

LIGHT EDGES IN 1-PLANAR GRAPHS WITH PRESCRIBED MINIMUM DEGREE¹

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Abstract

A graph is called 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. We prove that each 1-planar graph of minimum degree $\delta \geq 4$ contains an edge with degrees of its endvertices of type $(4, \leq 13)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$. We also show that for $\delta \geq 5$ these bounds are best possible and that the list of edges is minimal (in the sense that, for each of the considered edge types there are 1-planar graphs whose set of types of edges contains just the selected edge type).

Keywords: light edge, 1-planar graph.

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1. INTRODUCTION

The research on graph theory particularly deals with structural properties of graphs. The knowledge of local graph structure is interesting by itself as well as in study of other graph properties. A typical example is a classical consequence of Euler polyhedral formula: every planar graph contains a vertex of degree at most 5. This result further developed into theory of unavoidable configurations widely used in proofs of results on graph colourings (notably, the Four Colour Theorem). Among several milestones on the way from Euler formula to modern structural theory of planar graphs, an important position has the theorem of Kotzig [14] which states that each 3-connected planar graph contains an edge with weight (that is, the sum of degrees of its endvertices) at most 13, and at most 11 if

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the graph has minimum degree at least 4; in addition, the bounds 13 and 11 are sharp. This result was generalized in many different directions: various authors have studied the number of light edges in several families of plane graphs (see [1, 3, 13]) or projective plane graphs ([16]), the existence of light edges in graphs embedded in higher surfaces ([7, 9, 10]) or in graphs with given number of edges ([8]); the survey of research in the area of light configurations can be found in survey papers [11] and [12].

The aim of this paper is to investigate light edges in certain nonplanar graphs which can be drawn in the plane in such a way that each edge is crossed by at most one other edge; such graphs are called *1-planar*. These graphs were first introduced by Ringel [15] in connection with simultaneous vertex-face colouring of plane graphs. The local properties of 1-planar graphs were studied in [4] where, among other results, the following analogy of Kotzig theorem, was proved: each 3-connected 1-planar graph contains an edge such that each its endvertex has degree at most 20, and this bound is best possible. Other results on light edges in 1-planar graphs with prescribed minimum degree and girth can be found in [5] and [6]. However, the full analogy of Kotzig theorem concerning the weight of light edges in the family of 1-planar graphs of minimum degree at least 3 is still not known. In this paper, we prove such a partial analogy for 1-planar graphs of minimum degree at least 4 and present examples of 1-planar graphs of minimum degree at least 5 for which our result is best possible.

2. PRELIMINARIES

In this paper we consider simple connected graphs. We use the standard graph theory terminology [2]. The degree of a vertex v in a graph G is denoted by $\deg_G(v)$. Similarly, the size of a face f in a plane graph G is denoted by $\deg_G(f)$. A vertex of degree k (at least k , at most k) is called a k -vertex ($\geq k$ -vertex, $\leq k$ -vertex, respectively). Similarly, a face of size r (at least r , at most r) is called an r -face ($\geq r$ -face, $\leq r$ -face, respectively).

Given a 1-planar graph G , let $D(G)$ denote a 1-planar drawing of the graph. Referring to the notation from [4], we denote by $D(G)^\times$ the *associated plane graph* of $D(G)$, that is, a plane graph obtained by replacing each crossing in $D(G)$ by a new 4-vertex (called *false* in what follows). All other vertices of $D(G)^\times$ will be called *true*. All edges and faces of $D(G)^\times$ incident to a false vertex will be called *false*, all other elements will be called *true*.

Given an edge $uv \in E(D(G)^\times)$ with endvertices of degree a and b , respectively, we say that uv is of type (a, b) ; similarly, we say that a 3-face f is of type (a, b, c) if its vertices have degrees a , b and c , respectively. For type entries, we will also use the entries $\geq k$ or $\leq k$ if the corresponding vertices are of degree at

least k or at most k . Finally, the symbol \otimes in edge/face type indicates that the corresponding vertex is a false vertex.

