

PROPERTIES OF SET-VALUED STOCHASTIC INTEGRALS

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Abstract

We introduce set-valued stochastic integrals driven by a square-integrable martingale and by a semimartingale. We investigate properties of both integrals.

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1. INTRODUCTION

Many studies have been devoted to the stochastic control theory described by stochastic equations dependent on random control parameters. This theory has been naturally connected with the theory of stochastic inclusions (i.e., equations with set-valued operators). An appropriate set-valued model, describing such dynamical systems, needs some special type of set-valued integrals generalizing stochastic integrals.

Although a deterministic set-valued integration theory has been extensively developed by many authors, the stochastic case seems to appear rather seldom in literature. The first results dealing with set-valued stochastic integrals with respect to a standard Brownian motion have been given by G. Boscan in [3], B. Gelman and J. Gliklikh in [8]. Some properties of set-valued stochastic integrals of $\{\mathcal{F}_t\}_{t \geq 0}$ -progressively measurable set-valued functions with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian motion can be found in [12]. The most general theory of set-valued stochastic integrals driven by a cylindrical Brownian motion on a Hilbert space has been studied by Hiai in [11]. Some properties of set-valued stochastic integrals can be found also in [14, 15].

In the paper, we consider some properties of integrals $\int G_\tau dM_\tau$ and $\int G_\tau dZ_\tau$, where G means a set-valued predictable process taking values in the space of closed subsets of \mathbb{R}^n , while M is a square-integrable martingale and Z is a semimartingale.

2. ASSUMPTIONS

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space satisfying the usual hypothesis: (i) \mathcal{F}_0 contains all P -null sets of \mathcal{F} , (ii) $\mathcal{F}_t = \bigcap_{u \geq t} \mathcal{F}_u$, all t , $0 \leq t < \infty$. By a stochastic process on (Ω, \mathcal{F}, P) we mean a collection $x = (x_t)_{t \geq 0}$ of n -dimensional random variables $x_t : \Omega \rightarrow \mathbb{R}^n$, $t \geq 0$. The process x is adapted if x_t belongs to \mathcal{F}_t for each $t \geq 0$. A stochastic process x is cádlág if it has right continuous sample paths with left limits. Similarly, a stochastic process x is said to be cáglád if it has left continuous sample paths with right limits. The family of all adapted cádlág (cáglád) processes is denoted by $D [L]$.

Let $\mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ denote the smallest σ -algebra on $\mathbb{R}_+ \times \Omega$ with respect to which every cáglád adapted process is measurable in $\lambda \otimes P$. (λ means the Lebesgue measure on \mathbb{R}_+). It is known that $\mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ is generated by a class of all subsets of $\mathbb{R}_+ \times \Omega$ of the form $\{0\} \times F_0$ and $(s, t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $s < t$ in \mathbb{R}_+ . A stochastic process x is predictable if x is $\mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ -measurable. The family of all such processes is denoted by \mathcal{P} .

Let X be a Banach space and let $cl(X)$, $comp(X)$ and $conv(X)$ denote spaces of all nonempty closed, compact, compact and convex, respectively, subsets of X . Let us denote by $\text{dist}(a, A)$ the distance of $a \in X$ from a set $A \subset X$. Put $\bar{h}(A, B) = \sup_{a \in A} \text{dist}(a, B)$ and $H(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$ for $A, B \in cl(X)$.

Consider a set-valued stochastic process $G = (G_t)_{t \geq 0}$ with values in $cl(\mathbb{R}^n)$, i.e., a family of \mathcal{F} -measurable set-valued mappings $G_t : \Omega \rightarrow cl(\mathbb{R}^n)$, each $t \geq 0$.

We call G predictable if G is $\mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ -measurable. We denote the family of all such processes also by \mathcal{P} .

For a semimartingale Z let L_Z denote a set of all predictable processes x integrable with respect to Z (see [16]).

A set $K \subset L_Z$ is called decomposable, if for every $g, h \in K$ and $A \in \mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ one has $\mathbb{1}_A h + \mathbb{1}_B g \in K$, where $B = (\mathbb{R}_+ \times \Omega) \setminus A$ and $\mathbb{1}_A$ denotes the characteristic function of A .

3. SET-VALUED STOCHASTIC INTEGRAL DRIVEN BY A SQUARE-INTEGRABLE MARTINGALE

Let M be a right continuous and square-integrable martingale. We can define the unique measure μ_M on \mathcal{P} – a Doléans-Dade measure of M as follows. Let $(s, t] \times A$ be a rectangle in $\mathbb{R}_+ \times \Omega$ with A being \mathcal{F}_s -measurable, $0 \leq s \leq t$. We define a set function λ_M

$$\lambda_M((s, t] \times A) = E(\mathbb{1}_A(M_t - M_s)^2)$$

and extend this function to a unique σ -finite measure μ_M on \mathcal{P} (see e.g., [5] Section 2.4).

Let \mathcal{L}^2 denote the space $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu_M, \mathbb{R}^n)$.

Denote by \mathcal{M}_0^2 the space of all square-integrable martingales $M = (M_t)_{t \geq 0}$ such that $M_0 = 0$ a.s. \mathcal{M}_0^2 can be endowed with a norm $\|M\| = (EM_\infty^2)^{\frac{1}{2}}$. We also put $(M, N) = E\{M_\infty \cdot N_\infty\}$ for $M, N \in \mathcal{M}_0^2$. Then \mathcal{M}_0^2 becomes a Hilbert space, ([16]).

