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ITERATED NEIGHBORHOOD GRAPHS

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Abstract

The *neighborhood graph* $N(G)$ of a simple undirected graph $G = (V, E)$ is the graph (V, E_N) where $E_N = \{\{a, b\} \mid a \neq b, \{x, a\} \in E \text{ and } \{x, b\} \in E$ for some $x \in V\}$. It is well-known that the neighborhood graph $N(G)$ is connected if and only if the graph G is connected and non-bipartite.

We present some results concerning the *k-iterated neighborhood graph* $N^k(G) := N(N(\ldots N(G)))$ of G. In particular we investigate conditions for G and k such that $N^k(G)$ becomes a complete graph.

Keywords: neighborhood graph, 2-step graph, neighborhood completeness number.

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1. Introduction and Definitions

All graphs considered here are undirected and finite without loops and multiple edges.

Definition. The *neighborhood graph* $N(G)$ of a graph $G = (V, E)$ is the graph (V, E_N) where $E_N = \{\{a, b\} \mid a \neq b, \{x, a\} \in E \text{ and } \{x, b\} \in E$ for some $x \in V\}$.

Several aspects of neighborhood graphs were investigated in the last thirty years (cf. [1–3, 5, 6, 9–14, 16]). Some of these papers use the notation 2-*step graph* or *competition graph* instead of neighborhood graph. As the latter name indicates, the neighborhood graph $N(G)$ of an undirected graph G is closely related to the competition graph $C(D)$ of a digraph D. Surveys of competition graphs can be found in Kim [7], Lundgren [8] and Roberts [15].

With $d_G(x, y)$ and $d(x : G)$ we denote the distance of $x, y \in V$ in G and the degree of $x \in V$ in G, respectively. Further we use the neighborhood sets $N_G(x) = \{z \in V \mid \{x, z\} \in E\}$ and $N_G(x, y) = N_G(x) \cap N_G(y)$. Definitions not explicitly given here can be found in [4].

First, we summarize some simple results on neighborhood graphs from the literature mentioned above.

Proposition 1. *Let $G = (V, E)$ be a connected graph and $N(G) = (V, E_N)$ its neighborhood graph. Then the following hold:*

(a) *$N(G)$ has at most two connected components.*

(b) *$N(G)$ is connected if and only if G is non-bipartite.*

(c) *If G is 2-connected and non-bipartite, then $N(G)$ is also 2-connected and non-bipartite.*

(d) *For each $n \geq 5$ and $p \geq 2$ with $2p \leq n$ there is a p-connected, non-bipartite graph G with n vertices, such that the neighborhood graph $N(G)$ has connectivity 2.*

(e) *For the path P_n with n vertices: $N(P_n) \cong P_{\lceil \frac{n}{2} \rceil} \cup P_{\lfloor \frac{n}{2} \rfloor}$.*

(f) *For the cycle C_n with n vertices: $N(C_{2k+1}) \cong C_{2k+1}$, $N(C_{2k}) \cong C_k \cup C_k$ (for $k \geq 3$) and $N(C_4) \cong P_2 \cup P_2$.*

(g) *For the complete graph K_n with n vertices: $N(K_n) \cong K_n, n \neq 2$ (note that $G = C_{2n+1}$ and $G = K_n, n \neq 2$, are the only connected graphs with $N(G) \cong G$ (cf. Brigham and Dutton [3])).*

(h) *For the complete bipartite graph $K_{m,n}$ with $m + n$ vertices:*
$N(K_{m,n}) \cong K_m \cup K_n$.

(i) *For the wheel W_n with $n + 1$ vertices: $N(W_n) \cong K_{n+1}$.*

Properties (e)–(i) lead to the question what happens if the construction of the neighborhood graph is iterated:

Definition. For a positive integer $k \in \mathbb{N}^+$, the *k-iterated neighborhood graph* $N^k(G)$ of a graph G is the neighborhood graph of $N^{k-1}(G)$, where $N^0(G) := G$.

In this paper we consider the following problems:

Problem 1. What is the structure of $N^k(G)$, for large k?

Problem 2. Under which conditions $N^k(G) \cong K_n$, for sufficiently large k?

Problem 3. If G fulfils the conditions mentioned in Problem 2, what is the minimum k such that $N^k(G) \cong K_n$?

The answers of Problems 1 and 2 follow from the results of Exoo and Harary [5]; we discuss these problems in the (short) Section 2. Section 3 contains the main results of this paper. There we determine the minimum k mentioned in Problem 3 for a certain class of graphs and give upper bounds for k being better than those from [5].

2. THE STRUCTURE OF $N^k(G)$ FOR LARGE k

Summarizing the results of Lemma 1–3 of [5] we obtain immediately the following theorem solving Problem 2. Here we present another (short) proof using arguments which prepare several ideas used in Section 3.

Theorem 2. *Let $G = (V, E)$ be a graph with $n > 1$ vertices. Then there exists $k \in \mathbb{N}$ with $N^k(G) \cong K_n$ if and only if G is connected, non-bipartite and $G \ncong C_{2p+1}$ (for $p > 1$).*

Proof. Let $n = |V| > 1$. If G is an odd cycle C_{2p+1}, $p > 1$, or bipartite or not connected then, by Proposition 1 (b) and (f), $N^k(G) \ncong K_n$ for all $k \in \mathbb{N}$. Therefore the three conditions (connected, non-bipartite and $G \ncong C_{2p+1}$, $p > 1$) are necessary for the existence of $k \in \mathbb{N}$ with $N^k(G) \cong K_n$.