3. MAIN RESULT

Theorem 1. *Every 1-planar graph of minimum degree $\delta \geq 4$ contains an edge of type $(4, \leq 13)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$.*

Proof. In the proof of our result we use the classical strategy. Suppose there is a counterexample G to Theorem 1. Consider a 1-planar drawing $D(G)$ of G . Note that G contains only edges of type $(4, \geq 14)$ or $(5, \geq 10)$ or $(6, \geq 9)$ or $(\geq 7, \geq 8)$. We proceed by the Discharging method on the associated plane graph $D(G)^\times$. Assigning the initial charge $c(v) = \deg_{D(G)^\times}(v) - 4$ to every vertex $v \in V(D(G)^\times)$ and $c(f) = \deg_{D(G)^\times}(f) - 4$ to every face $f \in F(D(G)^\times)$, we obtain, according to the Euler polyhedral formula,

$$\sum_{v \in V(D(G)^\times)} (\deg(v) - 4) + \sum_{f \in F(D(G)^\times)} (\deg(f) - 4) = \sum_{x \in V(D(G)^\times) \cup F(D(G)^\times)} c(x) = -8.$$

Then, we redistribute locally the initial charge of elements of $D(G)^\times$ by a set of rules in such a way that the total sum remains the same (negative). After application of these rules, the initial charge is transformed to a new charge $\bar{c} : V(D(G)^\times) \cup F(D(G)^\times) \rightarrow \mathbb{Q}$. Finally, it is shown that the function \bar{c} is nonnegative, yielding that the sum of all new charges is also nonnegative, a contradiction.

Discharging Rules:

Rule 1: Every 5-vertex sends $\frac{1}{4}$ to every incident false 3-face.

Rule 2: Every 6-vertex sends $\frac{1}{3}$ to every incident false 3-face.

Rule 3: Every 7-vertex sends $\frac{1}{2}$ to every incident false 3-face.

Rule 4: Every 8-vertex sends $\frac{1}{2}$ to every incident 3-face.

Rule 5: Every 9-vertex sends

- $\frac{1}{3}$ to every incident true 3-face of type $(9, \geq 9, \geq 9)$,
- $\frac{1}{2}$ to any other incident true 3-face,
- $\frac{2}{3}$ to every incident false 3-face of type $(9, 6, \otimes)$,
- $\frac{1}{2}$ to every incident false 3-face of type $(9, \geq 7, \otimes)$.

Rule 6: For $10 \leq k \leq 13$, every k -vertex sends

- $\frac{1}{2}$ to every incident true 3-face,
- $\frac{3}{4}$ to every incident false 3-face of type $(k, 5, \otimes)$,
- $\frac{2}{3}$ to every incident false 3-face of type $(k, 6, \otimes)$,
- $\frac{1}{2}$ to every incident false 3-face of type $(k, \geq 7, \otimes)$.

Rule 7: For $l \geq 14$, every l -vertex sends

- $\frac{1}{2}$ to every incident true 3-face,
- 1 to every incident false 3-face of type $(l, 4, \otimes)$,
- $\frac{3}{4}$ to every incident false 3-face of type $(l, 5, \otimes)$,
- $\frac{2}{3}$ to every incident false 3-face of type $(l, 6, \otimes)$,
- $\frac{1}{2}$ to every incident false 3-face of type $(l, \geq 7, \otimes)$.

We collect these Rules in a compact table:

Rule	deg(x)	True 3-faces		False 3-faces			
1	5	-		1/4			
2	6	-		1/3			
3	7	-		1/2			
4	8	1/2		1/2			
5	9	$(9, \geq 9, \geq 9)$	other.	$(9, \geq 7, \otimes)$	$(9, 6, \otimes)$		
		1/3	1/2	1/2	2/3		
6	$10 \leq k \leq 13$	1/2		$(k, \geq 7, \otimes)$	$(k, 6, \otimes)$	$(k, 5, \otimes)$	
				1/2	2/3	3/4	
7	$l \geq 14$	1/2		$(l, \geq 7, \otimes)$	$(l, 6, \otimes)$	$(l, 5, \otimes)$	$(l, 4, \otimes)$
				1/2	2/3	3/4	1

After application of Rules we have to consider several cases.

Let f be an r -face ($r \geq 4$). The initial charge of f will not change by using the Rules, hence, the new charge $\bar{c}(f)$ remains nonnegative. By the Rules the charge is redistributed only from vertices to incident 3-faces. Now, we will check all types of 3-faces:

Claim 2. *After the application of Rules, the charge of every true 3-face is non-negative.*

Proof. Let $f = \{abc\}$ be a true 3-face. Note that f can be incident to at most one vertex of degree 4, 5, 6 or 7.

