf{T}^{-},$$

$$\mathbf{U} = [\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-}.$$

Then the expression

$$(\mathbf{X}', \mathbf{H}') \begin{pmatrix} \mathbf{Q}, & \mathbf{R} \\ \mathbf{S}, & \mathbf{U} \end{pmatrix}$$

can be rewritten as

$$\big(\mathbf{X}'\mathbf{T}^{-} - (\mathbf{I} - \mathbf{D})\mathbf{H}'[\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-}\mathbf{H}\mathbf{X}'\mathbf{T}^{-}; (\mathbf{I} - \mathbf{D})\mathbf{H}'[\mathbf{H}(\mathbf{I} - \mathbf{D})\mathbf{H}']^{-}\big)$$

and

$$\left[(\mathbf{X}', \mathbf{H}')\begin{pmatrix}\mathbf{Q}, & \mathbf{R}\\ \mathbf{S}, & \mathbf{U}'\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\right]^{-}$$

$$= \Big\{\mathbf{D} + (\mathbf{I}-\mathbf{D})\mathbf{H}'[\mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{H}']^{-}\mathbf{H}(\mathbf{I}-\mathbf{D})\Big\}^{-}.$$

Further, with respect to formula (2) we have

$$\Big\{\mathbf{D} + (\mathbf{I}-\mathbf{D})\mathbf{H}'[\mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{H}']^{-}\mathbf{H}(\mathbf{I}-\mathbf{D})\Big\}^{-}$$

$$= \mathbf{D}^{-} - \mathbf{D}^{-}(\mathbf{I}-\mathbf{D})\mathbf{H}'\Big[\mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{H}' + \mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{D}^{-}(\mathbf{I}-\mathbf{D})\mathbf{H}'\Big]^{-}\mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{D}^{-}$$

and using $\mathbf{D}\mathbf{D}^{-}\mathbf{H}' = \mathbf{H}'$ we obtain

$$\mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{H}' + \mathbf{H}(\mathbf{I}-\mathbf{D})\mathbf{D}^{-}(\mathbf{I}-\mathbf{D})\mathbf{H}'$$

$$= \mathbf{H}\mathbf{H}' + \mathbf{H}\mathbf{D}^{-}\mathbf{H}' - 2\mathbf{H}\mathbf{D}\mathbf{D}^{-}\mathbf{H}' = \mathbf{H}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}'.$$

Now the expression for the BLUE of the vector $\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}\boldsymbol{\beta}$ is obvious.

The statement on an arbitrary choice of a g-inverse is a consequence of the relationship

$$\mathcal{P}\left\{\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} \in \mathcal{M}\left[\begin{pmatrix}\boldsymbol{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix} + \begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix}(\mathbf{X}', \mathbf{H}')\right]\right\} = 1.$$

■

Theorem 4.7. *Let in the universal model* (3) *the null hypothesis be considered. If* $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$, *then the test statistic is*

$$R_1^2 - R_0^2 = \left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right)'\left[\mathbf{H}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}'\right]^{-}\left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right),$$

where $\mathbf{H}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}' = \mathrm{Var}(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h})$.

The statistic $R_1^2 - R_0^2$ *has the central chi-squared distribution when the null hypothesis is true and the noncentral chi-squared distribution when the null hypothesis is not true; the parameter of noncentrality is*

$$\delta = (\mathbf{H}\boldsymbol{\beta}^* + \boldsymbol{h})' \left[\mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}'\right]^- (\mathbf{H}\boldsymbol{\beta}^* + \boldsymbol{h}),$$

where $\boldsymbol{\beta}^*$ *is an actual value of the parameter* $\boldsymbol{\beta}$*. The degrees of freedom are*

$$\text{rank}\left[\mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}'\right].$$

Proof. With respect to Lemma 4.1 as a g-inverse of both matrices

$$\text{Var}\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right) \quad \text{and} \quad \text{Var}\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)$$

the matrix $\mathbf{W}^- = (\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-$ can be chosen and at the same time the quadratic forms

$$\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)' \left[\text{Var}\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)\right]^- \left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)$$

and

$$\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)' \left[\text{Var}\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)\right]^- \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)$$

are invariant with respect to the choice of the g-inverse. Let

$$\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = \widehat{\mathbf{X}\boldsymbol{\beta}} - \mathbf{X}\boldsymbol{a},$$

where

$$\boldsymbol{a} = (\mathbf{D}^- - \mathbf{I})\mathbf{H}' \left[\mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}'\right]^- \left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right).$$

Then

$$\left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)' \mathbf{W}^- \left(\boldsymbol{Y} - \widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}}\right)$$

$$= \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)' \mathbf{W}^- \left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right) - 2\left(\boldsymbol{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}\right)' \mathbf{W}^- \mathbf{X}\boldsymbol{a} + \boldsymbol{a}'\mathbf{X}'\mathbf{W}^-\mathbf{X}\boldsymbol{a}.$$

In the following consideration the matrix $\mathbf{W}^+$ will be used instead of $\mathbf{W}^-$. Thus we can proceed in simpler way.

