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ON A CHARACTERIZATION OF SYMMETRIC BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract

All the symmetric balanced incomplete block (SBIB) designs have been characterized and a new generalized expression on parameters of SBIB designs has been obtained. The parameter b has been formulated in a different way which is denoted by $b_i, i = 1, 2, 3$, associating with the types of the SBIB design D_i. The parameters of all the designs obtained through this representation have been tabulated while corresponding them with the suitable formulae for the number of blocks b_i and the expression S_i for the convenience of practical users for constructional methods of certain designs, which is the main theme of this paper.

Keywords: symmetric balanced incomplete block (SBIB) design.

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1. Introduction

A balanced incomplete block (BIB) design is an arrangement of v symbols in b sets, each containing $k(< v)$ symbols such that every symbol occurs at most once in a set, every symbol occurs in exactly r sets, and every pair of symbols occurs together in λ sets. The parameters of a BIB design satisfy

$$vr = bk, \quad \lambda(v-1) = r(k-1). \tag{1.1}$$

A BIB design is said to be symmetric if $v = b$, and consequently $r = k$. It is well known (cf. [13]) that in a symmetric BIB design any two sets have exactly λ symbols in common. Besides a necessary condition for the existence of an SBIB design is that $r - \lambda$ be a perfect square when v is even. The non-existence of certain SBIB designs was demonstrated by Chowla and Ryser [1] and Shrikhande [14]. Many authors have done much of the work on these designs for the last so many decades, for example, refer to Hussain [4], Szekers [17], Zaidi [19].

In this paper we propose a new generalized expression on parameters of SBIB designs and obtain all the possible parameters of these designs, and tabulate them, where $v \le 111$ and $k \le 55, \lambda \le 30$, in which there are some SBIB designs which are not seen elsewhere when compared to the tables of Collins [2], Kageyama [7], Mathon and Rosa [9], Raghavarao [13] and Takeuchi [18]. We may take the ranges of parameters to the higher values also if need be. In this attempt, we classify the SBIB designs into three types, where Type I is with $k = n\lambda$ for an integer $n \ge 2$, Type II is with $k = n\lambda + 1$ for an integer $n \ge 1$ and Type III is with $k = n\lambda + m$ for integers $n \ge 1$ and $m \ge 2$, which give an insight on all SBIB designs in a different way. This type of characterization is entirely distinct and found useful for constructional purposes of designs. Depending on these types we formulate the parameter b in each case. We consider different existing series available in literature and we show that how they can be fitted in, in the general expression established here. But as given in Fanning [3], proving an embedding theorem for certain quasi-symmetric designs, and by using that, a new series of SBIB designs with $v = 9(1 + 16 + 16^2 + \cdots + 16^m) + 16^{m+1}$, $k = 9(1 + 16 + 16^2 + \cdots + 16^{m-1})$ and $\lambda = 9(1 + 16 + 16^2 + \cdots + 16^{m-1}) + 3 \cdot 16^m$ which is of Type III. But the parameters that we obtain with this series are so large.

Consequently, when we enlist all possible designs obtained by the generalized expression, the parameters of the designs obtained by different series do exist in the above list itself. Besides in [5] Ionin found four families of SBIB designs, and when compared to them our generalized expression S_{11} is much easier and practicable. And in [6] Ionin interestingly gave a method of constructing certain SBIB designs. We refer Collins [2], Kageyama [7], Mathon and Rosa [9], Raghavarao [13] and Takeuchi [18], the parameters of the designs in these references are all in the list of parameters of designs we obtained here, but some parameters of designs are seen to be existing which are not mentioned in any one of these references [2, 7, 9, 13, 18]. Lastly Table 3.1 has been given in the third section, along with the suitable formulae for the number of blocks $b_i, i = 1, 2, 3$, which is the main theme of this paper, with all due references needed for the sake of practical users, while constructing the other designs using these SBIB designs or constructing these SBIB designs using other designs. To mention a practical purpose, the SBIB designs play a pivotal role in the construction of affine μ-resolvable BIB designs for $\mu \geq 1$. Mohan [10], by taking SBIB designs of Type II, gave a method of construction of affine μ-resolvable BIB designs by use of certain SBIB designs through a juxtaposition pattern of a matrix which was later called by Mohan [11, 12] as M_n-matrix, which played a pivotal role in the construction of designs. Kageyama and Mohan [8], while generalizing the above result, gave a method of construction of affine μ-resolvable BIB designs by taking all the three types of SBIB designs. Shrikhande and Singh [15] constructed SBIB designs of Type I only, by taking BIB designs with $\lambda = 1$ and $r = 2k + 1$. Shrikhande and Raghavarao [16] also gave a method of constructing all affine μ-resolvable BIB designs by using SBIB designs.