Case (T1). Let a be a 4-vertex; then b and c are ≥ 14 -vertices. By Rule 7 each of b and c sends $\frac{1}{2}$ to the face f and $\bar{c}(f) = 3 - 4 + 2 \cdot \frac{1}{2} = 0$.

Case (T2). Let a be a 5-vertex; then b and c are ≥ 10 -vertices. By Rule 6 (or 7) each of b and c sends $\frac{1}{2}$ to the face f , hence $\bar{c}(f) = 3 - 4 + 2 \cdot \frac{1}{2} = 0$.

Case (T3). Let a be a 6-vertex; then b and c are ≥ 9 -vertices. By Rule 5 (6 or 7) both vertices b and c send $\frac{1}{2}$ to the face f , $\bar{c}(f) = 3 - 4 + 2 \cdot \frac{1}{2} = 0$.

Case (T4). Let a be a ≥ 7 -vertex; then b and c are ≥ 8 -vertices. Let $\deg(a) \leq \deg(b) \leq \deg(c)$. If $\deg(a) \in \{7, 8\}$ then by Rule 4 (5, 6 or 7) both b and c send $\frac{1}{2}$ to the face f , therefore $\bar{c}(f) \geq 3 - 4 + 2 \cdot \frac{1}{2} = 0$. If $\deg(a) \geq 9$ then all three vertices send at least $\frac{1}{3}$ to f , that means $\bar{c}(f) \geq 3 - 4 + 3 \cdot \frac{1}{3} = 0$. \square

Claim 3. *After the application of Rules, the charge of every false 3-face is non-negative.*

Proof. Let $f = \{abc\}$ be a false 3-face with a false vertex c .

Case (F1). Let a be a 4-vertex; then b is a ≥ 14 -vertex. By Rule 7 the vertex b sends 1 to the face f , $\bar{c}(f) = 3 - 4 + 1 = 0$.

Case (F2). Let a be a 5-vertex; then b is a ≥ 10 -vertex. The vertex a sends $\frac{1}{4}$ to f by Rule 1 and the vertex b sends $\frac{3}{4}$ to f by Rule 6 (or 7), hence $\bar{c}(f) = 3 - 4 + \frac{1}{4} + \frac{3}{4} = 0$.

Case (F3). Let a be a 6-vertex; then b is a ≥ 9 -vertex. The vertex a sends $\frac{1}{3}$ to f by Rule 2 and the vertex b sends $\frac{2}{3}$ by Rule 5 (6 or 7) to the face f , therefore $\bar{c}(f) = 3 - 4 + \frac{1}{3} + \frac{2}{3} = 0$.

Case (F4). Let a be a ≥ 7 -vertex; then b is a ≥ 8 -vertex. Both vertices a and b send $\frac{1}{2}$ to the face f by Rule 3 (4, 5, 6 or 7). Hence $\bar{c}(f) = 3 - 4 + 2 \cdot \frac{1}{2} = 0$. \square

We can conclude that all faces have nonnegative value of new charge. Next we consider vertices of $D(G)^\times$. We can see that 4-vertices (false or true) are not influenced by any Rule, so their charge remains zero.

For $k \in \{4, 5, 6, 7\}$ we denote by s_k the minimal degree of a vertex adjacent to a k -vertex in G . Note that $s_4 = 14$, $s_5 = 10$, $s_6 = 9$, and $s_7 = 8$. Furthermore call a false 3-face of type $(k, \geq s_k, \otimes)$ a *k-small 3-face*.

Each edge of type $(\otimes, \geq s_k)$ can be incident to at most one k -small 3-face (otherwise two edges of type (k, \otimes) from two k -small 3-faces sharing the same edge of type $(\otimes, \geq s_k)$ correspond in G to an edge of forbidden type (k, k)). Now,

we determine $S_k^n(x)$, the number of k -small 3-faces incident to the n -vertex x ($n \geq s_k$). In following the Lemmas 4–7 we give an upper bound on $S_k^n(x)$, for $k \in \{4, 5, 6, 7\}$.