For a given $M \in \mathcal{M}_0^2$ let $L_M^2 = \{f \in \mathcal{P} : E\{\int_0^\infty f_s^2 d[M, M]_s\} < \infty\}$. Similarly as above, L_M^2 can be endowed with a norm $\|f\|_M = (\int_{\mathbb{R}_+ \times \Omega} f^2 d\mu_M)^{\frac{1}{2}}$. It can be easily verified that for $M \in \mathcal{M}_0^2$ and $f \in L_M^2$ given above one has:

$$\left\| \int_0^\infty f_t dM_t \right\|^2 = E \int_0^\infty f_t^2 d[M, M]_t = \int_{\mathbb{R}_+ \times \Omega} f^2 d\mu_M = \|f\|_M^2,$$

where $\|\cdot\|$ is a norm in \mathcal{M}_0^2 defined above. From the above equality we deduce.

Proposition 3.1.

- (i) A mapping $\mathcal{J}_M : L_M^2 \ni f \rightarrow \mathcal{J}_M(f) = \int f_t dM_t \in \mathcal{M}_0^2$ is a linear isometry for every $M \in \mathcal{M}_0^2$ and maps closed and convex subsets of L_M^2 into closed and convex subsets of \mathcal{M}_0^2 .
- (ii) Let $M \in \mathcal{M}_0^2$ and $m \in L_M^2$ be given. For a set $K_M^m = \{f \in L_M^2 : f \leq |m| : \mu_M\text{-a.e.}\}$ sets K_M^m and $\mathcal{J}_M(K_M^m)$ are convex and weakly compact subsets of L_M^2 and \mathcal{M}_0^2 , respectively.

Definition 3.2. For a square-integrable martingale M and a predictable set-valued process G , we define a set $\mathcal{S}_M(G)$ by

$$\mathcal{S}_M(G) := \left\{ f \in L_M^2 : f(t, \omega) \in G(t, \omega) \ \mu_M\text{-a.e.} \right\}.$$

A predictable set-valued process G is integrable with respect to a martingale M , or simply M -integrable, if the set $\mathcal{S}_M(G)$ is nonempty. It is M -integrably bounded if there exists a process $m \in L_M^2$ such that $H(G, \{0\}) \leq m$ μ_M -a.e.

Lemma 3.3. Let M be a square-integrable martingale, $M_0 = 0$. Suppose that G is an M -integrably bounded and predictable set-valued process. Then

- (i) the set $\mathcal{S}_M(G)$ is a nonempty closed and bounded subset of L_M^2 ,
- (ii) when G takes on convex values, $\mathcal{S}_M(G)$ is convex and weakly compact in L_M^2 .

Proof. (i) By virtue of Kuratowski and Ryll-Nardzewski measurable selection theorem ([2]), there exists a predictable process f such that $f(t, \omega) \in G(t, \omega)$ μ_M -a.e. This, together with $\|f\|_M \leq \|m\|_M < \infty$ implies that $f \in \mathcal{S}_M(G)$ and thus the set $\mathcal{S}_M(G)$ is nonempty.

Boundedness is evident. Suppose (f^n) is a sequence of $\mathcal{S}_M(G)$ converging in the $\|\cdot\|_M$ -topology to some $f \in L_M^2$. By virtue of Kunita-Watanabe inequality ([16]) one obtains

$$\begin{aligned} E \left(\int_0^t \text{dist}(f_\tau, G_\tau) d[M, M]_\tau \right) &\leq E \left(\int_0^t |f_\tau - f_\tau^n| d[M, M]_\tau \right) \\ &\leq \left\| \left(\int_0^t |f_\tau - f_\tau^n|^2 d[M, M]_\tau \right)^{1/2} \right\|_{L^2} \cdot \left\| \left(\int_0^t d[M, M]_\tau \right)^{1/2} \right\|_{L^2} \\ &= \|f^n - f\|_M \cdot \|\mathbb{I}_{[0,t]}\|_M. \end{aligned}$$

The last term tends to zero with $n \rightarrow \infty$. Therefore, $f \in \mathcal{S}_M(G)$.

(ii) Suppose G takes on convex values and assume $\mathcal{S}_M(G)$ is not convex. Then there are $f, g \in \mathcal{S}_M(G)$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ such that:

$$E \left(\int_0^t \text{dist}((\alpha f + \beta g)_\tau, G_\tau) d[M, M]_\tau \right) > 0.$$

Then there is a set $C \subset [0, t] \times \Omega$ with $\mu_M(C) > 0$, such that $\text{dist}((\alpha f + \beta g)_t, G_t) > 0$ for $(t, \omega) \in C$. On the other hand, there are sets $C_1, C_2 \subset [0, t] \times \Omega$ with $\mu_M(C_1) = \mu_M(C_2) = \mu_M([0, t] \times \Omega)$, such that $f_t(\omega), g_t(\omega) \in G_t(\omega)$ for $(t, \omega) \in C_1 \cap C_2$. Hence, by the convexity of values of G one has $\alpha f_t(\omega) + \beta g_t(\omega) \in G_t(\omega)$ for $(t, \omega) \in C_1 \cap C_2$. Let us observe that $\mu_M(C \cap C_1 \cap C_2) > 0$ because of $\mu_M([0, t] \times \Omega \setminus C_1 \cap C_2) = 0$ and $\mu_M(C) > 0$. Thus for $(t, \omega) \in C \cap C_1 \cap C_2$ one has $0 < \text{dist}(\alpha f_t(\omega) + \beta g_t(\omega), G_t(\omega)) = 0$. Contradiction.