2. Main results

Since $k > \lambda$ holds in an SBIB design, we have $k = n\lambda + m$, where n is the quotient and m is the remainder, $0 \leq m < n$. Then all the SBIB designs can be classified into three types, namely, Type I is with $k = n\lambda$ for an integer $n \geq 2$; Type II is with $k = n\lambda + 1$ for an integer $n \geq 1$; Type III is with $k = n\lambda + m$ for integers $n \geq 1$ and $m \geq 2$.

Theorem 2.1. *In an SBIB design of Type I, where $k = n\lambda, k \geq 2$ and $n \geq 2$, $b = r + n(r - \lambda) - (n - 1)$ holds.*

Proof. Since $k = n\lambda$, the parameters of the given SBIB design satisfy $v = b, r = k = n\lambda, \lambda$. From (1.1), $\lambda(v - 1) = n\lambda(n\lambda - 1)$. Therefore $v = n^2\lambda - n + 1 = n^2\lambda - n + n\lambda - n\lambda + 1 = n\lambda + n(n\lambda - \lambda) - (n - 1) = k + n(k - \lambda) - (n - 1)$ or $r + n(r - \lambda) - (n - 1) = b$. ■

Theorem 2.2. *In an SBIB design, if $b = r + n(r - \lambda) - (n - 1)$ holds, then either $k = n\lambda$ or $k = \lambda + 1$.*

Proof. Let the parameters of the SBIB design be $v = b, r = k, \lambda$. Then $v - 1 = r(r - 1)/\lambda$, i.e., $v = (r^2 - r + \lambda)/\lambda$. Thus we have $b = (r^2 - r + \lambda)/\lambda$. But given that $b = r + n(r - \lambda) - (n - 1)$. Now equating these values of b, we have $(r^2 - r + \lambda)/\lambda = r + n(r - \lambda) - (n - 1)$, which yields $r^2 - r(n\lambda + \lambda + 1) + (n\lambda^2 + n\lambda) = 0$. This is a quadratic equation on r that gives $r = [n\lambda + \lambda + 1 \pm \sqrt{(n\lambda + \lambda + 1)^2 - 4(n\lambda^2 + n\lambda)}]/2 = [n\lambda + \lambda + 1 \pm \sqrt{(n\lambda - \lambda - 1)^2}]/2$. Hence on simplification $r = n\lambda = k$ or $r = k = \lambda + 1$. Thus, when $k = n\lambda$ we obtain the parameters from (1.1), i.e., $v = n^2\lambda - n + 1 = b, r = k = n\lambda, \lambda$. On the other hand, when $k = \lambda + 1$, from (1.1), $\lambda(v - 1) = r(k - 1) = r\lambda$, i.e., $v - 1 = r$. Therefore $v = r + 1 = k + 1 = \lambda + 2$. Thus we have parameters as $v = b = \lambda + 2, r = k = \lambda + 1, \lambda$, which obviously yield an irreducible SBIB design. ■

Theorem 2.3. *In an SBIB design, it is of Type II where $k = n\lambda + 1$ if and only if $b = r + n(r - \lambda)$ holds for $n \geq 1$.*

Proof. (Necessity) Let the parameters of the given SBIB design be $v = b, r = k, \lambda$. Since $k = n\lambda + 1$, from (1.1), $\lambda(v - 1) = n\lambda(n\lambda + 1)$. Therefore $v - 1 = n(n\lambda + 1)$. Thus $v = n^2\lambda + n + 1 = n^2\lambda + n + 1 + n\lambda - n\lambda = n\lambda + 1 + n(n\lambda + 1) - n\lambda = k + nk - n\lambda = r + n(r - \lambda) = b$.

(Sufficiency) From (1.1), we obtain $b = (r^2 - r + \lambda)/\lambda$. Equating this to the value of b from the assumption, we get $r^2 - r + \lambda = r\lambda + nr\lambda - n\lambda^2$. This gives $r = n\lambda + 1$ or λ. But r cannot be equal to λ in an SBIB design. Hence $r = k = n\lambda + 1$. ■

Remark 2.1. An SBIB design of Type II has parameters $v = b = n^2\lambda + n + 1, r = k = n\lambda + 1, \lambda$.