Lemma 4. *For an n -vertex x ($n \geq s_k$), $S_k^n(x) \leq \lfloor \frac{2n}{3} \rfloor$.*

Proof. Each edge of type (k, n) can be incident to at most two k -small 3-faces. For $i = 1, 2$, let m_i denote the number of k -vertices y adjacent to x , where the edge xy is incident to exactly i k -small 3-faces. The number of k -small 3-faces incident to x is $S_k^n(x) = m_1 + 2m_2$. Let y be a k -vertex adjacent to x . If the edge xy is incident to exactly one k -small 3-face α , then α covers exactly two edges incident to x . We see that no edge incident to α can be incident to another k -small 3-face incident to x . If the edge xy is incident to exactly two k -small 3-faces, say α, β , then α, β cover exactly three edges incident to x . Again, no edge incident to α, β can be incident to another k -small 3-face incident to x . From the number of edges incident to x we have $2m_1 + 3m_2 \leq n$, which gives $\frac{4}{3}m_1 + 2m_2 \leq \frac{2n}{3}$. Hence $m_1 + 2m_2 \leq \frac{4}{3}m_1 + 2m_2 \leq \frac{2}{3}n$, which gives $S_k^n(x) \leq \lfloor \frac{2n}{3} \rfloor$. □

Lemma 5. *Let x be an n -vertex ($n \geq s_k$) incident to 3-faces only, then*

$$S_k^n(x) \leq \begin{cases} \lfloor \frac{n}{2} \rfloor - 1, & n \equiv 2 \pmod{4}, \\ \lfloor \frac{n}{2} \rfloor, & n \not\equiv 2 \pmod{4}. \end{cases}$$

Proof. If an n -vertex x is incident to 3-faces only, then x can be adjacent to at most $\lfloor \frac{n}{2} \rfloor$ false vertices, hence $S_k^n(x) \leq \lfloor \frac{n}{2} \rfloor$. Now, consider an n -vertex where $n = 4m + 2$. Since $\lfloor \frac{n}{2} \rfloor = 2m + 1$ we have at most $2m + 1$ false vertices. If the number of false vertices incident to x is at most $2m$, then there are at most $2m$ k -small 3-faces incident to x . If x is incident to exactly $2m + 1$ false vertices, then since no two k -vertices can be adjacent in G , it follows that the number of k -vertices incident to x is at most $\lfloor \frac{2m+1}{2} \rfloor = m$. Hence the vertex x can be incident to at most $2m = \lfloor \frac{n}{2} \rfloor - 1$ k -small 3-faces. Due to this fact, $S_k^n(x) \leq \lfloor \frac{n}{2} \rfloor - 1$ for all $n = 4m + 2$. □

Lemma 6. *Let x be an n -vertex ($n \geq s_k$) incident to exactly one ≥ 4 -face. Then $S_k^n(x) \leq \lceil \frac{n}{2} \rceil$.*

Proof. If an n -vertex x is incident to exactly one ≥ 4 -face, then x can be adjacent to at most $\lceil \frac{n}{2} \rceil$ false vertices, hence $S_k^n(x) \leq \lceil \frac{n}{2} \rceil$. □

Lemma 7. *Let x be an n -vertex ($n \geq s_k$) incident to exactly two ≥ 4 -faces. Then $S_k^n(x) \leq \lceil \frac{n}{2} \rceil + 1$.*

Proof. If an n -vertex x is incident to exactly two ≥ 4 -faces, then x can be adjacent to at most $\lceil \frac{n}{2} \rceil + 1$ false vertices, hence $S_k^n(x) \leq \lceil \frac{n}{2} \rceil + 1$. □

Claim 8. *After the application of Rules, the charge of every vertex is nonnegative.*

Proof. Let x be a k -vertex ($k \geq 5$). Note that vertices send charge only to incident 3-faces and x can be incident with at most $2\lfloor \frac{k}{2} \rfloor$ false 3-faces.

Case (V1). Let $k = 5$. The vertex x is involved only in Rule 1 and sends a charge only to incident false 3-faces. Since the degree of x is odd, it follows that x is incident to at most four false 3-faces and $\bar{c}(x) \geq 5 - 4 - 4 \cdot \frac{1}{4} = 0$.

