Discussiones Mathematicae
Probability and Statistics 21 (2001) 63–75

SELECTION THEOREMS FOR STOCHASTIC SET-VALUED INTEGRALS

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Abstract

Some special selections theorems for stochastic set-valued integrals with respect to the Lebesgue measure are given.

Keywords: stochastic set-valued integrals, nonanticipated stochastic processes, diagonal convexity, selections.

2000 Mathematics Subject Classiffication: 93E03, 93C30.

1. Introduction

There are many works on the stochastic optimal control theory dealing with dynamical systems described by integral stochastic equations depending on random control parameters. Although the deterministic optimal control theory is extensively developed in connection with set-valued analysis, the multivalued approach in the stochastic case is still not very popular. The results of the paper deal with stochastic set-valued integrals with respect to the Lebesgue measure. Such integrals were, among other, considered in the author's paper [1]. Selection theorems presented in Section 3 of the paper apear in connection with investigations into the weak compactnes of solutions sets of stochastic differential inclusions [3]. Similarly as in [1], throughout the paper we consider stochastic set-valued integrals defined by Aumann's procedure for nonanticipative set-valued mappings that are assumed to be p-intergrable bounded. They are defined on the space $[0,T]\times\Omega$, where $(\Omega,\mathcal{F},P)$ is a given complete probability space with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ of sub-σ-algebras of $\mathcal{F}$. In what follows, we shall denote this system by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and call it a filtred probability space. Throughout the paper, we will assume that the following usual hypotheses are satisfied:

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$,

(ii) $\mathcal{F}_t = \bigcap\limits_{u>t} \mathcal{F}_u$ for $t \in [0,T]$, i.e., the filtration $(\mathcal{F}_t)_{0\le t\le T}$ is right continuous.

As usual, we shall consider $[0,T]\times\Omega$ as a measurable space with the product σ-algebra $\beta_T \otimes \mathcal{F}$, where β_T is the Borel σ-algebra of subsets of $[0,T]$. An n-dimensional stochastic process, understood as a function $x : [0,T]\times\Omega \to \mathbb{R}^n$, with $(\mathcal{F}, \beta(\mathbb{R}^n))$-measurable sections x_t, for each $t \in [0,T]$ is denoted by $(x_t)_{0\le t\le T}$. We shall assume that it is measurable, with respect to the product σ-algebra $\beta_T \otimes \mathcal{F}$ and such that each section x_t is $(\mathcal{F}_t, \beta(\mathbb{R}^n))$-measurable. Such processes are said to be $\mathcal{F}_t$-nonanticipative. In what follows, we shall also consider stochastic processes with values in the space $\mathbb{R}^{n\times m}$ of all $n\times m$-type matrices. Such processes will be denoted as $n\times m$-dimensional ones. In particular, an n-dimensional process $x = (x_t)_{0\le t\le T}$, can also be denoted as $1\times n$-dimensional one. We shall consider $\mathbb{R}^{n\times m}$ as a normed space with the norm $\|\cdot\|$ defined by $\|a\| = (\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2)^{1/2}$ for $a = (a_{ij})_{n\times m}$, with $a_{ij} \in \mathbb{R}$ for $i = 1, ..., n$, $j = 1, ..., m$.

Let $\mathcal{M}^p_{n\times m}(\mathcal{F}_t)$ denote the family of all (equivalence classes of) $n\times m$-dimensional $\mathcal{F}_t$- nonanticipative stochastic processes $(f_t)_{0\le t\le T}$ such that $\int_0^T \|f_t\|^p dt < \infty$, P-a.s. For $p \ge 1$, we denote

$$\mathcal{L}^p_{n\times m}(\mathcal{F}_t) = \left\{ (f_t)_{0\le t\le T} \in \mathcal{M}^p_{n\times m}(\mathcal{F}_t) : E\int_0^T \|f_t\|^p dt < \infty \right\}.$$

We shall consider $\mathcal{L}^p_{n\times m}(\mathcal{F}_t)$ as a Banach space with its usual norm $\|f\|_p$ defined by

$$\|f\|_p = \left(E\int_0^T \|f_t\|^p dt \right)^{1/p} \quad \text{for } f = (f_t)_{0\le t\le T} \in \mathcal{L}^p_{n\times m}(\mathcal{F}_t) \text{ with } p \ge 1.$$

Given $f = (f_t)_{0\le t\le T} \in \mathcal{L}^p(\mathcal{F}_t)$ denote by $\int_0^T f_t dt$ a stochastic integrals of f with respect to the Lebesgue measure dt on β_T. It is defined in the usual way as the Lebesgue integral $\int_0^T f_t(\omega) dt$ for fixed $\omega \in \Omega$ and is understood as $\mathcal{F}$-measurable $n\times m$-dimensional random variable. In particural, we can also define, an $n\times m$-dimensional $\mathcal{F}_t$-nonanticipative stochastic process

$(\int_0^t f_\tau d\tau)_{0\le t\le T}$. It is easy to see that $\int_0^T f_t dt \in L^p(\Omega,\mathcal{F},P,\mathbb{R}^{n\times m})$. In what follows, the space $L^p(\Omega,\mathcal{F},P,\mathbb{R}^{n\times m})$ will be denoted by $L^p_{n\times m}(\Omega,\mathcal{F},P)$.