Finally, convexity and closedness of $\mathcal{S}_M(G)$ imply its weak compactness because $\mathcal{S}_M(G)$ is a subset of a weakly compact set $K_M^m \subset L_M^2$.

Lemma 3.4. *Let M be a square-integrable martingale, $M_0 = 0$. For every M -integrably bounded and predictable set-valued process G , the set $\mathcal{S}_M(G)$ is decomposable.*

Proof. Let us observe that for every $f, g \in L_M^2$, $A \in \mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ and $B = (\mathbb{R}_+ \times \Omega) \setminus A$ one has $\mathbb{1}_A f + \mathbb{1}_B g \in L_M^2$. Furthermore, $\int (\mathbb{1}_A f + \mathbb{1}_B g)_t dM_t = \int (\mathbb{1}_A f_t) dM_t + \int (\mathbb{1}_B g_t) dM_t$. Let $f, g \in \mathcal{S}_M(G)$ and $A \in \mathcal{P}(\{\mathcal{F}_t\}_{t \geq 0})$ be given. We obtain

$$\text{dist}((\mathbb{1}_A f + \mathbb{1}_B g), G) \leq \text{dist}(f, G) + \text{dist}(g, G) \quad \mu_M\text{-a.e.}$$

Then for each $t > 0$

$$\begin{aligned} & E \left(\int_0^t \text{dist}((\mathbb{1}_A f + \mathbb{1}_B g)_\tau, G_\tau) d[M, M]_\tau \right) \\ & \leq E \left(\int_0^t \text{dist}(f_\tau, G_\tau) d[M, M]_\tau \right) + E \left(\int_0^t \text{dist}(g_\tau, G_\tau) d[M, M]_\tau \right) = 0. \end{aligned}$$

Therefore, $\mathbb{1}_A f + \mathbb{1}_B g \in \mathcal{S}_M(G)$.

Definition 3.5. Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrable and predictable set-valued process.

A set-valued stochastic integral $\int G_\tau dM_\tau$ of G with respect to M is defined by $\int G_\tau dM_\tau = \{\int g_\tau dM_\tau : g \in \mathcal{S}_M(G)\}$.

For a fixed $0 \leq s < t < \infty$ we also define $\int_s^t G_\tau dM_\tau = \{\int_s^t g_\tau dM_\tau : g \in \mathcal{S}_M(G)\}$.

Proposition 3.6. *Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrably bounded and predictable set-valued process. Then*

- (i) $\int G_\tau dM_\tau = \mathcal{J}_M(\mathcal{S}_M(G))$
- (ii) $\int_s^t G_\tau dM_\tau = \mathcal{J}_M(\mathbb{1}_{(s,t]} \mathcal{S}_M(G))_t$, for $0 \leq s < t < \infty$, where \mathcal{J}_M is a linear mapping defined in Proposition 3.1.

Proof. Condition (i) follows immediately from the definition of $\int G_\tau dM_\tau$. On the other hand, by definitions of $\int_s^t G_\tau dM_\tau$ and $\mathcal{J}_M(f)_t$ one obtains

$$\begin{aligned} \int_s^t G_\tau dM_\tau &= \{\mathcal{J}_M(f)_t - \mathcal{J}_M(f)_s : f \in \mathcal{S}_M(G)\} \\ &= \{\mathcal{J}_M(\mathbb{1}_{[0,t]}f)_t - \mathcal{J}_M(\mathbb{1}_{[0,s]}f)_t : f \in \mathcal{S}_M(G)\} \\ &= \{\mathcal{J}_M(\mathbb{1}_{(s,t]}f)_t : f \in \mathcal{S}_M(G)\} \\ &= \mathcal{J}_M(\mathbb{1}_{(s,t]}\mathcal{S}_M(G))_t \quad \text{for } 0 \leq s < t < \infty. \end{aligned}$$

Proposition 3.7. *Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrably bounded and predictable set-valued process. Then*

- (i) $\int G_\tau dM_\tau$ is a nonempty closed and bounded subset of \mathcal{M}_0^2 ,
- (ii) when G takes on convex values, $\int G_\tau dM_\tau$ is convex and weakly compact in \mathcal{M}_0^2 ,
- (iii) $\int_s^t G_\tau dM_\tau$ is a nonempty closed and bounded subset of L^2 , each $0 \leq s < t < \infty$,
- (iv) when G takes on convex values, $\int_s^t G_\tau dM_\tau$ is convex and weakly compact in L^2 for each $0 \leq s < t < \infty$.

Proof. The result follows from the properties of $\mathcal{S}_M(G)$ and linear mapping \mathcal{J}_M , because of conditions (i) and (ii) of Proposition 3.6.