Case (V2). Let $k = 6$. The vertex x sends $\frac{1}{3}$ to every incident false 3-face by Rule 2. It follows that $\bar{c}(x) \geq 6 - 4 - 6 \cdot \frac{1}{3} = 0$.

Case (V3). Let $k = 7$. The vertex x sends $\frac{1}{2}$ to every incident false 3-face by Rule 3. Again since the degree of x is odd, it follows that x is incident to at most six false 3-faces and $\bar{c}(x) \geq 7 - 4 - 6 \cdot \frac{1}{2} = 0$.

Case (V4). Let $k = 8$. The vertex x sends $\frac{1}{2}$ to every incident 3-face by Rule 4. It follows $\bar{c}(x) \geq 8 - 4 - 8 \cdot \frac{1}{2} = 0$.

Case (V5). Let $k = 9$. Let a (b , b' , c and d) denote the number of ≥ 4 -faces (true 3-faces of type $(9, \geq 9, \geq 9)$, true 3-faces of other types, false 3-faces of type $(\geq 7, 9, \otimes)$ and false 3-faces of type $(6, 9, \otimes)$, respectively) incident to x . Note that $a + b + b' + c + d = 9$, which gives $d = 9 - a - b - b' - c$.

Now, we determine the new charge of the vertex x . Precisely, $\bar{c}(x) = 9 - 4 - \frac{1}{3} \cdot b - \frac{1}{2} \cdot b' - \frac{1}{2} \cdot c - \frac{2}{3} \cdot d = 5 - \frac{1}{2}(b' + c) - \frac{1}{3}b - \frac{2}{3}(9 - a - b - b' - c) = -1 + \frac{2a}{3} + \frac{1}{6}(b' + c) + \frac{1}{3}b$. To ensure the nonnegativity of $\bar{c}(x)$ we need $-1 + \frac{2a}{3} + \frac{1}{6}(b' + c) + \frac{1}{3}b \geq 0$, that implies $b' + c \geq 6 - 4a - 2b$, hence $d = 9 - a - b - b' - c \leq 3 + 3a + b$. Clearly, if $d \leq 3 + 3a + b$ then $\bar{c}(x) \geq 0$. We consider several cases:

Let $a \geq 1$, then $6 \leq 3 + 3a + b$. From Lemmas 4, 6 and 7 it follows that $d \leq 6$, in total $d \leq 6 \leq 3 + 3a + b$.

Let $a = 0$; then $3 \leq 3 + 3a + b$. Clearly, if $d \leq 3$ then $\bar{c}(x) \geq 0$. From Lemma 5 it follows that $d \leq 4$. It remains to consider the case $d = 4$. Since x is a 9-vertex incident to 3-faces only, x can be adjacent to at most four false vertices. Each edge joining a false vertex and x can be incident to at most one 6-small 3-face. Since $d = 4$ we have $b = 1$; consequently, $b' = 0$ and $c = 4$. So, it holds $\bar{c}(x) = 9 - 4 - \frac{1}{3} \cdot 1 - \frac{1}{2} \cdot 4 - \frac{2}{3} \cdot 4 = 0$.

Case (V6). Let $10 \leq k \leq 13$. Let a (b , c , d and e) denote the number of ≥ 4 -faces (true 3-faces, false 3-faces of type $(\geq 7, k, \otimes)$, false 3-faces of type $(6, k, \otimes)$ and false 3-faces of type $(5, k, \otimes)$, respectively) incident to x . Note that $a + b + c + d + e = k$, which yields $d + e = k - a - b - c$.

Now, we determine the new charge of vertex the x . By computing $\bar{c}(x) = k - 4 - \frac{1}{2} \cdot b - \frac{1}{2} \cdot c - \frac{2}{3} \cdot d - \frac{3}{4} \cdot e \geq k - 4 - \frac{1}{2}(b + c) - \frac{3}{4}(d + e) = k - 4 - \frac{1}{2}(b + c) - \frac{3}{4}(k - a - b - c) = \frac{k}{4} - 4 + \frac{3a}{4} + \frac{1}{4}(b + c)$. To ensure the nonnegativity of $\bar{c}(x)$ we need $\frac{k}{4} - 4 + \frac{3a}{4} + \frac{1}{4}(b + c) \geq 0$, which implies $b + c \geq 16 - k - 3a$, hence

$d + e = k - a - b - c \leq 2k + 2a - 16$. Clearly, if $d + e \leq 2k + 2a - 16$ then $\bar{c}(x) \geq 0$. We consider several cases:

Let $a \geq 1$. From Lemmas 6, 7 and 4 it follows that $d + e \leq \lfloor \frac{2k}{3} \rfloor$. But, for $10 \leq k \leq 13$, $\lfloor \frac{2k}{3} \rfloor \leq 2k - 14 \leq 2k + 2a - 16$, so $\bar{c}(x) \geq 0$ in this case.