We shall also consider $\mathcal{F}_t$-nonanticipative set-valued processes defined as multifunctions $\mathfrak{F}:[0,T]\times\Omega\to Cl(\mathbb{R}^{n\times m})$ that are $\beta_T\otimes\mathcal{F}$-measurable and such that each section $\mathfrak{F}_t$ is $\mathcal{F}_t$-measurable for $t\in[0,T]$. Recall that for a given measurable space $(\mathcal{X},\Sigma)$ and a separable metric space (X,ρ) a multifunction $H:\mathcal{X}\to Cl(X)$, where $Cl(X)$ denotes the family of all nonempty closed subsets of X, is said to be Σ-measurable if for every $A\in Cl(X)$ one has $H^-(A):=\{z\in\mathcal{X}: H(z)\cap A\neq\emptyset\}\in\Sigma$.

In what follows, by $\mathcal{L}^p_{s-v}(\mathcal{F}_t,\mathbb{R}^{n\times m})$ we shall denote the family of all (equivalence classes of) all $\mathcal{F}_t$- nonanticipative set-valued processes $\mathfrak{F}$: $[0,T]\times\Omega\to Cl(\mathbb{R}^{n\times m})$ such that $E\int_0^T \| \mathfrak{F}_t\|^p dt<\infty$, $p\ge 1$, where $\|\mathfrak{F}_t\| = \sup\{|f_t| : f_t\in\mathfrak{F}_t\}$ and $\mathfrak{F}=(\mathfrak{F}_t)_{0\le t\le T}$.

Let us observe that $\mathcal{F}_t$-nonanticipativity of stochastic processes or set-valued stochastic processes can be equivalently defined as Σ-measurability, where Σ-algebra on $[0,T]\times\Omega$ is defined by $\Sigma=\{Z\in\beta_T\otimes\mathcal{F}: Z_t\in\mathcal{F}_t$ for $t\in[0,T]\}$, where Z_t denotes the t-section of a set Z.

Hence, in particular, by the Kuratowski and Ryll-Nardzewski measurable selection theorem ([2], Theorem II 3.10) it follows that for a given $\mathfrak{F}\in\mathcal{L}^p_{s-v}(\mathcal{F}_t,\mathbb{R}^{m\times m})$ the set $S^p(\mathfrak{F})$ of all $f\in\mathcal{L}^p_{n\times m}(\mathcal{F}_t)$ such that $f_t(\omega)\in\mathfrak{F}_t(\omega)$ for a.e. $(t,\omega)\in[0,T]\times\Omega$ is nonempty. For such $\mathfrak{F}\in\mathcal{L}^p_{s-v}(\mathcal{F}_t,\mathbb{R}^{n\times m})$ we define a set-valued stochastic integral $\int_0^T\mathfrak{F}_t dt$ by setting $\int_0^T\mathfrak{F}_t dt=\{\int_0^T f_\tau dt : (f_\tau)_{0\le t\le T}\in S^p(\mathfrak{F})\}$. We can also define $\int_s^t\mathfrak{F}_\tau d\tau$ for fixed $0\le s<t\le T$ by taking $\int_0^T \mathbb{I}_{[s,t]}(\tau)\mathfrak{F}_\tau d\tau$, where $\mathbb{I}_{[s,t]}$ denotes the characteristic function for $[s,t]$. It is easy to see that $\int_s^t\mathfrak{F}_\tau d\tau\subset L^p_{n\times m}(\Omega,\mathcal{F},P)$ for fixed $0\le s<t\le T$.

2. Set-valued stochastic processes with convex and diagonally convex values

Given filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$ satisfying usual hypotheses we shall consider set valued stochastic processes $\mathfrak{F},\mathcal{G}\in\mathcal{L}^P_{s-v}$ $(\mathcal{F}_t,\mathbb{R}^{n\times m})$ such that $\mathfrak{F}_t(\omega)$ and $D(\mathcal{G}_t)(\omega)$ are convex subsets of $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$, respectively, where $D(\mathcal{G}_t)(\omega)=\{u\cdot u^T : u\in\mathcal{G}_t(\omega)\}$ for fixed $t\in[0,T]$ and $\omega\in\Omega$, where u^T denotes the transposition of u. In what follows, we call $\mathfrak{F}$ as a set-valued stochastic process with convex values, whereas $\mathcal{G}$ are said to be a stochastic set-valued process with diagonally convex values.

We have the following properties of stochastic multiprocesses $D(\mathcal{G})$.

Proposition 1. *Let* $\mathcal{G} \in \mathcal{L}^{2p}_{s-v}(\mathcal{F}_t, \mathrm{IR}^{n\times m})$, $p \geq 1$. *Then* $D(\mathcal{G}) \in \mathcal{L}^{p}_{s-v}$ $(\mathcal{F}_t, \mathrm{IR}^{n\times n})$.

Proof. Let $l : \mathrm{IR}^{n\times m} \to \mathrm{IR}^{n\times n}$ be defined by $l(u) = u \cdot u^T$ for $u \in \mathrm{IR}^{n\times m}$. We have $D(\mathcal{G}_t)(\omega) = l(\mathcal{G}_t(\omega))$ for $t \in [0,T]$ and $\omega \in \Omega$. Therefore, by the continuity of l, a set-valued mapping $D(\mathcal{G})$ is $\mathcal{F}_t$-nonanticipative (equivalenty Σ-measurable). Furthermore, for every $v_t \in D(\mathcal{G}_t)$, $P.-a.s.$ and $t \in [0,T]$ we have $\|v_t\|^p \leq (\|u_t\|^p \cdot \|u_t^T\|^p) \leq m_t^{2p}$, $P-a.s$ for $t \in [0,T]$, where $m_t = \|\mathcal{G}_t\|$. We have of course $(m_t)_{0\leq t\leq T} \in L^{2p}([0,T]\times\Omega, \mathrm{IR})$. ∎

Proposition 2. *For every* $\mathcal{G} \in \mathcal{L}^{2p}(\mathcal{F}_t, \mathrm{IR}^{n\times m})$ *one has* $S^p(D(\mathcal{G})) = D(S^{2p}(\mathcal{G}))$, *where* $D(S^{2p}(\mathcal{G})) = \{g \cdot g^T : g \in S^{2p}(\mathcal{G})\}$.