Proposition 3.8. *Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrably bounded and predictable set-valued process. Then $\int_s^t G_\tau dM_\tau$ is \mathcal{F}_s -decomposable for every $0 \leq s < t < \infty$.*

Proof. Let $0 \leq s < t < \infty$ be fixed, and $A \in \mathcal{F}_s$ and $J^1, J^2 \in \int_s^t G_\tau dM_\tau$ be given. Then there are $f^1, f^2 \in \mathcal{S}_M(G)$ such that $J^1 = \mathcal{J}_M(\mathbb{I}_{(s,t]}f^1)_t$ and $J^2 = \mathcal{J}_M(\mathbb{I}_{(s,t]}f^2)_t$. But $\mathbb{I}_A J^1 = \mathbb{I}_A \mathcal{J}_M(\mathbb{I}_{(s,t]}f^1)_t = \mathcal{J}_M(\mathbb{I}_{(s,t]}(\mathbb{I}_{(s,t] \times A} f^1))_t$ and $\mathbb{I}_B J^2 = \mathbb{I}_B \mathcal{J}_M(\mathbb{I}_{(s,t]}f^2)_t = \mathcal{J}_M(\mathbb{I}_{(s,t]}(\mathbb{I}_{(s,t] \times B} f^2))_t = \mathcal{J}_M(\mathbb{I}_{(s,t]}(\mathbb{I}_{\beta_s^t} f^2))_t$ where $B = \Omega \setminus A$ and $\beta_s^t = ([0, s] \cup (t, \infty)) \times \Omega \cup (s, t] \times B$ is a complement of $(s, t] \times A$ to $\mathbb{R}_+ \times \Omega$. Therefore

$$\mathbb{I}_A J^1 + \mathbb{I}_B J^2 = \mathcal{J}_M\left(\mathbb{I}_{(s,t]}(\mathbb{I}_{(s,t] \times A} f^1 + \mathbb{I}_{\beta_s^t} f^2)\right)_t.$$

By virtue of Lemma 3.4, $\mathcal{S}_M(G)$ is decomposable, and therefore $\mathbb{I}_{(s,t] \times A} f^1 + \mathbb{I}_{\beta_s^t} f^2 \in \mathcal{S}_M(G)$ because $(s, t] \times A$ is a predictable rectangle and β_s^t is its complement. Thus, $\mathbb{I}_A J^1 + \mathbb{I}_B J^2 \in \int_s^t G_\tau dM_\tau$.

Theorem 3.9. *Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrably bounded and predictable set-valued process. Then*

$$\text{dist}_{L^2} \left(\int_0^t f_\tau dM_\tau, \int_0^t G_\tau dM_\tau \right) = \left(E \int_0^t \text{dist}^2(f_\tau, G_\tau) d[M, M]_\tau \right)^{1/2}$$

for $f \in L_M^2$ and $t \geq 0$.

Proof. Immediately from Theorem 2.2 of [10] for each $t \geq 0$ one has

$$\begin{aligned} & \text{dist}_{L^2}^2 \left(\int_0^t f_\tau dM_\tau, \int_0^t G_\tau dM_\tau \right) \\ &= \inf \left\{ E \left(\int_0^t (f_\tau - g_\tau) dM_\tau \right)^2 : g \in \mathcal{S}_M(G) \right\} \\ &= \inf \left\{ \int_{[0,t] \times \Omega} |f_\tau - g_\tau|^2 d\mu_M : g \in \mathcal{S}_M(G) \right\} \\ &= \int_{[0,t] \times \Omega} \text{dist}^2(f_\tau, G_\tau) d\mu_M = E \int_0^t \text{dist}^2(f_\tau, G_\tau) d[M, M]_\tau. \end{aligned}$$

Theorem 3.10. *Let M be a square-integrable martingale, $M_0 = 0$, and let F, G be M -integrably bounded and predictable set-valued processes. Then*

$$H_{L^2} \left(\int_0^t G_\tau dM_\tau, \int_0^t F_\tau dM_\tau \right) \leq \left(E \int_0^t H^2(G_\tau, F_\tau) d[M, M]_\tau \right)^{1/2}$$

each $t \geq 0$, where H_{L^2} is the Hausdorff metric induced by the $L^2(\mathcal{F}_t)$ norm.

Proof. Let us observe that

$$\begin{aligned} & \bar{h}_{L^2}^2 \left(\int_0^t F_\tau dM_\tau, \int_0^t G_\tau dM_\tau \right) \\ &= \sup \left\{ \text{dist}_{L^2}^2 \left(k, \int_0^t G_\tau dM_\tau \right) : k \in \int_0^t F_\tau dM_\tau \right\} \\ &= \sup \left\{ E \left(\int_0^t \text{dist}^2(f_\tau, g_\tau) d[M, M]_\tau \right) : f \in \mathcal{S}_M(F) \right\} \\ &= E \left(\int_0^t \bar{h}^2(F_\tau, G_\tau) d[M, M]_\tau \right) \leq E \int_0^t H^2(F_\tau, G_\tau) d[M, M]_\tau. \end{aligned}$$

Therefore the result holds.

Theorem 3.11. *Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M -integrably bounded and predictable set-valued process. Then*

$$\left\| \int_0^t G_\tau dM_\tau \right\|_{L^2} = \left(E \int_0^t \|G_\tau\|^2 d[M, M]_\tau \right)^{1/2}$$

each $t \geq 0$, where $\|\cdot\|_{L^2}$ and $\|\cdot\|$ denote norms of sets in L^2 and \mathbb{R}^n , respectively.