Let $a = 0$. In this case, we prove the following inequality: $d + e \leq 2k - 16 = 2k + 2a - 16$. In parts, for $k = 10$ it means that $\bar{c}(x) \geq 0$ if $d + e \leq 4$. On the other hand from Lemma 5 for $k = 10$ it follows that $d + e \leq 4$. For $k \in \{11, 12, 13\}$ from Lemma 5 it follows that $d + e \leq \lfloor \frac{k}{2} \rfloor$. But, for $k \in \{11, 12, 13\}$, $\lfloor \frac{k}{2} \rfloor \leq 2k - 16$, so $\bar{c}(x) \geq 0$ in this case.

Case (V7). Let $k \geq 14$. Let a (b, c, d, e and f) denote the number of ≥ 4 -faces (true 3-faces, false 3-faces of type $(\geq 7, k, \otimes)$, false 3-faces of type $(6, k, \otimes)$, false 3-faces of type $(5, k, \otimes)$ and false 3-faces of type $(4, k, \otimes)$, respectively) incident to x . Note that $a + b + c + d + e + f = k$, which implies $d + e + f = k - a - b - c$.

Now, we determine the new charge of the vertex x : $\bar{c}(x) = k - 4 - \frac{1}{2} \cdot b - \frac{1}{2} \cdot c - \frac{2}{3} \cdot d - \frac{3}{4} \cdot e - f \geq k - 4 - \frac{1}{2}(b + c) - (d + e + f) = k - 4 - \frac{1}{2}(b + c) - (k - a - b - c) = a - 4 + \frac{1}{2}(b + c)$. To ensure the nonnegativity of $\bar{c}(x)$ we need $a - 4 + \frac{1}{2}(b + c) \geq 0$, which implies $b + c \geq 8 - 2a$, hence $d + e + f = k - a - b - c \leq k + a - 8$. Clearly, if $d + e + f \leq k + a - 8$ then $\bar{c}(x) \geq 0$. We consider several cases:

Let $a \geq 3$. From Lemma 4 it follows that $d + e + f \leq \lfloor \frac{2k}{3} \rfloor$. But, for $k \geq 14$, $\lfloor \frac{2k}{3} \rfloor \leq k - 5 \leq k + a - 8$, so $\bar{c}(x) \geq 0$ in this case.

Let $a = 2$. From Lemma 7 it follows that $d + e + f \leq \lceil \frac{k}{2} \rceil + 1$. But, for $k \geq 14$, $\lceil \frac{k}{2} \rceil + 1 \leq k - 6 = k + a - 8$, so $\bar{c}(x) \geq 0$ in this case.

Consider the case $a = 1$. From Lemma 6 it follows that $d + e + f \leq \lceil \frac{k}{2} \rceil$. But, for $k \geq 14$, $\lceil \frac{k}{2} \rceil \leq k - 7 = k + a - 8$, so $\bar{c}(x) \geq 0$ in this case.

Finally, let $a = 0$. Then $d + e + f \leq k - 8 = k + a - 8$. Clearly, if $d + e + f \leq k - 8$ then $\bar{c}(x) \geq 0$.

Particularly, for $k = 14$ it follows that $\bar{c}(x) \geq 0$ only if $d + e + f \leq 6$. On the other hand, from Lemma 5 for $k = 14$, it follows that $d + e + f \leq 6$.

For $k \geq 15$ we have $\lfloor \frac{k}{2} \rfloor \leq k - 8$, so $\bar{c}(x) \geq 0$ in this case. □

The proof of Theorem 1 follows from the above Claims. ■

Corollary 9. *Every 1-planar graph of minimum degree 5 contains an edge of type $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$.*

Corollary 10. *Every 1-planar graph of minimum degree 6 contains an edge of type $(6, \leq 8)$ or $(7, 7)$.*

As a next corollary we get a result proved in [5].