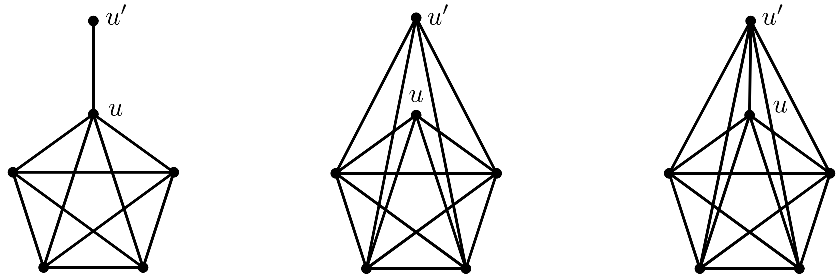
Now let G fulfil these conditions and $v \in V$ be a vertex with the degree $d(v : G) = p \geq 3$. Then the neighborhood $N_G(v)$ induces a p-clique K_p in the neighborhood graph $N^1(G)$.

We prove that for $k, p \in \mathbb{N}^+$ with $3 \leq p < n$ the existence of a p-clique K_p in $N^k(G)$ implies the existence of a $(p+1)$-clique K_{p+1} in $N^{k+2}(G)$.

By Proposition 1(b), $N^k(G)$ is connected. Since $p < n$, there is a vertex u in the p-clique K_p having a neighbor $u' \in V(G) \setminus V(K_p)$ in $N^k(G)$. Consequently, in $N^{k+1}(G)$ — in addition to K_p — the set $(V(K_p) \setminus \{u\}) \cup \{u'\}$ induces a second p-clique. Therefore, in $N^{k+2}(G)$ also the vertices u and u' are adjacent (in $N^{k+1}(G)$ they have common neighbors in $V(K_p) \setminus \{u\}$) and $V(K_p) \cup \{u'\}$ induces a $(p+1)$-clique (cf. Figure 1)). ■

Proposition 1 and Theorem 2 imply the following corollary, which solves Problem 1 (the result is established in [5] and also mentioned in [3]).

Corollary 3. *For an arbitrary graph $G = (V, E)$ and sufficiently large $k \in \mathbb{N}$, $N^k(G)$ consists of odd cycles and (possibly trivial) complete graphs.*



$N^k(G)$ contains a K_5 two 5-cliques in $N^{k+1}(G)$ $N^{k+2}(G)$ contains a K_6

Figure 1. An example with $p = 5$.

3. The Neighborhood Completeness Number

Now we turn to Problem 3. To determine the minimum k such that $N^k(G)$ is complete could be interesting in connection with graph algorithms; this motivates the definition:

Definition. For $G = (V, E)$ connected, non-bipartite and $G \ncong C_{2p+1}$ (for $p > 1$), we define the *neighborhood completeness number* of G by

$$cn(G) := \min\{k \in \mathbb{N} \mid N^k(G) \cong K_n\}.$$

The only result concerning the neighborhood completeness number can be found in [5]. Let G be a connected graph with n vertices which is neither bipartite nor an odd cycle. If C is a cycle of length $2k+1$ in G, d is the maximum least distance from a vertex not on C to a vertex on C and $r := \log_2 d$, then $N^{r+2k+1}(G) = K_n$. Hence

$$\text{(EH)} \qquad cn(G) \le r + 2k + 1.$$

The sharpness of this bound will be discussed at the end of Subsection 3.2. Before, in Subsection 3.1, we determine the neighborhood completeness number for a special class of graphs. This result is used in the following to improve the bound (EH) for $cn(G)$ for arbitrary non-bipartite graphs G.

3.1. A special class of graphs: *l*-cliques with a tail

Definition. For $l \ge 3$ and $s \ge 1$, let K_l^s be the graph (V, E) defined by

$V = \{1, 2, \ldots, l, l+1, \ldots, l+s\}$,

$E = \{\{i, j\} \mid 1 \le i < j \le l\} \cup \{\{l, l+1\}, \{l+1, l+2\}, \ldots, \{l+s-1, l+s\}\}$.

Hence, K_l^s consists of a complete graph K_l with l vertices and a "tail" of length s (cf. Figure 2). We start with a lemma describing several structural properties of $N^k(K_l^s)$, for $l \ge 3$.

We denote by $\langle v_1, v_2, \ldots, v_t\rangle = \langle v_1, v_2, \ldots, v_t\rangle_{N^k(G)}$ the subgraph of $N^k(G)$ induced by the vertices $v_1, v_2, \ldots, v_t \in V(N^k(G))$.