Theorem 2.4. *In a BIB design, if $k = n\lambda + 1$ and $b = r + n(r - \lambda)$, then it is an SBIB design.*

Proof. Let the parameters of the given BIB design be v, b, r, k, λ. From (1.1), we have $v = (rk - r + \lambda)/\lambda$. Therefore $b = (r/k)[(rk - r + \lambda)/\lambda] = (r^2k - r^2 + r\lambda)/(k\lambda)$. Given that $b = r + n(r - \lambda)$. Now equating the values of b, we have $(r^2k - r^2 + r\lambda)/(k\lambda) = r + n(r - \lambda)$. This shows a quadratic equation on r as $r^2(k-1) - r(nk\lambda + k\lambda - \lambda) + kn\lambda^2 = 0$. This gives $r = [\lambda(kn + k - 1) \pm \sqrt{\lambda^2(kn + k - 1)^2 - 4(k-1)kn\lambda^2}]/[2(k-1)]$, which shows $r = kn\lambda/(k-1)$ or $r = \lambda$. But given that $k = n\lambda + 1$, we get $r = k$. Hence $v = b$. Therefore it is an SBIB design. Note that $r > \lambda$ in an SBIB design. ∎

Theorem 2.5. *In an SBIB design, it is of Type III where $k = n\lambda + m$ for $n \geq 1$ and $m \geq 2$ if and only if $b = r + n(r - \lambda) + m(n-1) + t$ holds, where $t = [m(m-1) - \lambda(n-1)]/\lambda$ is an integer.*

Proof. (Necessity) Let the parameters of the given SBIB design be $v = b, r = k, \lambda$. Since $k = n\lambda + m$, we have $\lambda(v-1) = (n\lambda + m)(n\lambda + m - 1)$. Therefore $v = [(n\lambda+m)(n\lambda+m-1)+\lambda]/\lambda = [(n\lambda+m)(n\lambda+m-1+\lambda-\lambda)+\lambda]/\lambda = \{[n\lambda(n\lambda+m)+m(n\lambda+m)-(n\lambda+m)+\lambda(n\lambda+m)-\lambda(n\lambda+m)]+\lambda\}/\lambda$. Since $r = n\lambda + m$, we have $b = (rn\lambda + mn\lambda + m^2 - n\lambda - m + r\lambda - n\lambda^2 - m\lambda + \lambda)/\lambda = (r\lambda + rn\lambda - n\lambda^2 + mn\lambda - m\lambda + m^2 - n\lambda - m + \lambda)/\lambda = r+n(r-\lambda)+m(n-1)+[m(m-1)-\lambda(n-1)]/\lambda$. Let $[m(m-1)-\lambda(n-1)]/\lambda = t$. Then $b = r + n(r-\lambda) + m(n-1) + t$. Finally it will be shown that t is an integer. In the SBIB design, $r = k = n\lambda + m$ is a positive integer, where n, m, λ are all positive integers. Hence $n-1$ and $m-1$ are also integers since $m \geq 2$ and $n \geq 1$. From (1.1), we have $v - 1 = r(k-1)/\lambda$. Since $v \geq 2$, $v-1$ is a positive integer, so is $r(k-1)/\lambda$. Since $r = k = n\lambda + m$, it holds that $(n\lambda+m)(n\lambda+m-1)/\lambda$ is also a positive integer, i.e., $[(n\lambda+m)^2-(n\lambda+m)]/\lambda$ is a positive integer. Therefore $(n^2\lambda^2 + 2mn\lambda + m^2 - n\lambda - m)/\lambda$ is a positive integer. That is, $[m(m-1) - \lambda(n-1) + n^2\lambda^2 + 2mn\lambda - \lambda]/\lambda$ is a positive integer. Consequently $[m(m-1) - \lambda(n-1)]/\lambda + n^2\lambda + 2mn - 1$ is a positive integer. As the second part is a positive integer the first part should be an integer.

(Sufficiency) As is usual from (1.1), we obtain $b = (r^2 - r + \lambda)/\lambda$. Equating this to the value of b from the hypothesis, i.e., $b = r + n(r - \lambda) + m(n-1) + t$, where $t = [m(m-1) - \lambda(n-1)]/\lambda$ is an integer, we get $r^2 - r + \lambda = \lambda\{r + n(r-\lambda) + m(n-1) + [m(m-1) - \lambda(n-1)]/\lambda\}$. This gives a quadratic on r as $r^2 - r(n\lambda + \lambda + 1) + n\lambda^2 - mn\lambda + m\lambda - m^2 + m + n\lambda = 0$. On solving for r we get an equation $r(r - n\lambda - \lambda - 1) = (n\lambda + m)(m - \lambda - 1)$, i.e., $[r - (n\lambda + m)](r + m - \lambda - 1) = 0$, which shows that $r = n\lambda + m$, since the other case is not possible. ∎

Remark 2.2. Theorem 2.5 includes Theorems 2.1 and 2.3 when $m = 0$ and $m = 1$, respectively.