Proof. Let $v \in D(S^{2p}(\mathcal{G}))$. Then there exists $g \in S^{2p}(\mathcal{G})$ such that $v = g \cdot g^T$, or equivalenty $v_t(\omega) = g_t(\omega) \cdot g_t^T(\omega)$ a.e., for $(t,\omega) \in [0,T]\times\Omega$. Therefore $v_t(\omega) \in D(\mathcal{G}_t)(\omega)$ for a.e., (t,ω) and $v \in S^p(D(\mathcal{G}))$.

Conversely, if $v \in S^p(D(\mathcal{G}))$, then $v_t(\omega) \in l(\mathcal{G}_t(\omega))$ a.e., where l is such as in Proposition 1. Therefore, by ([2], Theorem II 3.12) there is $g \in S^{2p}(\mathcal{G})$ such that $v = g \cdot g^T \in D(S^{2p}(\mathcal{G}))$. ∎

Proposition 3. *Let* $\mathcal{G} \in \mathcal{L}^{p}_{s-v}(\mathcal{F}_t, \mathrm{IR}^{n\times m})$ *be convex valued. Then for fixed* $(t,\omega) \in [0,T]\times\Omega$, *every* $u, v \in \mathcal{G}_t(\omega)$ *and every* $\lambda \in [0,1]$ *there is a* $x_\lambda \in \mathcal{G}_t(\omega)$ *such that*

$$\lambda\|u\|^2 + (1-\lambda)\|v\|^2 = \|x_\lambda\|^2. \tag{1}$$

Proof. Let $(t,\omega) \in [0,T\times\Omega$ be fixed and let $u,v \in \mathcal{G}_t(\omega)$. If $\|u\| = \|v\|$ then for every $\lambda \in [0,T]$ we have $\lambda\|u\|^2 + (1-\lambda)\|v\|^2 = \|u\|^2$. Taking $z_\lambda = u$ we see that (1) is satisfied.

Assume $0 < \|u\| < \|v\|$ and let $z_\lambda = \lambda\|u\|^2 + (1-\lambda)\|v\|^2$ for fixed $\lambda \in [0,T]$. We have $\|u\|^2 < z_\lambda < \|v\|^2$. Taking now $r_\lambda = \sqrt{z_\lambda}$ we obtain $B_{\|u\|} \subset B_{r_\lambda} \subset B_{\|u\|})$ where $B_r = \{z \in \mathrm{IR}^{n\times m} : \|z\| \leq r\}$ for $r > 0$. Furthermore, we have $u \in \partial B_{\|u\|}$ and $v \in \partial B_{\|v\|}$, where $\partial B_r = \{z \in \mathrm{IR}^{n\times m} : \|z\| = r\}$ for $r > 0$. Denoting $l_{uv} = \{\lambda u + (1-\lambda)v : 0 \leq \lambda \leq 1\}$ we obtain $l_{uv} \cap \partial B_{r_\lambda} \neq \emptyset$ for every $\lambda \in [0,1]$. By the convexity of $\mathcal{G}_t(\omega)$ we also have $l_{uv} \cap \partial B_{r_\lambda} \subset \mathcal{G}_t(\omega)$. Therefore there is $x_\lambda \in \mathcal{G}_t(\omega)$ such that $\|x_\lambda\| = r_\lambda$. Thus (1) is satisfied. Similarly, we can see that also in the cases $0 < \|v\| < \|u\|$ and $\min(\|u\|,\|v\|) = 0$ there is $x_\lambda \in \mathcal{G}_t(\omega)$ such that (1) is satisfied. ∎

Immediately from Proposition 3 we obtain the following results.

Proposition 4. *If $\mathcal{G} \in \mathcal{L}^p_{s-v}(\mathcal{F}_t, \mathbb{R}^{1\times m})$ is convex valued then $\mathcal{G}$ is also diagonally convex valued.*

Proof. Let us observe that for every $u \in \mathcal{G}$ one has $u \cdot u^T = \|u\|^2$. Therefore for every $z_1, z_2 \in D(\mathcal{G}_t)(\omega)$ and $\lambda \in [0,T]$ there are $u, v \in \mathcal{G}_t(\omega)$ such that $\lambda z_1 + (1-\lambda)z_2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2$. By virtue of Proposition 3, for every $\lambda \in [0,1]$ there is $x_\lambda \in \mathcal{G}_t(\omega)$ such that $\lambda z_1 + (1-\lambda)z_2 = \|x_\lambda\|^2 = x_\lambda \cdot x_\lambda^T \in D(\mathcal{G}_t)(\omega)$.

■

3. Selection theorems

Assume we are given filtred probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\mathcal{F}_t)_{0\le t\le T}\widetilde{P})$ satisfying the usual hypotheses. We shall consider in this Section, point-valued and set-valued stochastic processes on one of the above filtered probability spaces. All set-valued processes are convex or diagonally convex valued. This assumption implies ([1], Proposition 2) that sets of all their nonanticipative selectors are convex.

We begin with the following basic selection theorem.