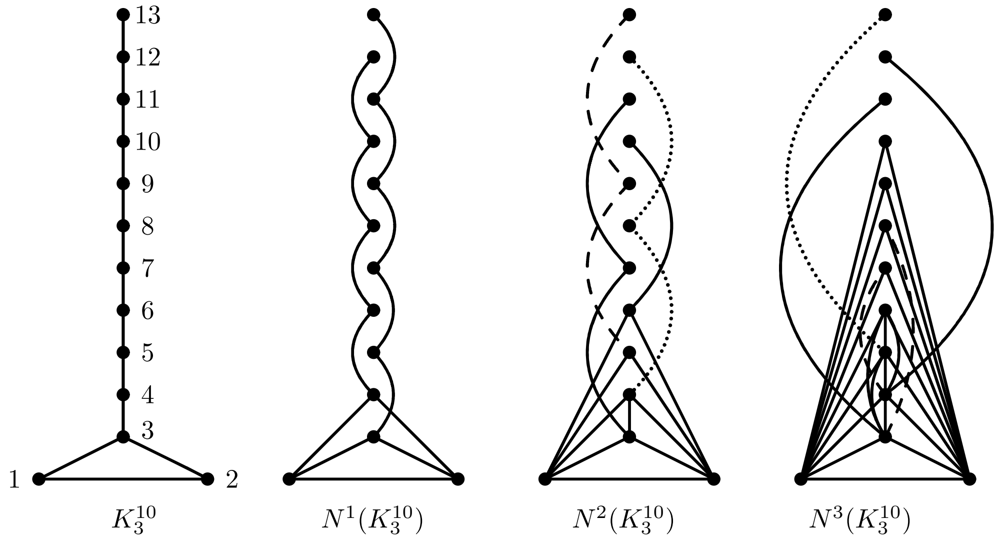


Figure 2. An example to Lemma 4.

Lemma 4. *Let $k, l, s \in I\!N$ with $l \geq 3$ and $s \geq 1$. Then the following hold for $N^k(K_l^s)$:*

(a) *If $2^k - 1 \leq s$, then there are exactly 2^k l-cliques containing the $(l-1)$-clique $\langle 1, 2, \ldots, l{-}1\rangle$, namely $\langle 1, 2, \ldots, l{-}1, l\rangle$, $\langle 1, 2, \ldots, l{-}1, l{+}1\rangle$, $\ldots$, $\langle 1, 2, \ldots, l-1, l+2^k-1\rangle$.*

(b) *If $2^k \leq s$, then all the edges between $\{1, 2, \ldots, l+2^k-1\}$ and $\{l+2^k, l+2^k+1, \ldots, l+s\}$ have the form $\{x, x+2^k\}$.*
These edges exist for all $x \in \{l, l+1, \ldots, l+\min\{2^k-1, s-2^k\}\}$.

(c) *If $2^k - 1 \leq s$, then $\langle l+2^k-1, l+2^k, \ldots, l+s\rangle$ is the union of the vertex disjoint paths $(y, y+2^k, y+2\cdot 2^k, y+3\cdot 2^k, \ldots)$, where $y \in \{l+2^k-1, l+2^k, \ldots, l+\min\{2^{k+1}-2, s-2^k\}\}$.*
(Therefore, these paths contain only edges of the form $\{x, x+2^k\}$, where $x \in \{l+2^k-1, l+2^k, \ldots, l+s-2^k\}$.)

(d) *If $k \geq 1$ and $2^{k-1} - 1 \leq s$, then $\langle 1, 2, \ldots, l+2^{k-1}-1\rangle$ is a maximal clique.*

Before proving Lemma 4, as an example we consider K_3^{10} (cf. Figure 2).

Note that the dashed edges $\{3, 8\}$ and $\{4, 7\}$ in $N^3(K_3^{10})$ (and corresponding edges in $N^k(K_3^{10})$ $(k > 3)$ will be of no account in our investigations. In reference to the Lemma, these edges connect a vertex of the maximum clique of $N^k(K_3^{10})$ (cf. (d)) with a vertex from the set $\{2^{k-1}+l, 2^{k-1}+l+1, \ldots, 2^k+l-1\}$, which is contained in one of the triangles (i.e. l-cliques with $l = 3$, cf. (a)), but not in the maximum clique.

Obviously, in $N^{k+1}(K_3^{10})$ these edges "disappear" since they are included in the maximum clique of $N^{k+1}(K_3^{10})$.

Now we verify Lemma 4 by induction on k:

Proof. Let $n := l + s$.

$k = 0$.

(a) Because $N^0(K_l^s) = K_l^s$ there is exactly $2^0 = 1$ l-clique, namely $\langle 1, 2, \ldots, l\rangle$.
(b) The only edge between $\{1, 2, \ldots, l\}$ and $\{l+1, l+2, \ldots, n\}$ is $\{l, l+1\}$.
(c) $\langle l, l+1, \ldots, n\rangle$ is the path $(l, l+1, \ldots, n)$.
(d) Not applicable.

$k = 1$.

(a) There are $2^1 = 2$ l-cliques: $\langle 1, 2, \ldots, l-1, l\rangle$ and $\langle 1, 2, \ldots, l-1, l+1\rangle$.
(b) The edges between $\{1, 2, \ldots, l+1\}$ and $\{l+2, l+3, \ldots, n\}$ are $\{l, l+2\}$ and $\{l+1, l+3\}$.
(c) $\langle l+1, l+2, \ldots, n\rangle$ is the (disjoint) union of the paths $(l+1, l+3, l+5, \ldots)$ and $(l+2, l+4, l+6, \ldots)$.
(d) $\langle 1, 2, \ldots, l\rangle$ is a maximum — and, therefore, also maximal — clique.

$k \geq 2$.

Induction hypotheses: (a)–(d) are true for all $k' \leq k-1$.

For technical reasons and a better comprehension of the following, we formulate the induction hypotheses for $k' = k-1$ in detail.