Remark 2.3. There are different series of SBIB designs in literature. But all of them can be brought into the canopy of the generalized expression S_{11} below obtained in this paper, which we will show now.

Series	Parameters	Under which type
S_1	$v = b = n^2\lambda - n + 1, r = k = n\lambda, \lambda$	Type I
S_2	$v = b = n^2\lambda + n + 1, r = k = n\lambda + 1, \lambda$	Type II
S_3	$v = b = 4t + 3, r = k = 2t + 1, \lambda = t$	Type II, $n = 2$
S_4	$v = b = s^2 + s + 1, r = k = s + 1, \lambda = 1$	Type I, $n=s+1,\lambda$ (Type II, $n=s, \lambda=1$)
S_5	$v = b = (s + 1)(s^2 + 1), r = k = s^2 + s + 1,$ $\lambda = s + 1$	Type II, $n = s$, $\lambda = s + 1$
S_6	$v = b = 4m^2, r = k = 2m^2 + m, \lambda = m^2 + m$	Type III, $n = 1$, $m = m^2$
S_7	$v = b = 4m^2, r = k = 2m^2 - m, \lambda = m^2 - m$	Type III, $n = 1$, $m = m^2$
	S_6 and S_7 are complements to each other	
S_8	$v = b = 4m^2 - 1, r = k = 2m^2 - 1, \lambda = m^2 - 1$	Type II, $n = 2$
S_9	$v = b = 4m^2 - 1, r = k = 2m^2, \lambda = m^2$	Type I, $n = 2$ (Also Type III)
	S_8 and S_9 are complements to each other	
S_{10}	$v = b = t^2(t + 2), r = k = t(t + 1), \lambda = t, t \geq 2$	Type I, $n=t+1$ (Also Type III)
S_{11}	$v = b = n^2\lambda + n(2m - 1) + 1 + m(m - 1)/\lambda,$ $r = k = n\lambda + m, \lambda$	Type III
	(*The generalized expression of SBIB designs*)	

Note 2.1. The generalized expression S_{11} includes the two Types I and II besides all the other series S_3 to S_{10} are included.

Note 2.2. S_4 can also be shown under Type II, where $n = s, \lambda = 1, k = n\lambda + 1$ hold. In such a case the values of n, m and t are different.

Note 2.3. All the series S_3 to S_{10} are taken from Raghavarao [13].

Note 2.4. The well-known Bruck-Ryser-Chowla Theorem states that *If v, k, λ are integers satisfying $\lambda(v-1) = k(k-1)$ then for the existence of an SBIB design it is necessary that*

(a) *If v is even then $k - \lambda$ is a perfect square, and*

(b) *If v is odd then $z^2 = (k-\lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2$ has a nontrivial solution in integers x, y, z.*

This theorem can be written using the present theory as follows. For the existence of an SBIB design it is necessary that

(a) If $v = n^2\lambda + n(2m-1) + 1 + m(m-1)/\lambda$ is even, then $\lambda(n-1) + m$ is a perfect square, and

(b) If $v = n^2\lambda + n(2m-1) + 1 + m(m-1)/\lambda$ is odd, then $z^2 = [\lambda(n-1) + m]x^2 + (-1)^{(v-1)/2}\lambda y^2$ has a nontrivial solution in integers x, y, z, where integers $n \geq 1$ and $m \geq 0$.

When $m = 0$ and $m = 1$ we get the particular cases of Types I and II, respectively.

Remark 2.4. We have formulated the parameter b in the three types and we will denote them as follows for the tabulation purpose.

Type I A $= b = r + n(r-\lambda) - (n-1), r = k = n\lambda$

Type II $\quad$ B $= b = r + n(r - \lambda), r = k = n\lambda + 1$

Type III $\quad$ D $= b = r + n(r - \lambda) + m(n - 1) + t, r = k = n\lambda + m,$

where $t = [m(m - 1) - \lambda(n - 1)]/\lambda$ is an integer.

Remark 2.5. In the above three formulae A, B and D, the formulae A and B are the particular cases of the formula D only, i.e., when $m = 0$ and $m = 1$.

Remark 2.6. Even though there are excellent tables of SBIB designs existing, hereunder, we have given another table of SBIB designs along with the suitable formula for b. To which series the design belongs is also given. But all the designs fall in the generalized expression that has also been depicted in this table. Thus this table corresponds each design D to the concerned formulae for the number of blocks b_i and the concerned series S_i.