Theorem 1. *Let $\mathfrak{F} \in \mathcal{L}^1_{s-v}(\mathcal{F}_t, \mathbb{R}^{1\times n})$ and $\mathcal{G} \in \mathcal{L}^4_{s-v}(\mathcal{F}_t, \mathbb{R}^{n\times m})$ be convex- and diagonally convex-valued, respectively and let $(\varphi_t)_{0\le t\le T} \in \mathcal{L}^1_{1\times n}(\widetilde{\mathcal{F}_t})$ and $(\psi_t)_{0\le t\le T} \in \mathcal{L}^4_{n\times m}(\widetilde{\mathcal{F}_t})$. Assume that for every $0 \le s < t \le T$ the following conditions are satisfied*

$$\widetilde{E}\int_s^t \varphi_\tau d\tau \in E\int_s^t \mathfrak{F}_\tau d\tau \tag{2}$$

and

$$\widetilde{E}\int_s^t \psi_\tau \cdot \psi_\tau^T d\tau \in E\int_s^t D(\mathcal{G}_\tau) d\tau, \tag{3}$$

where E and $\widetilde{E}$ denote the mean value operators with respect to P and $\widetilde{P}$, respectlively. Then there are $f \in S^1(\mathfrak{F})$ and $g \in S^4(\mathcal{G})$ such that $\widetilde{E}\int_s^t \varphi_\tau d\tau = E\int_s^t f_\tau d\tau$ and $\widetilde{E}\int_s^t \psi_\tau \cdot \psi_\tau^T d\tau = E\int_s^t g_\tau \cdot g_\tau^T d\tau$ for every $0 \le s < t \le T$.

Proof. Assume that (2) is satisfied and let $\varepsilon > 0$. Select $\delta_\varepsilon \in (0,1]$ such that $\sup_{0\le t\le T} \widetilde{E}\int_t^{t+\delta_\varepsilon} \|\varphi_\tau\| d\tau < \varepsilon/4$ and $\sup_{0\le t\le T} E\int_t^{t+\delta_\varepsilon} \| \mathfrak{F}_\tau \| d\tau < \varepsilon/4$. Let $\tau_0^\varepsilon = 0$ and $\tau_k^\varepsilon = k \cdot \delta_\varepsilon$ for $k = 1, ..., N_\varepsilon$, where N_ε is such that $(N_\varepsilon - 1)\delta_\varepsilon < T \le N_\varepsilon \delta_\varepsilon$. By that definition of set-valued integrals $\int_s^t \mathfrak{F}_\tau d\tau$

and $E(\int_s^t \mathfrak{F}_\tau d\tau)$ (see [2], pp. 49–57) we have $E(\int_s^t \mathfrak{F}_\tau d\tau) = \{E(\int_s^t f_\tau d\tau) : (f_t)_{0\le t\le T} \in S^1(\mathfrak{F})\}$. Therefore, by virtue of (2), for every $k = 1, ..., N_\varepsilon$ there is $f^k \in S^1(\mathfrak{F})$ such that

$$\widetilde{E}\int_{\tau_{k-1}^\varepsilon}^{\tau_k^\varepsilon} \varphi_\tau d\tau = E\int_{\tau_{k-1}^\varepsilon}^{\tau_k^\varepsilon} f_\tau^k d\tau$$

Define $f^\varepsilon = 1\!\!1_{[0,\tau_1^\varepsilon]} f^1 + ... + 1\!\!1_{(\tau_{N_\varepsilon-2}^\varepsilon, \tau_{N_\varepsilon-1}^\varepsilon]} f^{N_\varepsilon-1} + 1\!\!1_{(\tau_{N_\varepsilon-1}^\varepsilon, T]} f^{N_\varepsilon}$. It is clear that $f^\varepsilon \in S^1(\mathfrak{F})$ because $S^1(\mathfrak{F})$ is decomposable (see [6], p. 50). Now, for every fixed $0 \le s < t \le T$ there are positive integer numbers $1 \le r < l \le N_\varepsilon$ such that $s \in (\tau_{r-1}, \tau_r]$ and $t \in (\tau_{l-1}, \tau_l]$ or $s, t \in (\tau_{r-1}, \tau_r]$ or $s, t \in (\tau_{l-1}, \tau_l]$. Therefore, we have

$$\begin{aligned}
&\left|\widetilde{E}\int_s^t \varphi_\tau d\tau - E\int_s^t f_\tau^\varepsilon d\tau\right| \le \\
&\le \left|\widetilde{E}\int_s^{\tau_r^\varepsilon} \varphi_\tau d\tau - E\int_s^{\tau_r^\varepsilon} f_\tau d\tau\right| + \sum_{i=r+1}^{l-1}\left|\widetilde{E}\int_{\tau_{i-1}^\varepsilon}^{\tau_i^\varepsilon} \varphi_\tau d\tau - \right. \\
&\left. - E\int_{\tau_{i-1}^\varepsilon}^{\tau_i^\varepsilon} f_\tau^i d\tau\right| + \left|\widetilde{E}\textstyle\int_{\tau_{l-1}^\varepsilon}^t \varphi_\tau d\tau - E\int_{\tau_{l-1}^\varepsilon}^t f_\tau^l d\tau\right| \le \\
&\le \widetilde{E}\int_s^{\tau_r^\varepsilon} \|\varphi_\tau\| d\tau + \widetilde{E}\int_{\tau_{l-1}^\varepsilon}^t \|\varphi_\tau\| d\tau + \\
&+ E\int_s^{\tau_r^\varepsilon} |\mathfrak{F}_\tau|\, d\tau + E\int_{\tau_{l-1}^\varepsilon}^t |\mathfrak{F}_\tau|\, d\tau \le \varepsilon
\end{aligned}$$

or

$$\begin{aligned}
&\left|\widetilde{E}\int_s^t \varphi_\tau d\tau - E\int_s^t f_\tau^\varepsilon d\tau\right| \le \\
&\le \widetilde{E}\int_s^t \|\varphi_\tau\| d\tau + E\int_s^t |\mathfrak{F}_\tau|\, d\tau \le \varepsilon.
\end{aligned}$$