In $N^{k-1}(K_l^s)$ it holds:

(a′) If $2^{k-1} + l - 1 \leq n$, then there are exactly 2^{k-1} l-cliques over the $(l-1)$-clique $\langle 1, 2, \ldots, l-1\rangle$, namely $\langle 1, 2, \ldots, l-1, l\rangle, \langle 1, 2, \ldots, l-1, l+1\rangle, \ldots, \langle 1, 2, \ldots, l-1, 2^{k-1}+l-1\rangle$.
(b′) Between $\{1, 2, \ldots, 2^{k-1}+l-1\}$ and $\{2^{k-1}+l, 2^{k-1}+l+1, \ldots, n\}$ there are only edges of the form $\{x, x+2^{k-1}\}$.
These edges exist for all $x \in \{l, l+1, \ldots, \min\{2^{k-1}+l-1, n-2^{k-1}\}\}$.
(c′) $\langle 2^{k-1}+l-1, 2^{k-1}+l, \ldots, n\rangle_{N^{k-1}(K_l^s)}$ is the union of the vertex disjoint paths $(y, y+2^{k-1}, y+2\cdot 2^{k-1}, y+3\cdot 2^{k-1}, \ldots)$, where $y \in \{2^{k-1}+l-1, 2^{k-1}+l, \ldots, \min\{2^k+l-2, n-2^{k-1}\}\}$.
(Therefore, these paths contain only edges of the form $\{x, x+2^{k-1}\}$, where $x \in \{2^{k-1}+l-1, 2^{k-1}+l, \ldots, n-2^{k-1}\}$.)
(d′) If $2^{k-2}+l-1 \leq n$, then $\langle 1, 2, \ldots, 2^{k-2}+l-1\rangle_{N^{k-1}(K_l^s)}$ is a maximal clique.

Induction steps.

At first, we mention the following.

(∘) In $N^k(K_l^s)$, there exist the edges $\{x, x+2^k\}$ for each $x \in \{1, 2, \ldots, n-2^k\}$.

Verification of (∘).
For $x \geq l$, in $N^k(K_l^s)$ the existence of $\{x, x+2^k\}$ follows from the existence of the edges $\{x, x+2^{k-1}\}$, $\{x+2^{k-1}, (x+2^{k-1})+2^{k-1} = x+2^k\}$ in $N^{k-1}(K_l^s)$ (cf. the induction hypotheses (b′), (c′)), since, obviously, x and $x+2^k$ are common neighbors of $x+2^{k-1}$ in $N^{k-1}(K_l^s)$.

For $x \in \{1, 2, \ldots, l-1\}$, additionally to (b′) and (c′) also (a′) is needed to ensure $\{x, x+2^{k-1}\}, \{x+2^{k-1}, x+2^k\} \in E(N^{k-1}(K_l^s))$.

Now we show (a)–(d).

(a) Let $2^k + l - 1 \leq n$. Since the 2^{k-1} l-cliques $\langle 1, 2, \ldots, l-1, l\rangle$, $\langle 1, 2, \ldots, l-1, l+1\rangle, \ldots, \langle 1, 2, \ldots, l-1, 2^{k-1}+l-1\rangle$ from $N^{k-1}(K_l^s)$ (cf. (a′)) are complete subgraphs, they exist also in $N^k(K_l^s)$. Because of (a′) and (∘) in $N^{k-1}(K_l^s)$ each vertex $x \in \{l, l+1, \ldots, 2^{k-1}+l-1\}$ has at least the neighbors $1, 2, \ldots, l-1$ and $x + 2^{k-1}$. Hence, in $N^k(K_l^s)$ there are the l-cliques $\langle 1, 2, \ldots, l-1, 2^{k-1}+l\rangle$, $\langle 1, 2, \ldots, l-1, 2^{k-1}+l+1\rangle, \ldots, \langle 1, 2, \ldots, l-1, 2^k+l-1\rangle$. In $N^k(K_l^s)$, there are no other l-cliques over the $(l-1)$-clique $\langle 1, 2, \ldots, l-1\rangle$, since (a′), (b′) imply that, in $N^{k-1}(K_l^s)$, all neighbors x of the vertices $1, 2, \ldots, l-1$ are contained in $\{1, 2, \ldots, 2^{k-1}+l-1\}$ and, moreover, every vertex $x \in \{1, 2, \ldots, 2^{k-1}+l-1\}$ in the set $\{2^{k-1}+l, 2^{k-1}+l+1, \ldots, n\}$ has only the neighbor $y = x + 2^{k-1}$. Therefore, owing to $y = x + 2^{k-1} \leq 2^{k-1} + 2^{k-1} + l - 1 = 2^k + l - 1$, in $N^k(K_l^s)$, the l-cliques $\langle 1, 2, \ldots, l-1, l\rangle, \langle 1, 2, \ldots, l-1, l+1\rangle, \ldots, \langle 1, 2, \ldots, l-1, 2^k+l-1\rangle$ include all these neighbors y, which are the only possible candidates for building l-cliques containing the vertices $1, 2, \ldots, l-1$. This completes the proof of (a).

(b) Without loss of generality, let $2^k + l \leq n$, otherwise there is nothing to show. Because of (∘) it suffices to show that the edges of the form $\{x, x+2^k\}$, where $x \in \{l, l+1, \ldots, \min\{2^k+l-1, n-2^k\}\}$, are the only edges between the sets $\{1, 2, \ldots, 2^k+l-1\}$ and $\{2^k+l, 2^k+l+1, \ldots, n\}$.