3. Tabulation

The parametric ranges in the tables of Raghavarao [13] are $v \leq 100$, $k \leq 15, \lambda \leq 15$; in the tables of Takeuchi [18] are $v \leq 100, k \leq 30$, $\lambda \leq 14$; in the tables of Collins [2] are $v \leq 50, k \leq 23, \lambda \leq 11$; and in the table of Mathon and Rosa [9] are $r \leq 41$ and $k \leq v/2$. Whereas the limits of these ranges in the present work are $v \leq 111, k \leq 55$, $\lambda \leq 30$.

For the sake of tabulation we use the following notation. In the column 8, I indicates that it is an irreducible design, E indicates that the design exists and N indicates the non-existence of the design, and $-$ indicates that the existence of the design is not known. And s^* indicates that the design is the complement of the design of no. s. The reference tables are denoted by R: Raghavarao [13], T: Takeuchi [18], C: Collins [2], and MR: Mathon and Rosa [9], in the last column, where "a blank" indicates that the parameters are not shown in any one of the references R, T, C, MR. (Note that MR is indicated only when any of R, T, C is not shown as existence or non-existence of the design.) These designs are also shown because for the sake of formula of concerned b and the concerned series. Hereunder we enlist all the possible parameters notifying the status of each design in detail as follows:

Table 3.1. Parameters of SBIB designs with $v = b \leq 111, r = k \leq 55, \lambda \leq 30$

No.	$v = b$	$r = k$	λ	n	m	t	I/N/E	Formula	Series	Reference
1	3	2	1	2	0	-1	I	A	S_1, S_2, S_4, S_9	
2	4	3	2	1	1	0	I	B	S_2, S_5, S_6	R
3	5	4	3	1	1	0	I	B	S_2	R
4	6	5	4	1	1	0	I	B	S_2	R
5	7	3	1	3	0	-2	E	B	S_1, S_2, S_3, S_4	R, T, C
6	7	4	2	2	0	-1	5*	A	S_1	R
7	7	6	5	1	1	0	I	B	S_2	R
8	8	7	6	1	1	0	I	B	S_2	R
9	9	8	7	1	1	0	I	B	S_2	R
10	10	9	8	1	1	0	I	B	S_2	R
11	11	5	2	2	1	-1	E	B	S_2, S_3	R, T, C
12	11	6	3	2	0	-1	11*	A	S_1	R
13	11	10	9	1	1	0	I	B	S_2	R
14	12	11	10	1	1	0	I	B	S_2	R
15	13	4	1	4	0	-3	E	A	S_1, S_4	R, T
16	13	9	6	1	3	1	15*	D	S_{11}	R
17	13	12	11	1	1	0	I	B	S_2	
18	14	13	12	1	1	0	I	B	S_2	
19	15	7	3	2	1	-1	E	B	S_2, S_3, S_5, S_8	R, T, C
20	15	8	4	2	0	-1	19*	A	S_1, S_9	R
21	15	14	13	1	1	0	I	B	S_2	
22	16	6	2	3	0	-2	E	A	S_1, S_7, S_{10}	R, T
23	16	10	6	1	4	2	22*	D	S_6	R
24	16	15	14	1	1	0	I	B	S_2	
25	17	16	15	1	1	0	I	B	S_2	
26	18	17	16	1	1	0	I	B	S_2	
27	19	9	4	2	1	-1	E	B	S_2, S_3	R, T, C
28	19	10	5	2	0	-1	27*	A	S_1	R
29	19	18	17	1	1	0	I	B	S_2	
30	20	19	18	1	1	0	I	B	S_2	
31	21	5	1	5	0	-4	E	A	S_1, S_4	R, T, C
32	21	16	12	1	4	1	E	D	S_{11}	
33	21	20	19	1	1	0	I	B	S_2	
34	22	7	2	3	1	-2	N	B	S_2	R, T, C
35	22	15	10	1	5	2	N, 34*	D	S_{11}	
36	22	21	20	1	1	0	I	B	S_2	
37	23	11	5	2	1	-1	E	B	S_2, S_3	R, T, C

Table 3.1 (continued-1)