Select now a sequence $(\varepsilon_n)_{n=1}^\infty$ of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Taking to every $\varepsilon_n > 0$ a-selector $f^{\varepsilon_n} \in S^1(\mathfrak{F})$ defined above, we obtain a sequence $(f^{\varepsilon_n})_{n=1}^\infty$ of elements of $S^1(\mathfrak{F})$ such that $\widetilde{E}\int_s^t \varphi_\tau d\tau = \lim_{n\to\infty} E\int_s^t f_\tau^{\varepsilon_n} d\tau$ for every $0 \le s < t \le T$. It is easy to verify ([1], Proposition 2) that $S^1(\mathfrak{F})$ is a weakly compact subset of $\mathcal{L}^1_{1\times x}(\mathcal{F}_t)$. Therefore there are $f \in S(\mathfrak{F})$ and a subsecquence $(\varepsilon_{n_k})_{k=1}^{\infty}$ of $(\varepsilon_n)_{n=1}^{\infty}$ such that $E\int_s^t f_\tau d\tau = \lim_{n\to\infty} E\int_s^t f_\tau^{\varepsilon_{n_k}} d\tau$ for $0 \le s < t \le T$. Hence, and from the properties of a sequence $(f^{\varepsilon_{n_k}})_{k=1}^{\infty}$ it follows that $\widetilde{E}\int_s^t \varphi_\tau d\tau = E\int_s^t f_\tau d\tau$ for every $0 \le s < t \le T$.

In a similar way we can also prove the existence of $\sigma \in S^2(D(\mathcal{G}))$ such that $\widetilde{E}\int_s^t \psi_\tau \psi_\tau^T d\tau = E\int_s^t \sigma_\tau d\tau$, for every $0 \le s < t \le T$. But, by the definition of $D(\mathcal{G})$ we obtain that $\sigma_t(\omega) \in l(\mathcal{G}_t(\omega))$ for $(t,\omega) \in [0,T]$, where $l(u) = u \cdot u^T$ for $u \in \mathbb{R}^{n\times m}$. Hence and the continuity of l (by ([2], Theorem II 3.12) it follows the existence of $g \in S^4(\mathcal{G})$ such that $\sigma = l(g)$, i.e., that $\sigma = g \cdot g^T$. ■

Theorem 2. *Let (X,ρ) be a separable complete metric space and let $(\varphi_t(x))_{0\le t\le T} \in \mathcal{L}^1_{1\times m}(\widetilde{\mathcal{F}}_t), (\psi_t(x))_{0\le t\le T} \in \mathcal{L}^4_{n\times m}(\widetilde{\mathcal{F}}_t), (\Phi_t(x))_{0\le t\le T} \in \mathcal{L}^1_{s-v}(\mathcal{F}_t, \mathbb{R}^{1\times n})$ and $(\Psi_t(x))_{0\le t\le T} \in \mathcal{L}^4_{s-v}(\mathcal{F}_t, \mathbb{R}^{n\times m})$, for every $x \in X$, be such that*

(i) *$\varphi_t(\cdot)(\widetilde{\omega})$ and $\psi_t(\cdot)(\widetilde{\omega})$ are continuous on X for fixed $t \in [0,T]$ and $\widetilde{\omega} \in \widetilde{\Omega}$,*

(ii) *$\Phi_t(\cdot)(\omega)$ and $\Psi_t(\cdot)(\omega)$ are lower semicontinuous on X for fixed $t \in [0,T]$ and $\omega \in \Omega$,*

(iii) *$(\Phi_t(x))_{0\le t\le T}$ and $\Psi_t(x))_{0\le t\le T}$ are convex and diagonally convex valued, respectively for fixed $x \in X$.*

If furthermore, $\widetilde{E}\int_s^t \varphi_\tau(x)d\tau \in E\int_s^t \Phi_\tau(x)d\tau$ and $\widetilde{E}\int_s^t \psi_\tau(x)\cdot\psi_\tau^T(x)d\tau \in E\int_s^t D(\Psi_\tau(x))d\tau$ for every $0 \le s < t \le T$ and $x \in X$, then for every $x \in X$ there are $f(x) \in S^1(\Phi(x))$ and $g(x) \in S^4(\Psi(x))$ such that $f_t(\cdot)(\omega)$ and $g_t(\cdot)(\omega)\cdot g_t^T(\cdot)(\omega)$ are continuos on X for fixed $t \in [0,T]$ and $\omega \in \Omega$ and $\widetilde{E}\int_s^t \varphi_\tau(x)d\tau = E\int_s^t f_\tau(x)d\tau$ and $\widetilde{E}\int_s^t \psi_\tau(x)\cdot\psi_\tau^T(x)d\tau = E\int_s^t g_\tau(x)\cdot g_\tau^T(x)d\tau$ for every $0 \le s < t \le T$ and $x \in X$.