In $N^{k-1}(K_l^s)$, between $z \in \{1, 2, \ldots, 2^{k-1}+l-1\}$ and $\{2^{k-1}+l, 2^{k-1}+l+1, \ldots, n\}$ there are only edges of the form $\{z, z+2^{k-1}\}$ (cf. (b′)). This implies, for the end vertices of such edges, $z \in \{l, l+1, \ldots, 2^{k-1}+l-1\}$ and $z + 2^{k-1} \in \{2^{k-1}+l, 2^{k-1}+l+1, \ldots, 2^k+l-1\}$.

Now let $x+2^k \in \{2^k+l, 2^k+l+1, \ldots, n\}$ with $x \in \{l, l+1, \ldots, 2^k+l-1\}$ and assume $y \in \{1, 2, \ldots, 2^k+l-1\} \setminus \{x\}$ is another neighbor of $x+2^k$ in $N^k(K_l^s)$. Then, in $N^{k-1}(K_l^s)$, there are vertices z and z' such that z is a common neighbor of x and $x+2^k$, as well as z' is a common neighbor of y and $x+2^k$. Clearly, $x + 2^k > 2^{k-1} + l - 1$ and, consequently, owing to (b′) and (c′) this implies $z = x+2^k-2^{k-1}$ or $z = x+2^k+2^{k-1}$. Since z is also a neighbor of x in $N^{k-1}(K_l^s)$, the only possibility is $z = x+2^k-2^{k-1} = x+2^{k-1} \in \{2^{k-1}+l, 2^{k-1}+l+1, \ldots, n\}$.

Analogously, we obtain $z' = x + 2^{k-1}$. Consequently, $z = z' = x + 2^{k-1}$ has the three pairwise distinct neighbors $x, y, x+2^k$ in $N^{k-1}(K_l^s)$, in contradiction to $z \geq 2^{k-1} + l$ and (b′) and (c′), what excludes other neighbors than $z - 2^{k-1}$,

$z + 2^{k-1}$. Thus (b) holds.

(c) Due to ($\circ$), the existence (and, obviously, the disjointness) of the paths $(y, y + 2^k, y + 2 \cdot 2^k, y + 3 \cdot 2^k, \ldots)$ is clear, for all $y \in \{2^k + l - 1, 2^k + l, \ldots, \min\{2^{k+1} + l - 2, n - 2^k\}\}$.

Assume, there are $x, x' \in \{2^k + l - 1, 2^k + l, \ldots, n\}$ with $x < x', x' \neq x + 2^k$, and $\{x, x'\} \in E(N^k(K_l^s))$. Then, in $N^{k-1}(K_l^s)$, there must be a common neighbor z of x and x'.

If $z \leq 2^{k-1} + l - 1$, then (because of (b$'$)) the only edge in $N^{k-1}(K_l^s)$ between z and vertices in $\{2^{k-1} + l, 2^{k-1} + l + 1, \ldots, n\}$ is the edge $\{z, z + 2^{k-1}\}$. This implies the contradiction $x = z + 2^{k-1} = x'$.

If $z > 2^{k-1} + l - 1$, then (because of (b$'$) and (c$'$)) $x < x'$ induces $x = z - 2^{k-1}$ and $x' = z + 2^{k-1}$ and, therefore, $x' = x + 2 \cdot 2^{k-1} = x + 2^k$ incompatible with the assumption.

(d) Let $2^{k-1} + l - 1 \leq n$. In $N^{k-1}(K_l^s)$ the vertices $2, 3, \ldots, 2^{k-1} + l - 1$ are common neighbors of 1 (because of (a$'$)). Hence, $\langle 2, 3, 4, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is a clique. Analogously, we obtain that $\langle 1, 3, 4, 5, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is a clique. Because, in $N^{k-1}(K_l^s)$, the vertex 3 is a common neighbor of the vertices 1 and 2, it follows $\{1, 2\} \in E(N^k(K_l^s))$, and $\langle 1, 2, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is a clique.

Assume, the clique $\langle 1, 2, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is not maximal.

In $N^k(K_l^s)$, let $z \geq 2^{k-1} + l$ be the smallest vertex being adjacent to all vertices $x \in \{1, 2, \ldots, 2^{k-1} + l - 1\}$.

In $N^{k-1}(K_l^s)$, it follows that z has to have a common neighbor with every vertex $x \in \{1, 2, \ldots, 2^{k-1} + l - 1\}$. The induction hypotheses (b$'$) and (c$'$) imply that there are at most two neighbors of z in $N^{k-1}(K_l^s)$, namely $z - 2^{k-1}$ and $z + 2^{k-1}$.

In $N^{k-1}(K_l^s)$, because of (b$'$) and $z + 2^{k-1} > (2^{k-1} + l - 1) + 2^{k-1}$, the vertex $z + 2^{k-1}$ has no neighbor in the set $\{1, 2, \ldots, 2^{k-1} + l - 1\}$. Therefore, $z - 2^{k-1}$ is adjacent to all vertices $x \in \{1, 2, \ldots, 2^{k-1} + l - 1\}$. Since $z - 2^{k-1}$ cannot be adjacent to itself, this implies $z - 2^{k-1} \geq 2^{k-1} + l$. Hence, $z - 2^{k-1} > 2^{k-2} + l - 1$ and $\langle 1, 2, \ldots, 2^{k-2} + l - 1, z - 2^{k-1}\rangle_{N^{k-1}(K_l^s)}$ is a clique in $N^{k-1}(K_l^s)$. This contradicts the maximality of the clique $\langle 1, 2, \ldots, 2^{k-2} + l - 1\rangle_{N^{k-1}(K_l^s)}$ (cf. (d$'$)).