No.	$v=b$	$r=k$	λ	n	m	t	I/N/E	Formula	Series	Reference
38	23	22	21	1	1	0	I	B	S_2	
39	23	12	6	2	0	-1	37*	A	S_1	
40	24	23	22	1	1	0	I	B	S_2	
41	25	9	3	3	0	-2	E	A	S_1	R, T, C
42	25	16	10	1	6	3	41*	D	S_{11}	
43	25	24	23	1	1	0	I	B	S_2	
44	26	25	24	1	1	0	I	B	S_2	
45	27	13	6	2	1	-1	E	B	S_2, S_3	R, T, C
46	27	26	25	1	1	0	I	B	S_2	
47	27	14	7	2	0	-1	45*	A	S_1	
48	28	27	26	1	1	0	I	B	S_2	
49	29	8	2	4	0	-3	N	A	S_1	T, C
50	29	21	15	1	6	2	49*	D	S_{11}	
51	29	28	27	1	1	0	I	B	S_2	
52	30	29	28	1	1	0	I	B	S_2	
53	31	6	1	6	0	-5	E	A	S_1	R, T, C
54	31	10	3	3	1	-2	E	B	S_2	R, T, C
55	31	15	7	2	1	-1	E	B	S_2, S_3	R, T, C
56	31	16	8	2	0	-1	55*	A	S_1	
57	31	21	14	1	7	3	54*	D	S_{11}	
58	31	25	20	1	5	1	53*	D	S_{11}	
59	31	30	29	1	1	0	I	B	S_2	
60	32	31	30	1	1	0	I	B	S_2	
61	34	12	4	3	0	-2	N	A	S_1	T
62	34	22	14	1	8	4	N	D	S_{11}	
63	35	17	8	2	1	-1	E	B	S_2, S_3, S_8	T, C
64	35	18	9	2	0	-1	63*	A	S_1, S_9	
65	36	15	6	2	3	0	E	D	S_7	R, T
66	36	21	12	1	9	6	65*	D	S_6	
67	37	9	2	4	1	-3	E	B	S_2	R, T, C
68	37	28	21	1	7	2	67*	D	S_{11}	
69	39	19	9	2	1	-1	E	B	S_2, S_3	T, C
70	39	20	10	2	0	-1	69*	A	S_1	
71	40	13	4	3	1	-2	E	B	S_2, S_5	R, T, C
72	40	27	18	1	9	4	71*	D	S_{11}	
73	41	16	6	2	4	1	E	D	S_{11}	T, C, MR
74	41	25	15	1	10	6	73*	D	S_{11}	MR

Table 3.1 (continued-2)

No.	$v=b$	$r=k$	λ	n	m	t	I/N/E	Formula	Series	Reference
75	43	7	1	7	0	-6	N	A	S_1, S_4	R, T, C
76	43	15	5	3	0	-2	N	A	S_1	T, C, MR
77	43	21	10	2	1	-1	E	B	S_2, S_3	T, C
78	43	22	11	2	0	-1	77*	A	S_1	
79	43	28	18	1	10	5	N,76*	D	S_{11}	MR
80	43	36	30	1	6	1	N,75*	D	S_{11}	
81	45	12	3	4	0	-3	E	A	S_1, S_{10}	R, T
82	45	33	24	1	9	3	81*	D	S_{11}	
83	46	10	2	5	0	-4	N	A	S_1	T
84	46	36	28	1	8	2	N,83*	D	S_{11}	
85	47	23	11	2	1	-1	E	B	S_2, S_3	T
86	47	24	12	2	0	-1	85*	A	S_1	
87	49	16	5	3	1	-2	E	B	S_2	T, C, MR
88	49	33	22	1	11	5	87*	D	S_{11}	MR
89	51	25	12	2	1	-1	E	B	S_2, S_3	T
90	51	26	13	2	0	-1	89*	A	S_1	
91	52	18	6	3	0	-2	N	A	S_1	T
92	52	34	22	1	12	6	N,91*	D	S_{11}	
93	53	13	3	4	1	-3	N	B	S_2	T
94	53	40	30	1	10	3	N,93*	D	S_{11}	
95	55	27	13	2	1	-1	E	B	S_2, S_3	T
96	55	28	14	2	0	-1	95*	A	S_1	
97	56	11	2	5	1	-4	E	B	S_2	T, MR
98	57	8	1	8	0	-7	E	A	S_1, S_4	R, T
99	58	19	6	3	1	-2	N	B	S_2	T
100	58	39	26	1	13	6	N,99*	D	S_{11}	
101	59	29	14	2	1	-1	E	B	S_2, S_3	T
102	59	30	15	2	0	-1	101*	A	S_1	
103	61	16	4	4	0	-3	E	A	S_1	T, MR
104	61	21	7	3	0	-2	N	A	S_1	T, MR
105	61	25	10	2	5	1	E	D	S_{11}	T
106	61	36	21	1	15	10	105*	D	S_{11}	
107	61	40	26	1	14	7	N,104*	D	S_{11}	MR
108	63	31	15	2	1	-1	E	B	S_2, S_3, S_8	MR
109	63	32	16	2	0	-1	108*	A	S_9	MR
110	64	28	12	2	4	0	E	D	S_7	T