Proof. Let $\mathcal{D}_X = \{x_1, x_2, ..., x_n, ...\}$ be a countable subset of X such that $\overline{\mathcal{D}}_X = X$. By virtue of Theorem 1, for every $i = 1, 2, ...$ there are $f^i \in S^1(\Phi(x_i))$ and $g^i \in S^4(\Psi(x_i))$ such that $E\int_s^t f_\tau^i d\tau = \widetilde{E}\int_s^t \varphi_\tau(x_i)d\tau$ and $E\int_s^t g_\tau^i \cdot (g_\tau^i)^T d\tau = \widetilde{E}\int_s^t \psi_\tau(x_i)\cdot\psi_\tau^T(x_i)d\tau$ for every $0 \le s < t \le T$. Let us define now multifunctions $\mathfrak{F}_\Phi^i$ and $\mathcal{G}_\Psi^i$ by setting

$$\mathfrak{F}^i_\Phi(t,\omega,x) = \begin{cases} \Phi_t(x)(\omega) & \text{for} \quad x \neq x_i \\ f^i_\tau(\omega) & \text{for} \quad x = x_i \end{cases}$$

and

$$\mathcal{G}^i_\Psi(t,\omega,x) = \begin{cases} \Psi_t(x)(\omega) & \text{for} \quad x \neq x_i \\ g^i_\tau(\omega) & \text{for} \quad x = x_i \end{cases}$$

It is clear that $\mathfrak{F}^i_\Phi$ and $\mathcal{G}^i_\Psi$ are convex and diagonally convex valued. Futhermore, $\mathfrak{F}^i_\Phi(\cdot,\cdot,x) \in \mathcal{L}^1_{s-v}(\mathcal{F}_t, \mathbb{R}^{1\times n})$ and $\mathcal{G}^i_\Psi(\cdot,\cdot,x) \in \mathcal{L}^4_{s-v}(\mathcal{F}_t, \mathbb{R}^{n\times m})$ for fixed $x \in X$. Finally, we can easily see that $\mathfrak{F}^i_\Phi(t,\omega,\cdot)$ and $\mathcal{G}^i_\Psi(t,\omega,\cdot)$ are lower semicontinuous for fixed $t \in [0,T]$ and $\omega \in \Omega$. Therefore, by ([4], Theorem 2) for every $i = 1,2,...$ there are $\sum \otimes \beta(X)$-measurable mappings $\varphi^i : [0,T] \times \Omega \times X \to \mathbb{R}^n$ and $\psi^i : [0,T] \times \Omega \times X \to \mathbb{R}^{n\times m}$ such that $\varphi^i(\cdot,\cdot,x) \in \mathcal{L}^1_{1\times n}(\mathcal{F}_t)$ and $\psi^i(\cdot,\cdot,x) \in \mathcal{L}^4_{n\times m}(\mathcal{F}_t)$ for fixed $x \in X$, $\varphi^i(t,\omega,\cdot)$ and $\psi^i(t,\omega,\cdot)$ are continous on X for fixed $t \in [0,T]$ and $\omega \in \Omega, \varphi^i(t,\omega,x) \in \Phi_t(x)(\omega), \psi^i(t,\omega,x) \in \Psi_t(x)(\omega), \varphi^i(t,\omega,x_i) = f^i_t(\omega)$ and $\psi^i(t,\omega,x_i) = g^i_\tau(\omega)$ for a.e. $(t,\omega) \in [0,T] \times \Omega$ and $i = 1,2,...$

Let $(U_i)_{i=1}^\infty$ be a countable open covering for (X,ρ) such that $x_i \in U_i$, each $i = 1,2,...$ and select a continuous locally finite partition of the unity $(P_i)_{i=1}^\infty$ subordinated to $(U_i)_{i=1}^\infty$. Define now processes $f(x) = (f_t(x))_{0\le t\le T}$ and $\sigma(x) = (\sigma_t)(x))_{0\le t\le T}$ by setting

$$f_t(x)(\omega) = \sum_{i=1}^\infty P_i(x)\varphi^i(t,\omega,x)$$

and

$$\sigma_t(x)(\omega) = \sum_{i=1}^\infty P_i(x) \cdot \psi^i(t,\omega,x)(\psi^i(t,\omega,x))^T$$

for $x \in X$ and $(t,\omega) \in [0,T] \times \Omega$. It is clear that $f(x) \in S^1(\Phi(x))$ and $\sigma(x) \in S^2(D(\Psi(x))$ for $x \in X$ because $(P_i)_{i=1}^\infty$ is locally finite and $\Phi_t(x)(\omega)$ and $D(\Psi(x)(\omega))$ are convex for $x \in X, t \in [0,T]$ and $\omega \in \Omega$. It is also clear that $f_t(\cdot)(\omega)$ and $\sigma_t(\cdot)(\omega)$ are continuous on X for fixed $t \in [0,T]$ and $x \in \Omega$.

Let $\Lambda_{st}(x) = \widetilde{E}\int_s^t \varphi_\tau(x)d\tau - E\int_s^t f_\tau(x)d\tau$ for $0 \le s < t \le T$ and $x \in X$. For every $x \in X$ and $0 \le s < t \le T$ we have

$$\Lambda_{st}(x) = \left(\sum_{i=1}^{\infty} P_i(x)\cdot \widetilde{E}\int_s^t \varphi_\tau(x)d\tau\right) - E\int_s^t f_\tau(x)d\tau$$

$$= \sum_{i=1}^{\infty} P_i(x)\left[\widetilde{E}\int_s^t \varphi_\tau(x)d\tau - E\int_s^t \varphi^i(\tau,\cdot,x)d\tau\right].$$