Proof. Let us observe that

$$\begin{aligned} \left\| \int_0^t G_\tau dM_\tau \right\|_{L^2} &= \sup \left\{ E \left| \int_0^t f_\tau dM_\tau \right|^2 : f \in \mathcal{S}_M(G) \right\} \\ &= \sup \left\{ \int_{[0,t] \times \Omega} f^2 d\mu_M : f \in \mathcal{S}_M(G) \right\}. \end{aligned}$$

On the other hand, Theorem 2.2 in [10] implies that:

$$\begin{aligned} &\sup \left\{ \int_{[0,t] \times \Omega} f^2 d\mu_M : f \in \mathcal{S}_M(G) \right\} \\ &= \int_{[0,t] \times \Omega} \|G\|^2 d\mu_M = E \int_0^t \|G_\tau\|^2 d[M, M]_\tau. \end{aligned}$$

Theorem 3.12. *Let M be a square-integrable martingale, $M_0 = 0$, and let F, G be M -integrably bounded and predictable set-valued processes. Then*

- (i) *for a given $f \in L_M^2$ one has $f \in \mathcal{S}_M(G)$ if and only if $\int_0^t f_\tau dM_\tau \in \int_0^t G_\tau dM_\tau$ for $t \geq 0$,*
- (ii) *$F_\tau(\omega) \subset G_\tau(\omega)$ μ_M -a.e. if and only if $\int_0^t F_\tau dM_\tau \subset \int_0^t G_\tau dM_\tau$ for $t \geq 0$,*
- (iii) *$G_\tau(\omega)$ is convex μ_M -a.e. if and only if $\int_0^t G_\tau dM_\tau$ is convex for each $t \geq 0$,*
- (iv) *$G_\tau(\omega)$ is a subspace μ_M -a.e. if and only if $\int_0^t G_\tau dM_\tau$ is a subspace for each $t \geq 0$,*
- (v) *$\overline{\text{co}}G = (\overline{\text{co}}G_t(\omega))_{t \geq 0}$ is a predictable set-valued process and $\int_0^t (\overline{\text{co}}G_\tau) dM_\tau = \overline{\text{co}} \int_0^t G_\tau dM_\tau$, $t \geq 0$.*

Proof. (i) Immediately from Theorem 3.9 one obtains

$$\text{dist}_{L^2} \left(\int_0^t f_\tau dM_\tau, \int_0^t G_\tau dM_\tau \right) = 0 \text{ for each } t \geq 0 \text{ if and only if}$$

$$E \int_0^t \text{dist}^2(f_\tau, G_\tau) d[M, M]_\tau = 0, \text{ each } t \geq 0, \text{ i.e.}$$

$$E \int_0^t \text{dist}^2(f_\tau, G_\tau) d[M, M]_\tau = 0 \text{ that is equivalent to } (f_t)_{t \geq 0} \in \mathcal{S}_M(G).$$

(ii) By virtue of Castaing's representation of the measurable selection theorem ([4]) there are sequences (g^n) and (f^n) of $\mathcal{S}_M(G)$ and $\mathcal{S}_M(F)$ such that $G_t(\omega) = cl\{g_t^1(\omega), g_t^2(\omega), \dots\}$ and $F_t(\omega) = cl\{f_t^1(\omega), f_t^2(\omega), \dots\}$, each $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Then $\mathcal{S}_M(F) \subset \mathcal{S}_M(G)$ implies $F_t(\omega) \subset G_t(\omega)$ each $(t, \omega) \in \mathbb{R}_+ \times \Omega$. It is clear that $F_t(\omega) \subset G_t(\omega)$ μ_M -a.e. implies $\mathcal{S}_M(F) \subset \mathcal{S}_M(G)$. Therefore (ii) holds.

(iii) It is clear that $\int_0^t G_\tau dM_\tau$ is convex if and only if $\mathcal{S}_M(G)$ is convex. Immediately from the proof of Lemma 3.3 (ii) it follows that convexity of $G_\tau(\omega)$ μ_M -a.e. implies convexity of $\mathcal{S}_M(G)$. On the other hand, by Castaing's representation theorem it follows that for every M -integrable and predictable set-valued processes G, F ; $\mathcal{S}_M(G) = \mathcal{S}_M(F)$ implies $G_t(\omega) = F_t(\omega)$ μ_M -a.e. But $\mathcal{S}_M(\overline{co}G) = \overline{co}\mathcal{S}_M(G)$ by Theorem 1.5. of [10]. Therefore, convexity of $\mathcal{S}_M(G)$ implies $G_t(\omega) = \overline{co}G_t(\omega)$ μ_M -a.e. and (iii) holds.

Finally, (iv) can be proved similarly as (iii) while (v) follows from the equality $\mathcal{S}_M(\overline{co}G) = \overline{co}\mathcal{S}_M(G)$ and Proposition 3.6.

Remark 3.13. The above properties are also true, when we take $M = W$, where W is a Wiener process. In this case, the set $\mathcal{S}_M(G)$ is defined as a set of all progressively measurable selectors g of G such that $E(\int_0^t g_s^2 ds) < \infty$.

4. SET-VALUED STOCHASTIC INTEGRAL DRIVEN BY A SEMIMARTINGALE

In this section, Z denotes a one-dimensional semimartingale, $Z_0 = 0$, having a canonical decomposition $Z = N + A$, where N is a local martingale, while A is a process with a path of finite variation on compacts. $[Z, Z]$ is a quadratic variation process of Z (see e.g., [16]).

