CONVEX UNIVERSAL FIXERS

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Abstract

In [1] Burger and Mynhardt introduced the idea of universal fixers. Let $G = (V, E)$ be a graph with $n$ vertices and $G'$ a copy of $G$. For a bijective function $\pi : V(G) \rightarrow V(G')$, define the prism $\pi G$ of $G$ as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{ u\pi(u) \mid u \in V(G) \}$. Let $\gamma(G)$ be the domination number of $G$. If $\gamma(\pi G) = \gamma(G)$ for any bijective function $\pi$, then $G$ is called a universal fixer. In [9] it is conjectured that the only universal fixers are the edgeless graphs $\overline{K_n}$.

In this work we generalize the concept of universal fixers to the convex universal fixers. In the second section we give a characterization for convex universal fixers (Theorem 6) and finally, we give an infinite family of convex universal fixers for an arbitrary natural number $n \geq 10$.

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1. Introduction

Let $G = (V, E)$ be an undirected graph. The neighborhood of a vertex $v \in V$ in $G$ is the set $N_G(v)$ of all vertices adjacent to $v$ in $G$. For a set $X \subseteq V$, the
open neighborhood $N_G(X)$ is defined as $\bigcup_{v \in X} N_G(v)$ and the closed neighborhood $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V$ is a dominating set of $G$ if $N_G[D] = V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in $G$.

The distance $d_G(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $uv$-path in $G$. A $uv$-path of length $d_G(u,v)$ is called $uv$-geodesic. A set $X \subseteq V$ is a convex set of $G$ if the vertices from all $ab$-geodesic belong to $X$ for every two vertices $a, b \in X$. A set $X \subseteq V$ is a convex dominating set if $X$ is convex and dominating. The convex domination number $\gamma_{\text{con}}(G)$ of a graph $G$ is equal to the minimum cardinality of a convex dominating set. The convex domination number was defined by Jerzy Topp from the Gdańsk University of Technology in a verbal communication with the first author. In [5], the first results concerning this topic were published and developed in [6] and [7].

**Definition 1.** Let $G = (V,E)$ be a graph and $G'$ a copy of $G$. For a bijective function $\pi : V(G) \to V(G')$, define the prism $\pi G$ of $G$ as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{ u\pi(u) \mid u \in V(G) \}$.

Notice that $M_\pi$ is a perfect matching of $\pi G$. It is clear that every permutation $\pi$ of $V(G)$ defines a bijective function from $V(G)$ to $V(G')$, so we will indistinctly use the matching $M_\pi$, the permutation $\pi$ of $V(G)$ or the associated bijection $\pi : V(G) \to V(G')$.

The graph $G$ is called a universal fixer if $\gamma(\pi G) = \gamma(G)$ for all permutations $\pi$ of $V(G)$.

The universal fixers were studied in [9] for several classes of graphs and it was conjectured that the edgeless graphs $K_n$ are the only universal fixers. In [2], [3] and [4] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers. This concept was also generalized for the other types of domination; in [10] the idea of paired domination in prisms was introduced.

We generalize the above definition for the convex domination: if $\gamma_{\text{con}}(\pi G) = \gamma_{\text{con}}(G)$ for all permutation $\pi$ of $V(G)$, then we say that $G$ is a convex universal fixer.

2. Convex Universal Fixers

From now on we assume that the graph $G = (V,E)$ is a connected undirected graph with $n$ vertices. For $x \in V(G)$, the copy of $x$ in $V(G')$ is denoted by $x'$. Recall that the diameter of a graph $G$, denoted by $\text{diam}(G)$, is defined to be the maximum distance between any two vertices $x, y \in V(G)$.

**Proposition 2.** Let $G$ be a connected undirected graph.
(1) If $\text{diam}(G) \leq 2$, then both $V(G)$ and $V(G')$ are convex dominating sets of $\pi G$ for any permutation $\pi$.

(2) If $\text{diam}(G) \geq 3$, then there exist permutations $\pi_1$ and $\pi_2$ such that $V(G)$ is not a convex dominating set of $\pi_1 G$ and $V(G')$ is not a convex dominating set of $\pi_2 G$.