Furthermore, for every $x_i \in \mathcal{D}_X$ one has

$$\widetilde{E}\int_s^t \varphi_\tau(x_i)d\tau - E\int_s^t \varphi^i(\tau,\cdot,x_i)d\tau =$$

$$= \widetilde{E}\int_s^t \varphi_\tau(x_i)d\tau - E\int_s^t f_\tau^i d\tau = 0$$

for every $0 \le s < t \le T$. Therefore, $\Lambda_{st}(x_i) = 0$ for every $0 \le s < t \le T$ and $x_i \in \mathcal{D}_X$. Thus $\widetilde{E}\int_s^t \varphi_\tau(x_i) = E\int_s^t f_\tau(x_i)d\tau$ for $0 \le s < t \le T$ and $x_i \in \mathcal{D}_X$. Hence, by the continuity of $\widetilde{E}\int_s^t \varphi_\tau(\cdot)d\tau$ and $E\int_s^t f_\tau(\cdot)d\tau$ on X and the equality $X = \overline{\mathcal{D}}x$, we also get $\widetilde{E}\int_s^t \varphi_\tau(x)d\tau = E\int_s^t f_\tau(x)d\tau$ for every $0 \le s < t \le T$ and $x \in X$. In a similar way, we obtain

$$\widetilde{E}\int_s^t \psi_\tau(x)\cdot\psi_\tau^T(x)d\tau - E\int_s^t \sigma_\tau(x)d\tau =$$

$$= \sum_{i=1}^{\infty} P_i(x)\left[\widetilde{E}\int_s^t \psi_\tau(x)\cdot\psi_\tau^T(x)d\tau - E\int_s^t \psi^i(\tau,\cdot,x)\cdot(\psi^i(\tau,\cdot,x))^T d\tau\right]$$

for $0 \le s < t \le T, x \in X$ and

$$\widetilde{E}\int_s^t \psi_\tau(x_i)\cdot\psi_\tau^T(x_i)d\tau - E\int_s^t \psi^i(\tau,\cdot,x_i)\cdot(\psi^i(\tau,\cdot,x_i)^T d\tau =$$

$$= \widetilde{E}\int_s^t \psi_\tau(x_i)\cdot\psi_\tau^T(x_i)d\tau - E\int_s^t g_\tau^i\cdot(g_\tau^i)^T d\tau = 0$$

for every $0 \le s < t \le T, x_i \in \mathcal{D}_X$ and $i = 1, 2, ...$ Therefore

$$\widetilde{E}\int_s^t \psi_\tau(x_i)\cdot\psi_\tau^T(x_i)d\tau = E\int_s^t \sigma_\tau(x_i)d\tau$$

for $0 \le s < t \le T$ and $x_i \in \mathcal{D}_X$, which implies that $\widetilde{E}\int_s^t \psi_\tau(x)\cdot\psi_\tau^T(x)d\tau = E\int_s^t \sigma_\tau(x)d\tau$ for $0 \le s < t \le T$ and $x \in X$. Now, similarly as in the proof of Theorem 1, we can select, for every fixed $x \in X$ a selector $g(x) \in S^4(\Psi(x))$ such that $\sigma(x) = g(x)\cdot g^T(x)$. ■

Theorem 3. *Let (X, ρ) be a separable metric space and let $\varphi \in \mathcal{L}^1_{1-n}(\widetilde{\mathcal{F}_t}), \psi \in \mathcal{L}^4_{n\times m}(\widetilde{\mathcal{F}}_t), \mathfrak{F} \in \mathcal{L}^1_{s-v}(\mathcal{F}_t, \mathbb{R}^{1\times n})$ and $\mathcal{G} \in \mathcal{L}^4_{s-v}(\mathcal{F}_t, \mathbb{R}^{n\times m})$. Assume $\mathfrak{F}$ and $\mathcal{G}$ are convex and diagonally convex valued, respectively. Suppose $\Phi : X \times \mathbb{R}^n \to \mathbb{R}$ and $\Psi : X \times \mathbb{R}^{n\times n} \to \mathbb{R}$ are continuous and such that $\Phi(x, \cdot)$ and $\Psi(x, \cdot)$ are linear for fixed $x \in X$. If furthermore,*

$$\widetilde{E}\int_s^t \Phi(x, \varphi_\tau)d\tau \in E\int_s^t \Phi(x, \mathfrak{F}_\tau)d\tau \tag{4}$$

$$\widetilde{E}\int_s^t \Psi(x, \psi_\tau, \cdot\psi_\tau^T)d\tau \in E\int_s^t \Psi(x, D(\mathcal{G}_\tau))d\tau \tag{5}$$

for every $0 \le s < t \le T$ and $x \in X$, then there are $f \in S^1(\mathfrak{F})$ and $g \in S^4(\mathcal{G})$ such that $\widetilde{E}\int_s^t \Phi(x, \varphi_\tau)d\tau = E\int_s^t \Phi(x, f_\tau)d\tau$ and $\widetilde{E}\int_s^t \Psi(x, \psi_\tau, \cdot\psi_\tau^T)d\tau = E\int_s^t \Psi(x, g_\tau\cdot g_\tau^T)d\tau$ for every $0 \le s < t \le T$ and $x \in X$.

Proof. Let us observe that set-valued mappings $\Phi(\cdot, \mathfrak{F}_t(\omega))$ and $\Psi(\cdot, D(\mathcal{G}_t(\omega)))$ are continuous on X, convex valued for fixed $t \in [0, T]$ and $\omega \in \Omega$ and such that $\Phi(x, \mathfrak{F}) \in \mathcal{L}^1_{s-v}(\mathcal{F}_t, \mathbb{R})$ and $\Psi(x, D(\mathcal{G})) \in \mathcal{L}^2_{s-v}(\mathcal{F}_t, \mathbb{R})$ for fixed $x \in X$. Therefore, by Theorem 2, relations (4) and (5) imply the existence for every $x \in X$ selectors $\hat{\varphi}(x) \in S^1(\Phi(x, \mathfrak{F}))$ and $\hat{\sigma}(x) \in S^2(\Psi(x, D(\mathcal{G}))$ such that $\hat{\varphi}(\cdot)$ and $\hat{\sigma}(\cdot)$ are continuous and satisfy $\widetilde{E}\int_s^t \Phi(x, \varphi_\tau)d\tau = E\int_s^t \hat{\varphi}_\tau(x)d\tau$ and $\widetilde{E}\int_s^t \Psi(x, \psi_\tau\cdot\psi_\tau^T)d\tau = E\int_s^t \Psi(x, \hat{\sigma}(x))d\tau$ for every $0 \le s < t \le T$ and $x \in X$. Define now multifunctions $\mathcal{K}$ and Q by setting