At first the validity of the equality $(\boldsymbol{Y}-\widehat{\mathbf{X}\boldsymbol{\beta}})'\mathbf{W}^+\mathbf{X}\boldsymbol{a}=0$ will be proved. Using the relationship (2), the identity

$$(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^+=(\mathbf{T}-\mathbf{X}\mathbf{P}_{\mathbf{H}'}\mathbf{X}')^+$$

can be rewritten as

$$\left(\mathbf{T}-\mathbf{X}\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\right)^+=\mathbf{T}^++\mathbf{T}^+\mathbf{X}\mathbf{P}_{\mathbf{H}'}\left(\mathbf{I}-\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\mathbf{T}^+\mathbf{X}\mathbf{P}_{\mathbf{H}'}\right)^+\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\mathbf{T}^+.$$

Then we obtain

$$\begin{aligned}\left(\boldsymbol{Y}-\widehat{\mathbf{X}\boldsymbol{\beta}}\right)'\mathbf{W}^+\mathbf{X}\boldsymbol{a}&=\boldsymbol{Y}'\left(\mathbf{I}-\mathbf{T}^+\mathbf{X}\mathbf{D}^-\mathbf{X}'\right)\\&\quad\times\left[\mathbf{T}^++\mathbf{T}^+\mathbf{X}\mathbf{P}_{\mathbf{H}'}\left(\mathbf{I}-\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\mathbf{T}^+\mathbf{X}\mathbf{P}_{\mathbf{H}'}\right)^+\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\mathbf{T}^+\right]\mathbf{X}\boldsymbol{a}=0,\end{aligned}$$

since

$$\left(\mathbf{I}-\mathbf{T}^+\mathbf{X}\mathbf{D}^-\mathbf{X}'\right)\mathbf{T}^+\mathbf{X}=\mathbf{0}.$$

Further

$$\begin{aligned}&\boldsymbol{a}'\mathbf{X}'\mathbf{W}^+\mathbf{X}\boldsymbol{a}\\&=\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)'\left[\mathbf{H}(\mathbf{D}^--\mathbf{I})\mathbf{H}'\right]^-\mathbf{H}(\mathbf{D}^--\mathbf{I})\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^+\\&\qquad\times\mathbf{X}(\mathbf{D}^--\mathbf{I})\mathbf{H}'\left[\mathbf{H}(\mathbf{D}^--\mathbf{I})\mathbf{H}'\right]^-\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)\\&=\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)'\left[\mathbf{H}(\mathbf{D}^+-\mathbf{P}_{\mathbf{H}'})\mathbf{H}'\right]^+\mathbf{H}(\mathbf{D}^+-\mathbf{P}_{\mathbf{H}'})\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^+\\&\qquad\times\mathbf{X}(\mathbf{D}^+-\mathbf{P}_{\mathbf{H}'})\mathbf{H}'\left[\mathbf{H}(\mathbf{D}^+-\mathbf{P}_{\mathbf{H}'})\mathbf{H}'\right]^+\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)\\&=\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)'\left[\mathbf{H}(\mathbf{D}^+-\mathbf{I})\mathbf{H}'\right]^+\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)\\&=\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)'\left[\mathbf{H}(\mathbf{D}^--\mathbf{I})\mathbf{H}'\right]^-\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right),\end{aligned}$$

since

$$\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}'\right)^{+}\mathbf{X}$$

$$=\mathbf{X}'\left[\mathbf{T}^{+}+\mathbf{T}^{+}\mathbf{X}\mathbf{P}_{\mathbf{H}'}\left(\mathbf{I}-\mathbf{P}_{\mathbf{H}'}\mathbf{D}\mathbf{P}_{\mathbf{H}'}\right)^{+}\mathbf{P}_{\mathbf{H}'}\mathbf{X}'\mathbf{T}^{+}\right]\mathbf{X}$$

$$=\mathbf{D}+\mathbf{D}\mathbf{P}_{\mathbf{H}'}\left(\mathbf{I}-\mathbf{P}_{\mathbf{H}'}\mathbf{D}\mathbf{P}_{\mathbf{H}'}\right)^{+}\mathbf{P}_{\mathbf{H}'}\mathbf{D}=(\mathbf{D}^{+}-\mathbf{P}_{\mathbf{H}'})^{+}.$$