**Proof.** (1) It is clear that $V(G)$ and $V(G')$ are dominating sets of $\pi G$. Let $x, y \in V(G)$. Since $d_{\pi G}(x, y) \leq d_G(x, y) \leq 2$, any $xy$-geodesic is contained in $G$, so $V(G)$ is a convex dominating set of $\pi G$. In a similar way, we can prove that $V(G')$ is a convex dominating set of $\pi G$.

(2) Let $x, y \in V(G)$ be such that $d_G(x, y) \geq 3$. Let $wz \in E(G')$ and consider a permutation $\pi_1$ such that $\pi_1(x) = w$ and $\pi_1(y) = z$. Then $xwzy$ is an $xy$-geodesic in $\pi_1 G$ with $z, v \not\in V(G')$. In a similar way, we can prove that there exists a permutation $\pi_2$ such that $V(G')$ is not a convex dominating set in $\pi_2 G$.

From the above proposition we have the following observation.

**Observation 3.** For any permutation $\pi$, $\gamma_{\text{con}}(\pi G) \leq n$ whenever $\text{diam}(G) \leq 2$.

If $D$ is a convex dominating set of $\pi G$, we define $D_1$ as $D \cap V(G)$ and $D_2$ as $D \cap V(G')$. Moreover, we write $D_1^c = V(G) - D_1$ and $D_2^c = V(G') - D_2$.

**Proposition 4.** Let $D$ be a convex dominating set of $\pi G$.

(1) If $\gamma_{\text{con}}(\pi G) < n$, then $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$.

(2) If $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$, then there exists at least one edge $x\pi(x) \in M_\pi$ with $x \in D_1$ and $\pi(x) \in D_2$.

**Proof.** (1) Suppose that $D_1 = \emptyset$. Then $D = D_2 \subseteq V(G')$. Since $|D| < n$, $V(G)$ is not dominated by $D$. Similarly, if $D_2 = \emptyset$, then $V(G')$ is not dominated by $D$.

(2) Let $x \in D_1$ and $\pi(y) \in D_2$. Since $D$ is convex, any $x\pi(y)$-geodesic should use the edge $x\pi(x)$ or the edge $y\pi(y)$.

**Lemma 5.** Suppose that $\text{diam}(G) \leq 2$. Let $D$ be a minimum convex dominating set of $\pi G$. If $D = D_1 \cup D_2$ with $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$, then we have the following statements:

(1) if $\pi(D_1) \subseteq D_2$, then $D_2$ is a convex dominating set of $G'$, and

(2) if $\pi^{-1}(D_2) \subseteq D_1$, then $D_1$ is a convex dominating set of $G$.

**Proof.** Assume that $\pi(D_1) \subseteq D_2$. Then, since $D$ is a dominating set of $\pi G$, every vertex of $D_2^c$ has a neighbor in $D_2$. Moreover, $\text{diam}(G') \leq 2$ and $d_{\pi G}(a, b) \leq 2$ for every two vertices $a, b \in D_2$, so the vertices from all $ab$-geodesics belong to $D_2$, because $D$ is convex. Thus $D_2$ is a convex dominating set of $G'$. Similarly, we can prove the second part of the lemma.
Our main result is the following.

**Theorem 6.** Let $G$ be a connected undirected graph. If $\gamma_{\text{con}}(G) = n$ and $\text{diam}(G) \leq 2$, then $\gamma_{\text{con}}(\pi G) = n$, that is, $G$ is a convex universal fixer.

**Proof.** By Observation 3, if $\text{diam}(G) \leq 2$, then $\gamma_{\text{con}}(G) \leq n$ for all permutations $\pi$. By contradiction, suppose that $\gamma_{\text{con}}(G) = n$ and $\gamma_{\text{con}}(\pi G) < n$. If $\text{diam}(G) = 1$, then $\gamma_{\text{con}}(G) < n$, so we can assume $\text{diam}(G) = 2$.