Therefore, the clique $\langle 1, 2, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is maximal and the proof of (d) is complete. ∎

Theorem 5. *For $l \geq 3$ and $s \geq 1$, $cn(K_l^s) = \lceil 1 + \log_2(s + 1)\rceil$.*

Proof. Let $n = l + s$. For $2^{k-1} + l - 1 \leq n$, from part (d) of Lemma 4 it follows that $\langle 1, 2, \ldots, 2^{k-1} + l - 1\rangle_{N^k(K_l^s)}$ is a maximal clique in $N^k(K_l^s)$.

This implies that $N^k(K_l^s)$ is complete if and only if $2^{k-1} + l - 1 \geq n$, which is equivalent to $k - 1 \geq \log_2(n - l + 1) = \log_2(s + 1)$, i.e. $k \geq 1 + \log_2(s + 1)$. Therefore, $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil$. ■

3.2. The general case

In this section, let $G = (V, E)$ be connected, non-bipartite and not an odd cycle. For the first definition we suppose that G contains an l-clique ($l \geq 3$).

Definition. Let K_l be an l-clique ($l \geq 3$) in $G = (V, E)$ and $\mathcal{W} = \{w_1, \ldots, w_q\}$ a system of paths in G such that $V \setminus V(K_l) \subseteq V(\mathcal{W}) := \bigcup_{i=1}^q V(w_i)$ and every path $w_i \in \mathcal{W}$ has exactly one end vertex v_i in common with K_l, for $i \in \{1, \ldots, q\}$. The subgraph $G_{K_l,\mathcal{W}} = K_l \cup w_1 \cup \cdots \cup w_q = (V, E')$ with $V = V(K_l) \cup V(w_1) \cup \cdots \cup V(w_q)$ and $E' = E(K_l) \cup E(w_1) \cup \cdots \cup E(w_q) \subseteq E$ will be referred to as a *K_l-path-covering* of G. The paths $w_1, \ldots, w_q$ are called *tails.*

Note that the tails are not necessarily disjoint. Moreover, they cover all vertices of $G - K_l$ (and, additionally, the end vertices $v_1, \ldots, v_q \in (\bigcup_{i=1}^q V(w_i)) \cap V(K_l))$ but not necessarily all edges of $G - K_l$ (cf. Figure 3).

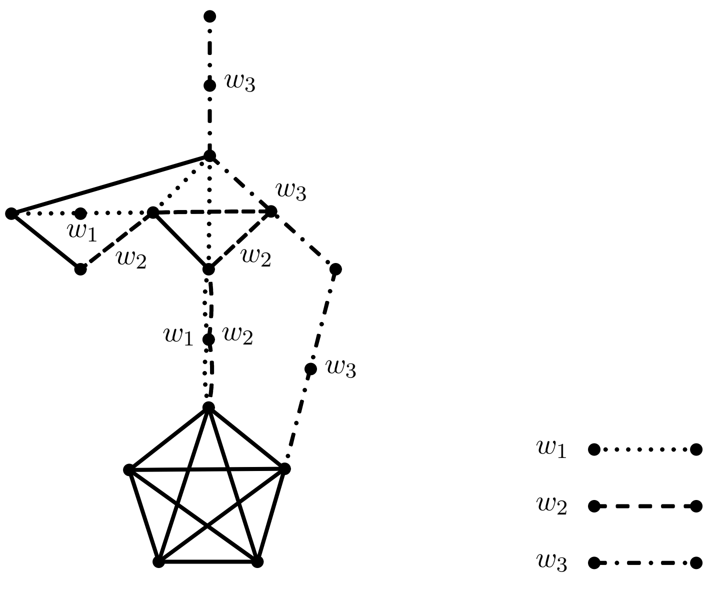


Figure 3. A K_5-path-covering $G_{K_5,\mathcal{W}} = K_5 \cup w_1 \cup w_2 \cup w_3$.

K_l-path-coverings are suitable auxiliaries to give an upper bound for the neighborhood completeness number of arbitrary graphs. In the case of connected graphs containing an l-clique ($l \geq 3$), this upper bound is the same as in the previous subsection.

Obviously, if the connected graph G contains an l-clique K_l ($l \geq 3$), then there is also a K_l-path-covering $G_{K_l,\mathcal{W}}$ in G and vice versa.

Theorem 6. *Let $G_{K_l,\mathcal{W}} = K_l \cup w_1 \cup \cdots \cup w_q$ be a K_l-path-covering of a graph $G = (V, E)$. If s is the maximum length of the tails $w_1, \ldots, w_q$, then $cn(G) \leq \lceil 1 + \log_2(s + 1) \rceil$.*

Proof. It suffices to show that $cn(G_{K_l,\mathcal{W}}) \leq \lceil 1 + \log_2(s + 1) \rceil$.