Finally

$$\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)=\mathbf{H}\mathbf{D}^{-}\mathbf{X}'\mathbf{T}^{-}\boldsymbol{\Sigma}\mathbf{T}^{-}\mathbf{X}\mathbf{D}^{-}\mathbf{H}'=\mathbf{H}(\mathbf{D}^{-}-\mathbf{I})\mathbf{H}'.$$

The other statements are obvious. ∎

Theorem 4.8. *Let in the universal model* (3) *the null hypothesis be considered. If* $\mathcal{M}(\mathbf{H}')\subset\mathcal{M}(\mathbf{X}'\mathbf{M}_{\boldsymbol{\Sigma}})$, *then the hypothesis need not be tested since in this case* $\mathcal{P}\{\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}=\mathbf{0}\}=1$.

Proof. This statement is implied by the fact that $\mathrm{Var}(\widehat{\mathbf{H}\boldsymbol{\beta}})=\mathbf{0}$. It is a consequence of the relationship

$$\mathcal{M}(\mathbf{H}')\subset\mathcal{M}(\mathbf{X}'\mathbf{M}_{\boldsymbol{\Sigma}})\quad\Rightarrow\quad\exists\mathbf{E}:\ \mathbf{H}'=\mathbf{X}'\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{E},$$

which implies

$$\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}\right)=\mathrm{Var}\left(\mathbf{E}'\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{X}\left[(\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{Y}\right)$$

$$=\mathbf{E}'\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{X}\left[(\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma}(\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma})}\mathbf{X}'\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{E}$$

$$=\mathbf{E}'\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{X}\left[(\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma}\mathbf{M}_{\boldsymbol{\Sigma}}\mathbf{E}=\mathbf{0}.$$

∎

If the null hypothesis is not true, the test statistic $R_1^2-R_0^2$ has the noncentral chi-squared distribution with f degrees of freedom and the parameter of noncentrality is

$$\delta=(\mathbf{H}\boldsymbol{\beta}^{*}+\boldsymbol{h})'\left[\mathrm{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}}+\boldsymbol{h}\right)\right]^{-}(\mathbf{H}\boldsymbol{\beta}^{*}+\boldsymbol{h}),$$

where $\boldsymbol{\beta}^*$ is a true value of the parameter $\boldsymbol{\beta}$. The power of the test at the point $\boldsymbol{\xi}$ is

$$p(\boldsymbol{\xi}) = \mathcal{P}\left\{R_1^2 - R_0^2 \geq \chi_f^2(0, 1-\alpha) | \; H_a : \mathbf{H}\boldsymbol{\beta} + \boldsymbol{h} = \boldsymbol{\xi} \neq \mathbf{0}\right\}.$$

Here $\chi_f^2(0, 1-\alpha)$ is $(1-\alpha)$-quantile of the central chi-squared distribution with f degrees of freedom.

The random variable $R_1^2 - R_0^2 \sim \chi_f^2(\delta)$ can be approximated by (cf. [2])

$$\chi_f^2(\delta) \approx c^2 \chi_g^2(0),$$

where

$$c^2 = \frac{f + 2\delta}{f + \delta}, \quad g = \frac{(f+\delta)^2}{f + 2\delta}.$$

Remark 4.9. It is to be pointed out that in practical computing it is necessary to be very careful since in some situations some derived formulae can be numerically unstable. For example, small numbers on the main diagonal of the matrix $\mathbf{\Sigma}$ can caused the covariance matrix $\mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}'$ numerically unstable. In practice it is useful to compute with both expressions

$$\operatorname{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right) = \mathbf{H}(\mathbf{D}^- - \mathbf{I})\mathbf{H}',$$

$$\operatorname{Var}\left(\widehat{\mathbf{H}\boldsymbol{\beta}} + \boldsymbol{h}\right) = \mathbf{H}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\mathbf{\Sigma}\mathbf{T}^-\mathbf{X}\mathbf{D}^-\mathbf{H}'$$

and to compare obtained results. If they are different, one can use for example the substitution $\mathbf{\Sigma} \to k\mathbf{\Sigma}$, $\mathbf{X} \to \sqrt{k}\mathbf{X}$, where $k > 0$ is a sufficiently large number, and to compute them once more. This substitution does not influence the result of the original covariance matrix.