Table 3.1 (continued-3)

No.	$v = b$	$r = k$	λ	n	m	t	I/N/E	Formula	Series	Reference
111	64	36	20	1	16	12	110*	D	S_6	
112	66	26	10	2	6	2	E	D	S_{11}	T, MR
113	66	40	24	1	16	10	112*	D	S_{11}	MR
114	67	12	2	6	0	-5	N	A	S_1	T
115	67	22	7	3	1	-2	N	B	S_2	T
116	67	33	16	2	1	-1	E	B	S_2, S_3	MR
117	67	34	17	2	0	-1	116*	A	S_1	MR
118	67	45	30	1	15	7	N,115*	D	S_{11}	
119	69	17	4	4	1	-3	E	B	S_2	T, MR
120	70	24	8	3	0	-2	E	A	S_1	MR
121	70	46	30	1	16	8	120*	D	S_{11}	MR
122	71	15	3	5	0	-4	E	A	S_1	T
123	71	21	6	3	3	-1	E	D	S_{11}	T, MR
124	71	35	17	2	1	-1	E	B	S_2, S_3	MR
125	71	36	18	2	0	-1	124*	A	S_1	MR
126	73	9	1	9	0	-8	E	A	S_1, S_4	R, T
127	75	37	18	2	1	-1	E	B	S_2, S_3	MR
128	75	38	19	2	0	-1	127*	A	S_1	MR
129	76	25	8	3	1	-2	N	B	S_2	T
130	77	20	5	4	0	-3	N	A	S_1	T
131	78	22	6	3	4	0	E	D	S_{11}	T, MR
132	79	13	2	6	1	-5	E	B	S_2	T, MR
133	79	27	9	3	0	-2	E	A	S_1	MR
134	79	39	19	2	1	-1	E	B	S_2, S_3	MR
135	79	40	20	2	0	-1	134*	A	S_1	MR
136	81	16	3	5	1	-4	–	B	S_2	T, MR
137	83	41	20	2	1	-1	E	B	S_2, S_3	MR
138	83	42	21	2	0	-1	137*	A	S_1	MR
139	85	21	5	4	1	-3	E	B	S_2, S_5	T, MR
140	85	28	9	3	1	-2	–	B	S_2	T, MR
141	85	36	15	2	6	1	–	D	S_{11}	MR
142	85	49	28	1	21	15	–,141*	D	S_{11}	MR
143	86	35	14	2	7	2	N	D	S_{11}	
144	86	51	30	1	21	14	N,143*	D	S_{11}	
145	87	43	21	2	1	-1	–	B	S_2, S_3	
146	87	44	22	2	0	-1	–,145*	A	S_1	
147	88	30	10	3	0	-2	N	A	S_1	T

Table 3.1 (continued-4)

No.	$v=b$	$r=k$	λ	n	m	t	I/N/E	Formula	Series	Reference
148	89	33	12	2	9	5	N	D	S_{11}	MR
149	91	10	1	10	0	−9	E	A	S_1, S_4	R, T
150	91	36	14	2	8	3	N	D	S_{11}	MR
151	91	45	22	2	1	−1	–	B	S_2, S_3	
152	91	46	23	2	0	−1	–,151*	A	S_1	
153	92	14	2	7	0	−6	N	A	S_1	T
154	93	24	6	4	0	−3	N	A	S_1	R
155	94	31	10	3	1	−2	N	B	S_2	MR
156	95	47	23	2	1	−1	–	B	S_2, S_3	
157	95	48	24	2	0	−1	–,156*	A	S_1	
158	96	20	4	5	0	−4	E	A	S_1, S_{10}	T
159	97	33	11	3	0	−2	–	A	S_1	MR
160	99	49	24	2	1	−1	–	B	S_2, S_3, S_8	
161	99	50	25	2	0	−1	–,160*	A	S_9	
162	100	45	20	2	5	0	–	D	S_7	
163	100	55	30	1	25	20	–,162*	D	S_6	
164	101	25	6	4	1	−3	E	B	S_2	MR
165	103	18	3	6	0	−5	N	A	S_1	R
166	103	34	11	3	1	−2	–	D	S_{11}	MR
167	103	51	25	2	1	−1	–	B	S_2, S_3	
168	103	52	26	2	0	−1	–,167*	A	S_1	
169	105	40	15	2	10	5	E	D	S_{11}	MR
170	106	15	2	7	1	−6	N	B	S_2	MR
171	106	21	4	5	1	−4	N	B	S_2	MR
172	106	36	12	3	0	−2	N	A	S_1	MR
173	107	53	26	2	1	−1	–	B	S_2, S_3	
174	107	54	27	2	0	−1	–,173*	A	S_1	
175	109	28	7	4	0	−3	E	A	S_1	MR
176	111	11	1	11	0	−10	N	A	S_1, S_4	MR
177	111	45	18	2	9	3	–	D	S_{11}	
178	111	55	27	2	1	−1	–	B	S_2, S_3	