So let $u, v \in V$ be arbitrary vertices of $G_{K_l,\mathcal{W}}$ and $t := \lceil 1 + \log_2(s + 1) \rceil$. Without loss of generality, let w_x and w_y be tails such that $u \in V(K_l) \cup V(w_x)$ and $v \in V(K_l) \cup V(w_y)$, respectively. (Note that also the special cases $u \in V(K_l) \setminus V(w_x)$ or $v \in V(K_l) \setminus V(w_y)$ or $w_x = w_y$ or $w_x \neq w_y$ and $V(w_x) \cap V(w_y) \neq \emptyset$ are possible.)

Since $K_l \cup w_x \cong K_l^{r_x}$, where $r_x \leq s$ denotes the length of the path w_x, by Theorem 5 it follows that $N^t(K_l \cup w_x)$ is complete. Consequently, due to Lemma 4(a), in $N^{t-1}(K_l \cup w_x)$ the vertex u has at least $l - 1$ neighbors in the vertex set $V(K_l)$. Clearly, the same holds for the vertex v in $N^{t-1}(K_l \cup w_y)$. Because of $l \geq 3$, in $N^{t-1}(K_l \cup w_x \cup w_y)$ the vertices u and v have at least $l - 2 \geq 1$ common neighbors (in $V(K_l)$). Therefore, they are adjacent in $N^t(G_{K_l,\mathcal{W}})$. So $N^t(G_{K_l,\mathcal{W}})$ is complete. ■

To obtain a class of graphs where the bound of Theorem 6 is sharp, we consider graphs $\widehat{G}$ having a K_l-path-covering with a longest tail w_i, such that only the end vertex $v_i \in V(K_l)$ of w_i has neighbors in $V(\widehat{G}) \setminus V(w_i)$; more precisely:

Corollary 7. *Let $\widehat{G}_{K_l,\mathcal{W}} = K_l \cup w_1 \cup \cdots \cup w_q$ be a K_l-path-covering of a graph $\widehat{G} = (V, E)$. If the length of the tail w_1 is equal to the maximum tail length s of $w_1, \ldots, w_q$ and all vertices of $V(w_1) \setminus V(K_l)$ except the end vertex, which has the degree one, have the degree two in $\widehat{G}$, then $cn(\widehat{G}) = \lceil 1 + \log_2(s + 1) \rceil$.*

Proof. If $w_1 = (u_1, u_2, \ldots, u_{s+1})$ and $V(K_l) \cap V(w_1) = \{u_1\}$, then $\widehat{G} = U \cup w_1$, where $U = \langle V(\widehat{G}) \setminus \{u_2, u_3, \ldots, u_{s+1}\} \rangle_{\widehat{G}}$. With $l := |V(\widehat{G})| - s$, the graph $\widehat{G}$ is isomorphic to an edge-deleted subgraph of K_l^s, i.e. to a subgraph containing all $l + s$ vertices of K_l^s. Because of $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil$, $cn(\widehat{G}) \geq cn(K_l^s)$ and Theorem 6 we obtain the assertion. ■

For graphs G containing an l-clique K_l ($l \geq 3$), Theorem 6 gives an upper bound for the neighborhood completeness number $cn(G)$. Now we consider graphs without such cliques. So let G be a triangle-free graph. The basic idea is the following:

Since G is non-bipartite and is not isomorphic to an odd cycle, there must be a vertex $v \in V(G)$ having a degree $d := d(v : G) \geq 3$. The neighborhood $N_G(v)$ of v in G induces a d-clique K_d in the neighborhood graph $N(G)$. Let $N(G)_{K_d,\mathcal{W}} = K_d \cup w_1 \cup \cdots \cup w_q$ be a K_d-path-covering of $N(G)$ and $\hat{s}$ be the maximum tail length of $N(G)_{K_d,\mathcal{W}}$.

Then, owing to Theorem 6,

$$(*) \qquad cn(G) = cn(N(G)) + 1 \le \lceil 1 + \log_2(\hat{s} + 1) \rceil + 1.$$

Following this idea, in Theorem 8 we give a bound for $cn(G)$ which uses only parameters of the graph G, not of its neighborhood graph $N(G)$. First, for a cycle C in G let $l(C)$ be the length of C and $s_{max}(C) := \max\{d_G(C, v) \mid v \in V\}$, where $d_G(C, v) := \min\{d_G(x, v) \mid x \in V(C)\}$, i.e. $s_{max}(C)$ is the maximum distance of any vertex in G from the cycle C.

Theorem 8. *Let $G = (V, E)$ be triangle-free, connected, non-bipartite and not an odd cycle. Moreover, let $s' := \min\left\{\frac{l(C)-1}{2} + \left\lceil \frac{s_{max}(C)}{2} \right\rceil \mid C \text{ is an odd cycle in } G\right\}$. Then, $cn(G) \le \lceil 2 + \log_2(s' + 1) \rceil$.*

Proof. Because of Theorem 6 and $(*)$, it suffices to show that there is a K_d-path-covering $(d \ge 3)$ of $N(G)$ with the maximum tail length $\hat{s} \le s'$.

Let $\widetilde{C}$ be an odd cycle in G such that $s' = \frac{l(\widetilde{C})-1}{2} + \left\lceil \frac{s_{max}(\widetilde{C})}{2} \right\rceil$, where s' is defined as above.

Moreover, let $\mathcal{W}_{\widetilde{C}} = \{\widetilde{w_1}, \dots, \widetilde{w_p}\}$ be a system of paths of length at most $s_{max}(\widetilde{C})$ in G such that $V \setminus V(\widetilde{C}) \subseteq V(\mathcal{W}_{\widetilde{C}}) := \bigcup_{i=1}^{p} V(\widetilde{w_i})$ and every path $\widetilde{w_i} \in \mathcal{W}_{\widetilde{C}}$ has exactly one end vertex v_i in common with $\widetilde{C}$, for $i \in \{1, \dots, p\}$.