Another problem can occur when degrees of freedom are computed. Here, e.g., expressions $\operatorname{rank}[(\mathbf{\Sigma} + \mathbf{X}\mathbf{X}')^-\mathbf{X}]$ or $\operatorname{rank}[(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')^-\mathbf{X}]$ can be numerically unstable.

5. EXAMPLE

Example 5.1. Let a linear part of high-speed lane be under consideration. One of the safety conditions is that rails are in the line. For the sake of simplicity let the problem be studied in plane only. An experiment for the verification that rails are in the straight line can be done, e.g., in the following way. Firstly, points X_i, $i = 1, \dots, 4$, are chosen elsewhere on rails. Then other points Z_1, Z_2, Z_3 are chosen around rails such that all distances $Z_i X_j$ and $Z_i Z_k$, $i, k = 1, 2, 3$, $i \neq k$, $j = 1, \dots, 4$, can be observed. Finally points X_i, Z_j are put into proper coordinates system (the map), see Figure 1. Let each distance be measured just once. Let distances $Z_i X_j$, $i, j = 1, 2$, and $Z_1 Z_2$ have been measured in previous experiment by Väisälä interferometer, i.e., the accuracy of measurement is practically $\sigma_1 = 0$. (cf. [5], p. 50). Let other distances be measured by optical range-finder with the accuracy $\sigma = 0.01$ m. The problem is to test a hypothesis that all points X_i, $i = 1, \dots, 4$, are located on a straight line.

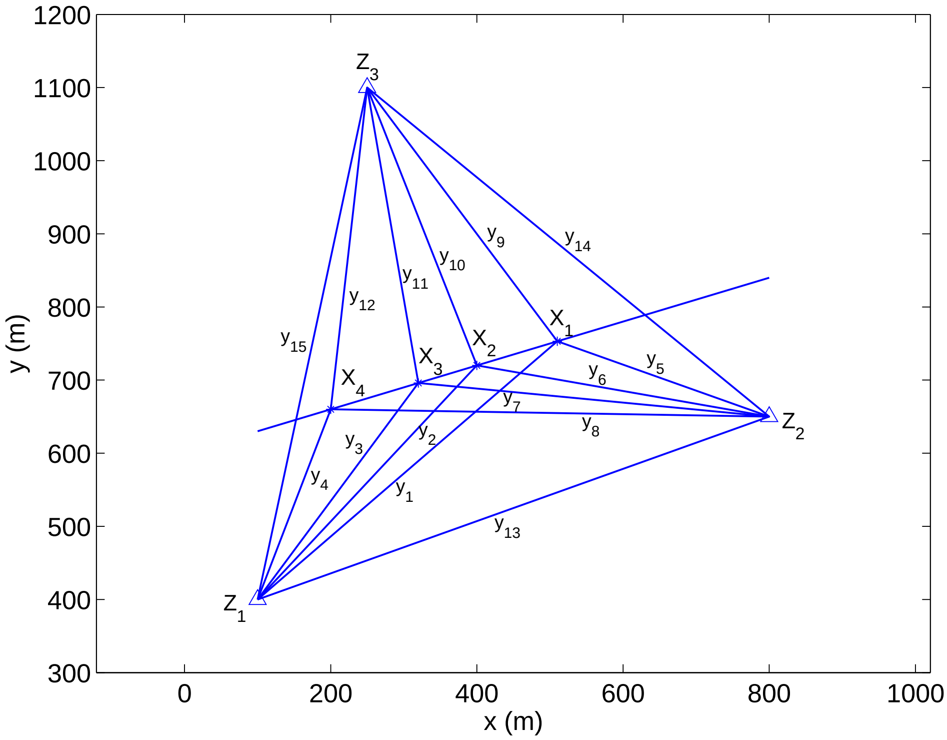

Figure 1. The design of the experiment

Let the notation

- $\boldsymbol{y} = (y_1, \ldots, y_{15})'$ ... a vector of observed distances $Z_iX_j,\ Z_iZ_k$, $i, k = 1, 2, 3,\ i \neq k,\ j = 1, \ldots, 4$,
- $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{14})'$ ... a vector of unknown coordinates of points

$$X_i = [\beta_{2i-1}, \beta_{2i}]',\ i = 1, \ldots, 4, \text{ and } Z_i = [\beta_{2i-1}, \beta_{2i}]',\ i = 5, 6, 7,$$