Remark 3.1. The irreducible designs where $v = 32$ are alone included. The irreducible design for higher values of v can also be included if need be. All the complementary designs are not included in the above table, which are inherently in the table. But some of them are mentioned with the suitable

formula of b and the suitable series. In fact, all these designs have been obtained from the new series S_{11}. For complete treatment on complements of designs see the forth-coming sequel of this paper.

Remark 3.2. There are 22 designs marked as $-$ (unknown) in the column 8. Among these designs a few might have been solved by giving plan of the design, but for a majority of these designs, solutions are still unknown and open for further research. And there are 102 designs, which are not enlisted in any one of R, T and C, but emerged out from the series S_{11}. Furthermore, among these 102 designs there are 34 designs, which are there in our parameteric range excluding the irreducible and complementary designs.

Remark 3.3. Some of the SBIB designs have been constructed from BIB designs with $\lambda = 1$, and all these SBIB designs are also useful in the construction of affine μ-resolvable BIB designs.

Remark 3.4. There exist no parameters of an SBIB design, which escape this classification, and this table is more useful than the earlier ones as we know the method of obtaining the parameters easily. This problem will be tackled further in a sequel to this paper to appear shortly. Some of the constructions are also given in that paper.

4. Computer programming for the SBIB designs in C++

Now we propose to construct parameters of all possible SBIB designs within the considered parametric ranges, by using a computer program. Thus, it gave the exhaustive list all designs in the present limitations and hence Remark 3.4 above.

```
#include<stdio.h>
#include<alloc.h>
main()
{
FILE *fp;
int n,c=0, *vect[300],numb=0,m,l,v1,v2,r,i,j,u=0,k,p,q,t;
float v,b;
clrscr();
fp=fopen("vinod.c","w");
fprintf(fp," v=b r=k l n m t\n");
fprintf(fp,"..............\n");
```

```
printf("     \n");
for(n=1;n<=11;n++)
{
for(m=0;m<=25;m++)
{
for(l=1;l<=30;l++)
{
v1=(((n*n)*l)+n*(2*m-1)+1);
v2=(m*(m-1))/l;
v=v1+v2;
b=v;
r=n*l+m;
t=(m*(m-1) - (l*(n-1)))/l;
k=r;
p=k/l;
q=k%l;
if ((v>= 3)&&(v<= 111)&&(r>= 2)&&(r<= 55)&&(k<v)&&(p==n)
&&(m==q)&&(l*(v-1)==r*(r-1)))
{
vect[numb]=(int *)malloc(6*sizeof(int));
*(vect[numb]+0)=(int)v;
*(vect[numb]+1)=(int)r;
*(vect[numb]+2)=l;
*(vect[numb]+3)=n;
*(vect[numb]+4)=m;
*(vect[numb]+5)=t;
numb++ ;
c++;
}
}
}
}
printf("%d\n",c);
for(i=0;i<numb-1;i++)
{
for(j=0;j<numb-1-i;j++)
{
if(*(vect[j]+0) >* (vect[j+1]+0))
```

```
{
swap((vect[j]+0),(vect[j+1]+0));
swap((vect[j]+1),(vect[j+1]+1));
swap((vect[j]+2),(vect[j+1]+2));
swap((vect[j]+3),(vect[j+1]+3));
swap((vect[j]+4),(vect[j+1]+4));
swap((vect[j]+5),(vect[j+1]+5));
}
}
}
for(i=0;i<numb;i++)
{
for(j=0;j<6;j++)
{
fprintf(fp,"%5d",*(vect[i]+j));
printf("%5d",*(vect[i]+j));
}
fprintf(fp,"\n");
printf("\n");
u++;
if(u==20)
{
print("press any key to continue ...\n");
getch();
clrscr();
u=0;
}
}
getch();
}
swap(int *q,int *p)
{
int t;
t=*q;
*q=*p;
*p=t;
}_
```

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