Let $D = D_1 \cup D_2$ be a minimum convex dominating set of $\pi G$ with $|D| < n$. From the first part of Proposition 4, we have that $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. In order to have a partition of $V(\pi G)$, we define the following subsets of vertices:

$$D_1^+ = \{u \in D_1 | \pi(u) \in D_2\}, \quad D_2^+ = \{u' \in D_2 | \pi^{-1}(u') \in D_1\} = \pi(D_1^+) \cup \pi^{-1}(\pi(D_1^+)),$$

$$D_1^- = \{u \in D_1 | \pi(u) \notin D_2\}, \quad D_2^- = \{u' \in D_2 | \pi^{-1}(u') \notin D_1\},$$

$$E_1 = \pi^{-1}(D_2^-), \quad E_2 = \pi(D_1^-),$$

$$F_1 = V(\pi G) - D_1 - E_1 \quad \text{and} \quad F_2 = \pi(F_1).$$

From the second part of Proposition 4, we have that $D_1^+ \neq \emptyset$ and $D_2^- \neq \emptyset$. If $\pi(D_1) \subseteq D_2$, then by Lemma 5, the set $D_2$ is a convex dominating set of $G'$, which is a contradiction since $\gamma_{\text{con}}(G') = n$. Therefore, $D_1^- \neq \emptyset$. In a similar way, $D_2^- \neq \emptyset$.

In consequence $E_1 \neq \emptyset$ and $E_2 \neq \emptyset$. Since $|D| < n$, $|D_1^+ \cup D_1^- \cup D_2^+ \cup D_2^-| < n$ and $|E_1 \cup F_2| = |D_1^- \cup D_2^-| < n$. Therefore, $F_1$ and $F_2$ are nonempty.

We claim that there are no edges between $E_1$ and $D_1$. Suppose $x \in D_1, y \in E_1$ and $xy \in E(G)$. Then $d_{\pi G}(x, \pi(y)) = 2$, and $x, \pi(y) \in D$ implies that $y \in D_1$, which leads us to a contradiction.

Let $x$ be a vertex in $D_1^+$ and $y \in E_1$. Since $\text{diam}(G) = 2$, $d_G(x, y) = 2$ and there exists a vertex $z \in F_1$ such that $xz \in E(G)$ and $yz \in E(G)$.

If $d_{\pi G}(x, \pi(y)) \geq 3$, then $xzy\pi(y)$ is an $x\pi(y)$-geodesic, which is not possible, since $D$ is a convex dominating set of $\pi G$ and $y, z \notin D$. Thus $d_{\pi G}(x, \pi(y)) = 2$.

But then there exists a vertex $w \in D$ such that $w$ is a common neighbor of $x$ and $\pi(y)$, a contradiction. Therefore, $\gamma_{\text{con}}(\pi G) = n$.

3. An Infinite Family of Convex Universal Fixers

Now we show that for an arbitrarily large $n$, there is a graph $G$ with $n$ vertices such that $G$ is a convex universal fixer. The following family $\mathcal{F}$ of graphs was defined in [8].

Let $G_1$ be the cycle of order five, $C_5^1 = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,1})$. For $i \geq 2$, the graph $G_i$ is obtained recursively from $G_{i-1}$ by adding a cycle graph $C_5^i = (v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}, v_{i,1})$ and for every vertex $v_{i,j}, j \in \{1, \cdots, 5\}$ of the
cycle $C^i_5$ we add edges $v_i,jv_{i,j-1}$ and $v_{i,j}v_{i,j+1}$ with $t \in \{1, \cdots, i-1\}$. The sums $j-1, j+1$ are done modulo five.

The authors denoted by $F$ the family of graphs $G$ obtained by adding to the graph $G_i$, $t \geq 2$ vertices $u_1, \ldots, u_t$ and edges $u_kv_{i,j}$, with $k \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, 5\}$.

![Diagram of a graph](image)

Figure 1. A graph belonging to the family $F$ with $n = 12$, $t = 2$ and $i = 2$.

The following result was proved in [8].

**Theorem 7.** If $G$ belongs to the family $F$, then $\gamma_{con}(G) = n$ and $diam(G) = 2$.

From the above theorem and our main result we can conclude the following

**Corollary 8.** For every natural number $n \geq 10$, there is a graph $G$ with $n$ vertices such that $G$ is a convex universal fixer.

4. Acknowledgments and Conjectures

We conclude this paper with the following two conjectures.

**Conjecture 9.** If $G$ is a convex universal fixer, then $\gamma_{con}(G) = n$ and $diam(G) = 2$.

**Conjecture 10.** If $G$ is a convex universal fixer, then the only minimum convex dominating sets of $\pi G$ are $V(G)$ and $V(G')$.

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