be used. The mentioned process of measurement can be modelled by

$$\boldsymbol{Y} \sim N_{15}\left(\boldsymbol{f}(\boldsymbol{\beta}), \boldsymbol{\Sigma}\right),$$

where, e.g.,

$$f_1(\boldsymbol{\beta}) = \sqrt{(\beta_9 - \beta_1)^2 + (\beta_{10} - \beta_2)^2}$$

and $\boldsymbol{\Sigma}$ is a diagonal matrix given by

$$\boldsymbol{\Sigma} = \text{Diag}\left\{0, 0, \sigma^2, \sigma^2, 0, 0, \underbrace{\sigma^2, \ldots, \sigma^2}_{6\times}, 0, \sigma^2, \sigma^2\right\}.$$

The linear version of the model can be written in the form

$$\boldsymbol{Y} - \boldsymbol{f}\left(\boldsymbol{\beta}^{(0)}\right) \sim N_{15}\left(\mathbf{X}\Delta\boldsymbol{\beta}, \boldsymbol{\Sigma}\right), \quad \Delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}^{(0)},$$

where $\boldsymbol{\beta}^{(0)}$ are approximate values of the vector $\boldsymbol{\beta}$ and

$$\mathbf{X} = \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial \boldsymbol{u}'}\Big|_{u=\beta^{(0)}}.$$

Let straight line p_i intersects points X_i and X_{i+1}, $i = 1, 2, 3$. Straight lines p_i can be expressed as

$$p_i:\ y = a_i + b_i x, \quad i = 1, 2, 3,$$

where

$$a_i = \beta_{2i} - \beta_{2i-1}\frac{\beta_{2i} - \beta_{2i+2}}{\beta_{2i-1} - \beta_{2i+1}},$$

$$b_i = \frac{\beta_{2i} - \beta_{2i+2}}{\beta_{2i-1} - \beta_{2i+1}}.$$

The problem is to test the null hypothesis

$$H_0: \ p_1 = p_2 = p_3$$

against the alternative hypothesis

$$H_a: \ \exists i \neq j : p_i \neq p_j, \ \ i, j \in \{1, 2, 3\}.$$

Straight lines p_1 and p_2 are identical if and only if $a_1 = a_2$ and $b_1 = b_2$, i.e.,

$$b_1 = b_2 = b \quad \Leftrightarrow \quad \frac{\beta_2 - \beta_4}{\beta_1 - \beta_3} = \frac{\beta_4 - \beta_6}{\beta_3 - \beta_5} = b$$

and

$$a_1 = a_2 \quad \Leftrightarrow \quad \beta_2 - \beta_1 b = \beta_4 - \beta_3 b \quad \Leftrightarrow \quad \frac{\beta_2 - \beta_4}{\beta_1 - \beta_3} = b.$$

Analogously for p_2, p_3 and p_1, p_3. Thus

$$p_1 = p_2 = p_3 \quad \Leftrightarrow \quad \boldsymbol{g}(\boldsymbol{\beta}) = \mathbf{0},$$

where

$$g_i(\boldsymbol{\beta}) = (\beta_{2i} - \beta_{2i+2})(\beta_{2i+1} - \beta_{2i+3}) - (\beta_{2i-1} - \beta_{2i+1})(\beta_{2i+2} - \beta_{2i+4}), \qquad i = 1, 2,$$

and

$$g_3(\boldsymbol{\beta}) = (\beta_2 - \beta_4)(\beta_5 - \beta_7) - (\beta_1 - \beta_3)(\beta_6 - \beta_8).$$

Linear version of the null hypothesis can be written as

$$H_0: \ \mathbf{H}\Delta\boldsymbol{\beta} = \mathbf{0},$$

where

$$\mathbf{H} = \frac{\partial \boldsymbol{g}(\boldsymbol{u})}{\partial \boldsymbol{u}'}\Big|_{u=\beta^{(0)}}$$

and the alternative hypothesis as

$$H_a: \ \mathbf{H}\Delta\boldsymbol{\beta} = \boldsymbol{\xi} \neq \mathbf{0}.$$

Let approximate values $\boldsymbol{\beta}^{(0)}$ have been chosen as (in meters):

$$Z_1^{(0)} = \begin{pmatrix} 100 \\ 400 \end{pmatrix}, \; Z_2^{(0)} = \begin{pmatrix} 800 \\ 650 \end{pmatrix}, \; Z_3^{(0)} = \begin{pmatrix} 250 \\ 1100 \end{pmatrix},$$

$$X_1^{(0)} = \begin{pmatrix} 200 \\ 660 \end{pmatrix}, \; X_2^{(0)} = \begin{pmatrix} 320 \\ 696 \end{pmatrix}, \; X_3^{(0)} = \begin{pmatrix} 400 \\ 720 \end{pmatrix}, \; X_4^{(0)} = \begin{pmatrix} 510 \\ 753 \end{pmatrix}.$$