Corollary 11. *Every 1-planar graph of minimum degree 7 contains an edge of type $(7, 7)$.*

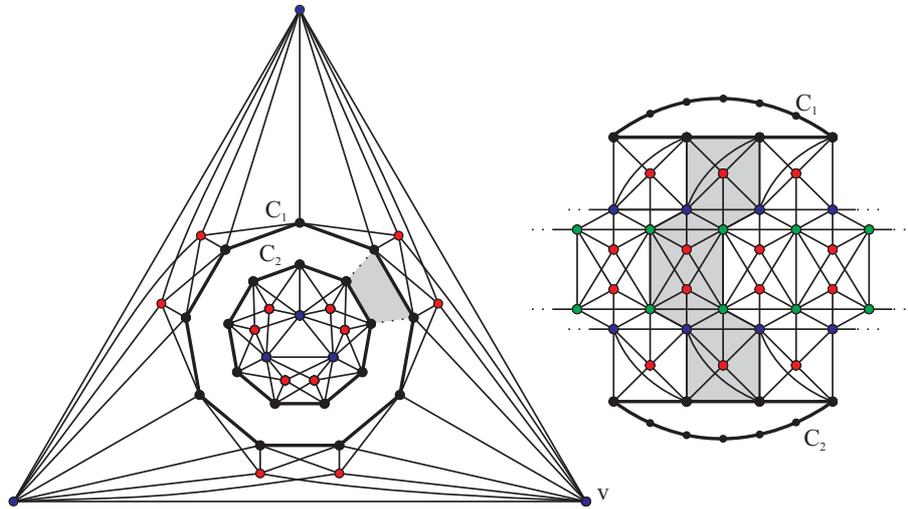


Figure 1. 1-planar graph A.

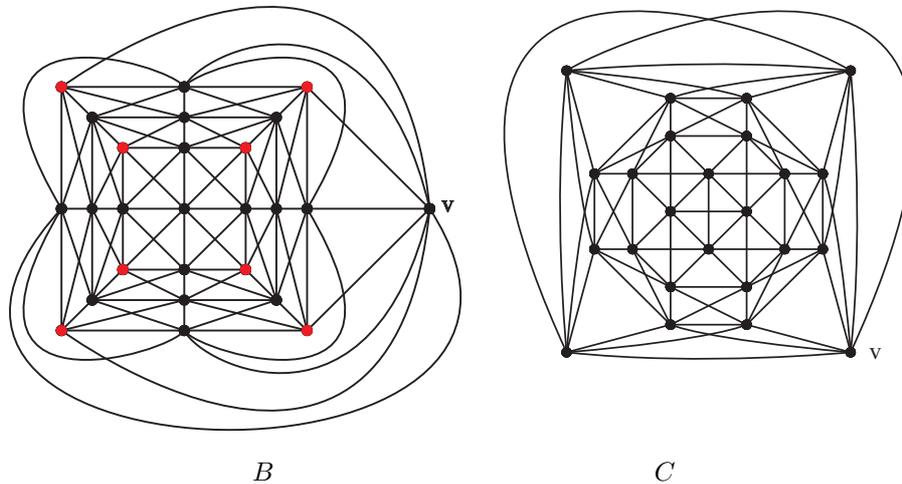
4. CONCLUDING REMARKS

Now we show that bounds 9 and 8 from Theorem 1 are best possible and that the list of edges is minimal in every case (in the sense that, for each $\delta \geq 4$ and each of the considered edge types there are 1-planar graphs whose set of types of edges contains just the selected edge type). We construct 1-planar graphs of minimum degree $\delta \geq 5$ such that the set of types of its edges contains only one edge type from the list. To show the sharpness of the above mentioned bounds, we use three graphs A, B and C from Figures 1, 2. All of them have a special vertex v . The graph A on Figure 1 is a 1-planar graph of minimum degree 5. In the middle of A , there are two 9-cycles C_1 and C_2 (drawn bold). The ring between C_1 and C_2 is filled with nine copies of the grey configuration from Figure 1. It is easy to see that the graph A has only edges of type $(5, 9)$, $(5, 10)$ and $(9, 10)$.