$$\mathcal{K}_t(\omega) = \mathfrak{F}_t(\omega) \cap \left\{ u \in \mathbb{R}^n : \sup_{x \in X} \operatorname{dist}(\hat{\varphi}_t(x)(\omega), \Phi(x,u)) = 0 \right\}$$

and

$$Q_t(\omega) = D(\mathcal{G}_t(\omega)) \cap \left\{ v \in \mathbb{R}^{n \times m} : \sup_{x \in X} \operatorname{dist}(\hat{\psi}_t(x)(\omega), \Psi(x,v)) = 0 \right\}$$

for $t \in [0,T]$ and $\omega \in \Omega$. By the continuity of mappings $\operatorname{dist}(\hat{\varphi}_t(\cdot)(\omega), \Phi(\cdot,u))$ and $\operatorname{dist}(\hat{\psi}(\cdot)(\omega), \Psi(\cdot,v))$ for fixed $(t,\omega) \in [0,T] \times \Omega, u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n \times m}$ and the separability of the metric space (X,ρ) we have

$$\mathcal{K}_t(\omega) = \mathfrak{F}_t(\omega) \cap \left\{ u \in \mathbb{R}^n : \sup_{x \in \mathcal{D}_X} \operatorname{dist}(\hat{\varphi}_t(x)(\omega), \Phi(x,u)) = 0 \right\}$$

and

$$Q_t(\omega) = D(\mathcal{G}_t(\omega)) \cap \left\{ v \in \mathcal{Z}(\mathbb{R}^n, \mathbb{R}^n) : \sup_{x \in \mathcal{D}_X} \operatorname{dist}(\hat{\psi}_t(x)(\omega), \Psi(x,v)) = 0 \right\}$$

where $\mathcal{D}_X$ is a countable subset of X such that $\overline{\mathcal{D}}_X = X$. Furthermore, let us observe that the functions

$$[0,T] \times \Omega \ni (t,x) \to \operatorname{dist}(\hat{\varphi}_t(x)(\omega), \Phi(x,u)) \in \mathbb{R}$$

and

$$[0,T] \times \Omega \ni (t,x) \to \operatorname{dist}(\hat{\psi}_t(x)(\omega), \Psi(x,v)) \in \mathbb{R}$$

are Σ-measurable for fixed $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n \times m}$.
Then, also functions

$$[0,T] \times \Omega \ni (t,x) \to \sup_{x \in \mathcal{D}X} \operatorname{dist}(\hat{\varphi}_t(x)(\omega), \Phi(x,u)) \in \mathbb{R}$$

and

$$[0,T] \times \Omega \ni (t,x) \to \sup_{x \in \mathcal{D}X} \operatorname{dist}(\hat{\psi}_t(x)(\omega), \Psi(x,v)) \in \mathbb{R}$$

are Σ-measurable for fixed $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n \times m}$. Now, similarly as in ([2], Theorem II 3.12) we can verify that $(\mathcal{K}_t)_{0 \le t \le T}$ and $(Q_t)_{0 \le t \le T}$ are $\mathcal{F}_t$-nonanticipative. Therefore, by virtue of Kuratowski and Ryll- Nardzewski measurable selection theorem, there are $\mathcal{F}_t$- nonanticipative selectors $f = (f_t)_{0\le t\le T}$ and $\sigma = (\sigma_t)_{0\le t\le T}$ for $(\mathcal{K}_t)_{0\le t\le T}$ and $(Q_t)_{0\le t\le T}$, respectively. By the definitions of $(\mathcal{K}_t)(\omega)$ and $(Q_t)(\omega)$ we have that $f \in S^1(\mathfrak{F})$ and $\sigma \in S^2(D(\mathcal{G}))$. Furthermore

$$\sup_{x\in X} \operatorname{dist}(\hat{\varphi}_t(x)(\omega), \Phi(x, f_t(\omega))) = 0$$

and

$$\sup_{x\in X} \operatorname{dist}(\hat{\psi}_t(x)(\omega), \Psi(x, \sigma_t(\omega))) = 0$$

a.e. on $[0,T]\times\Omega$. Hence and from properties of $\hat{\varphi}$ and $\hat{\psi}$, in particular it follows

$$\widetilde{E}\int_s^t \Phi(x,\varphi_\tau)d\tau = E\int_s^t \Phi(x, f_\tau)d\tau$$

and

$$\widetilde{E}\int_s^t \Psi(x,\psi_\tau\cdot\psi_\tau^T)d\tau = E\int_s^t \Psi(x,\sigma_\tau)d\tau$$

for every $0\le s<t\le T$ and $x\in X$. Similarly as in the proof of Theorem 2, we can verify that there exists $g\in S^4(\mathcal{G})$ such that $\sigma = g\cdot g^T$. Therefore, we also have

$$\widetilde{E}\int_s^t \Psi(x,\psi_\tau\cdot\psi_\tau^T)d\tau = E\int_s^t \Psi(x,g_\tau\cdot g_\tau^T)d\tau$$

for every $0\le s<t\le T$ and $x\in X$. ■

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Received 3 February 2001