Let \mathcal{H}^2 denotes a space of semimartingales with a norm

$$\|Z\|_{\mathcal{H}^2} = \left\| [N, N]_{\infty}^{1/2} \right\|_{L^2} + \left\| \int_0^{\infty} |dA_t| \right\|_{L^2}.$$

\mathcal{H}^2 is a Banach space (see [16]).

Let a predictable set-valued process $G = (G_t)_{t \geq 0}$ and a semimartingale $Z \in \mathcal{H}^2$ be given. Let

$$L_Z^2 = \left\{ f \in \mathcal{P} : E \left(\int_0^{\infty} f_s^2 d[N, N]_s \right) + E \left(\int_0^{\infty} |f_s| |dA_s| \right)^2 < \infty \right\}$$

L_Z^2 can be endowed with the \mathcal{H}^2 -norm defined above. Similarly as in Definition 3.2, by $\mathcal{S}_Z(G)$ we denote a set

$$\mathcal{S}_Z(G) := \{ f \in L_Z^2 : f(t, \omega) \in G(t, \omega), \lambda \otimes P\text{-a.e.} \}.$$

A predictable set-valued process $G = (G_t)_{t \geq 0}$ is integrable with respect to a semimartingale Z , or simply Z -integrable if $\mathcal{S}_Z(G)$ is a nonempty set. It follows immediately from the properties of stochastic integrals with respect to semimartingales (Theorem IV.32 of [16]) and Kuratowski and Ryll-Nardzewski's measurable selection theorem that every Z -integrably bounded and predictable set-valued stochastic process $G = (G_t)_{t \geq 0}$ is Z -integrable. Recall that a predictable set-valued stochastic process $G = (G_t)_{t \geq 0}$ is Z -integrably bounded if there exists a Z -integrable process m such that $H(G, \{0\}) \leq m$, $\lambda \otimes P$ -a.e.

A set-valued stochastic integral $\int G_{\tau} dZ_{\tau}$ is defined, as in Section 3, by formulas $\int G_{\tau} dZ_{\tau} = \{ \int g_{\tau} dZ_{\tau} : g \in \mathcal{S}_Z(G) \}$ and $\int_s^t G_{\tau} dZ_{\tau} = \{ \int_s^t g_{\tau} dZ_{\tau} : g \in \mathcal{S}_Z(G) \}$.

Investigating properties of a set $\mathcal{S}_Z(G)$ we cannot use the isometry property being so fruitful in Section 3, where the martingale case was investigated. For this reason not all results from Section 3 can be generalized to the semimartingale case. However, the following result holds by the same arguments as used in Section 3.

Proposition 4.1. *Let Z be an arbitrary semimartingale, ($Z_0 = 0$), and let G be a predictable and Z -integrable set-valued process. Then*

- (i) $\mathcal{S}_Z(G)$ is decomposable,
- (ii) if G is Z -integrably bounded, then $\mathcal{S}_Z(G)$ is a nonempty and bounded subset of L_Z^2 ,
- (iii) if G is Z -integrably bounded, then $\int G_\tau dZ_\tau$ is a nonempty and bounded subset of \mathcal{H}^2 ,
- (iv) $\int_s^t G_\tau dZ_\tau$ is \mathcal{F}_s -decomposable for every $0 \leq s < t < \infty$.

A result similar to Theorem 3.10 holds true also for a semimartingale case.

Theorem 4.2. *Let Z be an \mathcal{H}^2 -semimartingale decomposed into a sum $Z = M + A$, where M is a square-integrable martingale, A is a process with path of finite variation on compacts. Let g be a bounded Z -integrable predictable process and let G be a bounded Z -integrable predictable set-valued process. Then there exists a constant $K \geq 0$ such that*

$$\begin{aligned} & \text{dist}_{\mathcal{H}^2}^2 \left(\int g(t, \omega) dZ_t, \int G(t, \omega) dZ_t \right) \\ & \leq K \cdot \left\| \int \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) dZ_t \right\|_{\mathcal{H}^2}. \end{aligned}$$

Proof. We obtain

$$\begin{aligned}
I &= \text{dist}_{\mathcal{H}^2}^2 \left(\int g(t, \omega) dZ_t, \int G(t, \omega) dZ_t \right) \\
&= \inf_{f \in \mathcal{S}_Z(G)} \left\| \int g(t, \omega) dZ_t - \int f(t, \omega) dZ_t \right\|_{\mathcal{H}^2}^2 \\
&= \inf_{f \in \mathcal{S}_Z(G)} \left(\left\| \left(\int_0^\infty (g(t, \omega) - f(t, \omega))^2 d[M, M]_t \right)^{1/2} \right\|_{L^2} \right. \\
&\quad \left. + \left\| \int_0^\infty |g(t, \omega) - f(t, \omega)| |dA_t(\omega)| \right\|_{L^2} \right)^2.
\end{aligned}$$

Applying the Kunita-Watanabe inequality together with $(a+b)^2 \leq 2a^2 + 2b^2$ we get

$$\begin{aligned}
I &\leq 2 \cdot \inf_{f \in \mathcal{S}_Z(G)} \left\{ \int_\Omega \left(\int_0^\infty (g(t, \omega) - f(t, \omega))^2 d[M, M]_t \right) P(d\omega) \right. \\
&\quad \left. + \int_\Omega \left(\int_0^\infty (g(t, \omega) - f(t, \omega))^2 |dA_t(\omega)| \cdot \int_0^\infty |dA_t(\omega)| \right) P(d\omega) \right\}.
\end{aligned}$$