The graph B on Figure 2 is a 1-planar graph of minimum degree 6 having only edges of type $(6, 8)$ and $(8, 8)$. It is obtained from a cube graph subdividing each its edge with a new vertex, then inserting additional vertices into all faces, joining them with the subdivision vertices at face boundaries; finally, into each quadrangular face obtained in this way, a pair of crossing diagonals is inserted.

The graph C on Figure 2 is a 7-regular, 1-planar graph already constructed in [4].

To show that the list of edge types is complete for $\delta = 5$ we construct special graphs in the following way: we take $l > 1$ copies of the graph B or C and

Figure 2. 1-planar graphs B and C .

identify their special vertex v in one vertex u , thereby obtaining the graph H . Next, we take five copies $H^{(1)}, \dots, H^{(5)}$ of H , a new vertex w and we add new edges $wu^{(i)}$, $i = 1, \dots, 5$ (where $u^{(i)}$ is the counterpart of $u \in H$ in $H^{(i)}$). The resulting 1-planar graph is of minimum degree 5, and edge types from the list occur only in the copies of graphs B and C . For $\delta = 6$ we proceed similarly with joining copies of graph C to a 6-vertex w . From these constructions it follows that all types of edges are essential in the result.

For $\delta = 4$ we show that the bound, although probably not the best possible, cannot be less than 10: take the graph of Rhomb-Cubo-Octahedron and add to every 4-face $f = \{abcd\}$ two new vertices u and v joining them with vertices a, b, c, d by new edges (see Figure 3). By this construction, we obtain a 1-planar graph of minimum degree 4, with edges of type $(4, 10)$ and $(10, 10)$. By a similar construction as in the previous cases for larger δ we can construct 1-planar graphs of minimum degree 4 with only one desired type of edges.

Considering the infinite 1-planar graph S^\times which is obtained from the graph S of the square tiling of the plane using above described completion of 4-faces, we believe that the best upper bound in the edge type that involves a 4-vertex is less than 12 (as in S^\times there are only edges of types $(4, 12)$ and $(12, 12)$). It is an open question whether there exists a 1-planar graph of minimum degree 4 whose 4-vertices are adjacent only with ≥ 11 -vertices.

The previous corollaries yield the following analogy of Kotzig theorem:

Corollary 12. *Every 1-planar graph with minimum degree $\delta \geq 4$ contains an edge with weight at most 17. Moreover, if $\delta \geq 5$, then the bound is 14 and is best possible.*

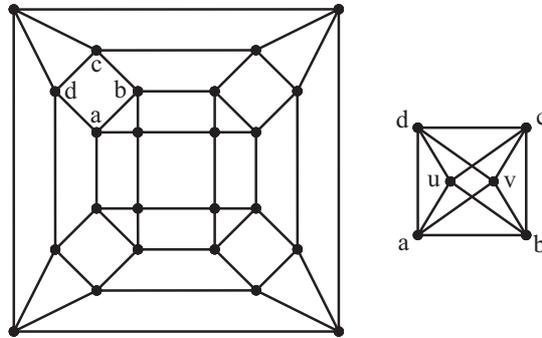


Figure 3. Rhomb-Cubo-Octahedron and a completion of a 4-face.

Since for all $n \in \mathbb{N}$ and $\delta = 1, 2$ the (1-planar) graphs $K_{1,n}$ and $K_{2,n}$ contain only edges with one endvertex of degree n , it shows that 1-planar graphs do not contain, in general, light edges. Thus, it remains to resolve the case $\delta = 3$ (note that the proof of Theorem in [4] uses the fact that the analyzed 1-planar graph is 3-connected, hence, it cannot be used for 1-planar graphs of minimum degree 3). To suggest the possible list of edge types in this case, take the graph of an icosahedron. In each 3-face $f = \{abc\}$ put three new vertices x, y, z . Then add new edges $ax, ay, az, bx, by, bz, cx, cy, cz$ preserving 1-planarity. The resulting graph is 1-planar of minimum degree 3, with edges of type $(3, 20)$ or $(20, 20)$. This rises a conjecture:

Conjecture 13. *Let G be a 1-planar graph of minimum degree 3. Then G contains an edge of type $(3, \leq 20)$ or $(4, \leq 13)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$.*

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