In this case, terms in linearized model are the following one:

$$\boldsymbol{f}\left(\boldsymbol{\beta}^{(0)}\right) = [\, 278.56777,\ 368.80347,\ 438.63424,\ 541.02588,\ 600.08333,\ 482.19913,\ 406.07881,\ 307.74827,\ 442.83180,\ 410.01951,\ 408.53396,\ 433.60005,\ 743.30344,\ 715.89105,\ 710.63352\,]',$$

the design matrix $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$, where

$$\mathbf{X}_1 = \begin{pmatrix}
5.99149, & 15.57786, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 11.45579, & 15.41325, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 14.32419, & 15.27913, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 17.62686 \\
-24.49320, & 0.40822, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & -21.85889, & 2.09481, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & -19.84974, & 3.47370, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & -16.53104 \\
-2.37602, & -20.90900, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 3.45697, & -19.95166, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 7.42125, & -18.80050, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 12.48615 \\
0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0
\end{pmatrix},$$

$$\mathbf{X}_2 = \begin{pmatrix} 0, & -5.99149, & -15.57786, & 0, & 0, & 0, & 0 \\ 0, & -11.45579, & -15.41325, & 0, & 0, & 0, & 0 \\ 0, & -14.32419, & -15.27913, & 0, & 0, & 0, & 0 \\ 15.17629, & -17.62686, & -15.17629, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 24.49320, & -0.40822, & 0, & 0 \\ 0, & 0, & 0, & 21.85889, & -2.09481, & 0, & 0 \\ 0, & 0, & 0, & 19.84974, & -3.47370, & 0, & 0 \\ 5.87137, & 0, & 0, & 16.53104, & -5.87137, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 2.37602, & 20.90900 \\ 0, & 0, & 0, & 0, & 0, & -3.45697, & 19.95166 \\ 0, & 0, & 0, & 0, & 0, & -7.42125, & 18.80050 \\ -16.66421, & 0, & 0, & 0, & 0, & -12.48615, & 16.66421 \\ 0, & -25.67527, & -9.16974, & 25.67527, & 9.16974, & 0, & 0 \\ 0, & -5.60619, & -26.16222, & 0, & 0, & 5.60619, & 26.16222 \\ 0, & 0, & 0, & 20.63193, & -16.88067, & -20.63193, & 16.88067 \end{pmatrix}$$

and the matrix $\mathbf{H} = (\mathbf{H}_1, \mathbf{0}_{3\times 6})$, where

$$\mathbf{H}_1 = \begin{pmatrix} 24, & -80, & -60, & 200, & 36, & -120, & 0, & 0 \\ 0, & 0, & 33, & -110, & -57, & 190, & 24, & -80 \\ 33, & -110, & -33, & 110, & -36, & 120, & 36, & -120 \end{pmatrix}.$$

Let the simulated data of observed distances be (in meters):

$$\boldsymbol{y} = [\ 278.56778,\ \ 368.80346,\ \ 438.63423,\ \ 541.02588,\ \ 600.07120,\ \ 482.18593,\ \ 406.08812,\ \ 307.74839,\ \ 442.82535,\ \ 410.02757,\ \ 408.53628,\ \ 433.59015,\ \ 743.31683,\ \ 715.89395,\ \ 710.64831\ ]'.$$

Let the risk of the test be $\alpha = 0.05$. The null hypothesis can be tested since $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$. Further $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{H}'}\mathbf{X}')$ and thus the hypothesis can be tested by the help of Theorem 4.5. Since

$$R_1^2(\boldsymbol{y}) - R_0^2(\boldsymbol{y}) = 18.82294, \quad \chi_2^2(0, 0.95) = 5.99146,$$

the null hypothesis can be rejected. The power of the test at the point

$$\widehat{\mathbf{H}\Delta\boldsymbol{\beta}}(\boldsymbol{y}) = \begin{pmatrix} -0.13668 \\ 0.06842 \\ -0.08530 \end{pmatrix}$$

is $p = 0.97956$.

Thus the null hypothesis that rails are in the line, i.e., all points $X_1, \dots, X_4$ lie one a straight line, can be rejected with the probability 98%.

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