Let μ_M denote the Doléans-Dade's measure associated with a martingale M and $c_A(\omega) = \int_0^\infty |dA_t(\omega)|$. Now we define a kernel of measure (a random measure) for Borel sets in \mathbb{R}_+ by

$$\alpha(\omega, dt) := c_A(\omega) |dA_t(\omega)|.$$

Afterwards we define a measure ν_A on a σ -algebra of predictable sets in $\mathbb{R}_+ \times \Omega$ by

$$\nu_A(B) = \int_\Omega \int_{\mathbb{R}_+} \mathbb{1}_B(\omega, t) \alpha(\omega, dt) P(d\omega)$$

and we obtain:

$$\begin{aligned}
 I &\leq 2 \cdot \inf_{f \in \mathcal{S}_Z(G)} \left\{ \int_{\Omega} \left(\int_0^{\infty} (g(t, \omega) - f(t, \omega))^2 d[M, M]_t \right) P(d\omega) \right. \\
 &\quad \left. + \int_{\Omega} \left(\int_0^{\infty} (g(t, \omega) - f(t, \omega))^2 \alpha(\omega, dt) \right) P(d\omega) \right\} \\
 &\leq 2 \cdot \inf_{f \in \mathcal{S}_Z(G)} \left\{ \int_{\Omega \times \mathbb{R}_+} (g(t, \omega) - f(t, \omega))^2 d\mu_M + \int_{\Omega \times \mathbb{R}_+} (g(t, \omega) - f(t, \omega))^2 d\nu_A \right\} \\
 &= 2 \cdot \inf_{f \in \mathcal{S}_Z(G)} \left(\int_{\Omega \times \mathbb{R}_+} (g(t, \omega) - f(t, \omega))^2 d\mu^* \right)
 \end{aligned}$$

where $\mu^* = \mu_M + \nu_A$.

Using Theorem 2.2 of [10] we deduce that the last term is equal to

$$\begin{aligned}
 &2 \cdot \int_{\Omega \times \mathbb{R}_+} \inf_{x \in G(t, \omega)} (g(t, \omega) - x)^2 d\mu^* \\
 &= 2 \cdot \int_{\Omega \times \mathbb{R}_+} \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) d\mu^* \\
 &= 2 \cdot \left(\int_{\Omega} \left(\int_0^{\infty} \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) d[M, M]_t \right) P(d\omega) \right. \\
 &\quad \left. + \int_{\Omega} \left(c_A(\omega) \cdot \int_0^{\infty} \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) |dA_t(\omega)| \right) P(d\omega) \right).
 \end{aligned}$$

Applying once again the Kunita-Watanabe inequality to the first component and Hölder inequality to the second one, we get

$$\begin{aligned}
I &\leq 2 \cdot \left(\left\| \left(\int_0^\infty \text{dist}_{\mathbb{R}^n}^4(g(t, \omega), G(t, \omega)) d[M, M]_t \right)^{1/2} \right\|_{L^2} \cdot \left\| \left(\int_0^\infty d[M, M]_t \right)^{1/2} \right\|_{L^2} \right. \\
&\quad \left. + \|c_A(\omega)\|_{L^2(\Omega)} \cdot \left(\int_\Omega \left(\int_0^\infty \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) |dA_t(\omega)| \right)^2 P(d\omega) \right)^{1/2} \right) \\
&\leq 2 \cdot \left(E[M, M]_\infty^{1/2} \cdot \left(\int_\Omega \int_0^\infty \text{dist}_{\mathbb{R}^n}^4(g(t, \omega), G(t, \omega)) d[M, M]_t P(d\omega) \right)^{1/2} \right. \\
&\quad \left. + \|c_A(\omega)\|_{L^2(\Omega)} \cdot \left(\int_\Omega \left(\int_0^\infty \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) |dA_t(\omega)| \right)^2 P(d\omega) \right)^{1/2} \right) \\
&\leq K \cdot \left\| \int \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) dZ_t \right\|_{\mathcal{H}^2}
\end{aligned}$$

with $K = 2 \cdot \max\{\|c_A(\omega)\|_{L^2(\Omega)}, E[M, M]_\infty^{1/2}\}$ what proves the result.

Corollary 4.3. *Let Z be an \mathcal{H}^2 -semimartingale decomposed into a sum $Z = N + A$, where N is a local martingale, A is a process with a path of finite variation on compacts. Let g be a bounded Z -integrable predictable process and let G be a bounded Z -integrable predictable set-valued process. Then there exists a constant $K \geq 0$ such that*

$$\begin{aligned}
&\text{dist}_{\mathcal{H}^2}^2 \left(\int g(t, \omega) dZ_t, \int G(t, \omega) dZ_t \right) \\
&\leq K \cdot \left(\left\| \int \text{dist}_{\mathbb{R}^n}^2(g(t, \omega), G(t, \omega)) dZ_t \right\|_{\mathcal{H}^2} \right).
\end{aligned}$$

Proof. The proof follows from the definition of \mathcal{H}^2 -norm, Theorem 4.2 and Corollary II.6.4 of [16].

Theorem 4.4. *Let Z be an \mathcal{H}^2 -semimartingale decomposed into a sum $Z = N + A$, where N is a local martingale, A is a process with a path of finite variation on compacts. Let F, G be bounded predictable and Z -integrable set-valued processes. Then there exists a constant $K \geq 0$ such that*

$$\begin{aligned} & H_{\mathcal{H}^2}^2 \left(\int G(t, \omega) dZ_t, \int F(t, \omega) dZ_t \right) \\ & \leq K \cdot \left\| \int H_{\mathbb{R}^n}^2(G(t, \omega), F(t, \omega)) dZ_t \right\|_{\mathcal{H}^2}. \end{aligned}$$

Proof. From the definition of the Hausdorff distance we have:

$$\begin{aligned} & H_{\mathcal{H}^2}^2 \left(\int G(t, \omega) dZ_t, \int F(t, \omega) dZ_t \right) \\ & = \max \left\{ \sup_{g \in \mathcal{S}_Z(G)} \text{dist}_{\mathcal{H}^2}^2 \left(\int g(t, \omega) dZ_t, \int F(t, \omega) dZ_t \right), \right. \\ & \quad \left. \sup_{f \in \mathcal{S}_Z(F)} \text{dist}_{\mathcal{H}^2}^2 \left(\int f(t, \omega) dZ_t, \int G(t, \omega) dZ_t \right) \right\}. \end{aligned}$$

From Lemma 4.3 applying two times Theorem 2.2 from [10] (for the functions "inf" and "sup", respectively) we get

$$\begin{aligned} & \sup_{g \in \mathcal{S}_Z(G)} \text{dist}_{\mathcal{H}^2}^2 \left(\int g(t, \omega) dZ_t, \int F(t, \omega) dZ_t \right) \\ & \leq K \cdot \left\| \int \bar{h}_{\mathbb{R}^n}^2(G(t, \omega), F(t, \omega)) dZ_t \right\|_{\mathcal{H}^2} \\ & \leq K \cdot \left\| \int H_{\mathbb{R}^n}^2(G(t, \omega), F(t, \omega)) dZ_t \right\|_{\mathcal{H}^2}, \end{aligned}$$

where: $\bar{h}(A, B) = \sup_{a \in A} \text{dist}(a, B)$.

A similar inequality holds for the second component and this completes the proof.

Now we present some real problems described by mathematical models containing semimartingale integrals.

Example 4.5. Suppose we have a model of a free-arbitrage market defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$. The capital of an investor (a writer of a contingent claim) is defined under a self-financing assumption by the relation:

$$\xi_t = \xi_0 + \int_0^t \theta_s dB_s + \int_0^t \gamma_s dS_s, t \in [0, T],$$

where (θ, γ) is an investor's strategy (hedge) process, while B and S are price processes of a bond (an asset with a predictable price) and a stock, respectively (see e.g., [6] for details).

A typical choice of the price processes is as follows: $B_t = \exp\{rt\}$, $r = \text{const}$, and $S_t = S_0 \exp\{-1/2\alpha^2 t + \alpha W_t\}$, where W is a standard Wiener process. The above problem can be rewritten to the form

$$\xi_t = \xi_0 + \int_0^t f_s dZ_s; t \in [0, T],$$

where $f_s = (\theta_s, \gamma_s)$ and $Z_s = (0, S_s) + (B_s, 0) = M_s + A_s$. Since the stochastic exponential S_s is a continuous martingale (see e.g., [16] Section 2.8) the above problem is modeled by a stochastic integral driven by a semimartingale.

Example 4.6. The stochastic integral described in Example 4.5 was driven by a Wiener process. Thus solution processes have been thought to be a processes with continuous paths. However, in real life there are many problems, in which we often experience a deviation from a "Gaussian mechanism". Such a situation is likely to happen for example in financial mathematics. Unfortunately, the assumption that stock prices evolve according to a geometric Brownian motion, as we have seen before, suffers from serious deficiencies. If the model is based on daily returns of a stock, statistical tests reject hypotheses about normality distribution made in the model of the Black and Scholes type.

It follows that real prices usually are characterized better by the so-called heavy tailed distributions, skewness property, effects of clusters and so on. Moreover, an empirical study of the German stock price data shows that paths should be modeled by a discontinuous process instead of a continuous one (see e.g., [7]).

Generalizations of the Gaussian model were proposed in many different manners. It was allowed in [1] that the price process has jumps and the resulting equation has the form (in a one dimensional case):

$$dS_t = \mu(t)S_{t-}dt + \sigma(t)S_{t-}dW_t + \beta S_{t-}dN_t,$$

where N is a point process counting the number of jumps of size β that the relative price S_t/S_{t-} had before time t .

Since (N_t) can be treated as a Poisson process with some intensity λ ([16]) then the above problem can be again rewritten equivalently as

$$S_t = S_0 + \int_0^t f_s dZ_s; \quad t \in [0, T],$$

with $f_s = (\mu(s)S_{s-}, \sigma(s)S_{s-}, \beta, \beta)$ and $Z_s = (0, W_s, N_s - \lambda s, 0) + (s, 0, 0, \lambda s) = M_s + A_s$.

If we consider controlled of financial problems connected with the models as these presented above, the set-valued integrals $\int F_s dZ_s$ driven by semimartingales appear, instead of a single valued $\int f_s dZ_s$, in a natural way. In such control models we put usually

$$F(s, \omega) = \bigcup_{u \in U} f(s, \omega, u),$$

where u denotes control parameter taken from a given set U of attainable controls.

Therefore the properties of such integrals can be useful in some financial investigations.

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