In the following, we investigate the subgraph $U := \widetilde{C} \cup \widetilde{w_1} \cup \dots \cup \widetilde{w_p}$ of G. Obviously, it suffices to prove the existence of a K_d-path-covering $(d \ge 3)$ of $N(U)$ with a maximum tail length $\hat{s} \le s'$.

For this end, let $v \in V(\widetilde{C}) \cap V(\widetilde{w_1})$ and $d := d(v : U) \ge 3$ be the degree of v in U.

Furthermore, let $K_d = \langle N_U(v) \rangle_{N(U)}$ be the d-clique induced in the neighborhood graph $N(U)$ by the neighborhood $N_U(v)$ of v in U.

At first we verify that the distance of each vertex $u \in V$ from K_d in $N(U)$ is at most s', i.e.

$$(**) \qquad \hat{s} = \max\{d_{N(U)}(K_d, u) \mid u \in V\} \le s',$$

where $d_{N(U)}(K_d, u) := \min\{d_{N(U)}(x, u) \mid x \in V(K_d)\}$.

Let $v' \in V$ be a vertex with $d_{N(U)}(K_d, v') = \hat{s}$. If $v' \in N_U(v)$, then $d_{N(U)}(K_d, v') = 0$ and there is nothing to prove.

If $v' \in V(\widetilde{C}) \setminus N_U(v)$, then in $\langle V(\widetilde{C}) \rangle_U$ there is path of even length $t \le l(\widetilde{C}) - 1$ from one vertex in $N_U(v) \cap V(\widetilde{C})$ to the vertex v'; therefore $\hat{s} \le \frac{t}{2} \le \frac{l(\widetilde{C})-1}{2} \le s'$.

Now let $v' \in V(\mathcal{W}_{\widetilde{C}}) \setminus (V(\widetilde{C}) \cup N_U(v))$; in detail, let $v' \in V(\widetilde{w_j}) \setminus (V(\widetilde{C}) \cup N_U(v))$, where $j \in \{1, 2, \dots, p\}$.

Then it is easy to see that in U there is a path of (even) length at most $(l(\widetilde{C}) - 1) + l(\widetilde{w_j}) \le (l(\widetilde{C}) - 1) + s_{max}(\widetilde{C})$ from v' to one of the vertices in

$V(\widetilde{C}) \cap N_U(v)$. Therefore, in $N(U)$ there is a path of length at most $\frac{l(\widetilde{C})-1}{2} + \left\lceil \frac{s_{max}(\widetilde{C})}{2} \right\rceil = s'$ from K_d to v' and $(**)$ is true.

Because of $(**)$ in $N(U)$ there exists a system $\mathcal{W} = \{w_1, \ldots, w_q\}$ of paths of maximum length $\hat{s} \le s'$ such that $N(U)_{K_d,\mathcal{W}} = K_d \cup w_1 \cup \cdots \cup w_q$ is a K_d-path-covering of $N(U)$ which has a maximum tail length $\hat{s} \le s'$; this completes the proof. ■

We conjecture that the bound given in Theorem 8 is sharp for many graphs C_q^s consisting of a cycle C of odd length $l(C) = q$ and a tail w of length $l(w) = s$. The computation of $cn(C_q^s)$ for a set of pairs (q, s) lead to

Conjecture 9. *If $q \ge 3$ is odd and $s \ge 1$, then $cn(C_q^s) = \lceil 1 + \log_2(s + q - 2) \rceil$.*

For $q = 3$, Theorem 5 proves the conjecture, because of $K_3^s = C_3^s$ and $n-2 = s+1$. In the case $q > 3$ for C_q^s due to $l(C) = q$ odd and $s_{max}(C) = s$ it follows $s' = \frac{l(C)-1}{2} + \lceil \frac{s_{max}(C)}{2} \rceil = \frac{q-1}{2} + \lceil \frac{s}{2} \rceil$. For s even (i.e. $n = q + s$ odd) we obtain $s' = \frac{q+s-1}{2} = \frac{n-1}{2}$ and for s odd (i.e. n even) $s' = \frac{q+s}{2} = \frac{n}{2}$.
Therefore,

$$\lceil 2 + \log_2(s' + 1) \rceil = \begin{cases} \lceil 2 + \log_2(\frac{n+1}{2}) \rceil & \text{if } n \text{ is odd,} \\ \lceil 2 + \log_2(\frac{n+2}{2}) \rceil & \text{if } n \text{ is even,} \end{cases}$$
$$= \begin{cases} \lceil 1 + \log_2(n + 1) \rceil & \text{if } n \text{ is odd,} \\ \lceil 1 + \log_2(n + 2) \rceil & \text{if } n \text{ is even.} \end{cases}$$

Provided that Conjecture 9 is true, for all odd $q > 3$ and all $s \ge 1$ the bound in Theorem 8 is sharp for C_q^s if and only if

$$\lceil \log_2(n - 2) \rceil = \begin{cases} \lceil \log_2(n + 1) \rceil & \text{if } n \text{ is odd,} \\ \lceil \log_2(n + 2) \rceil & \text{if } n \text{ is even,} \end{cases}$$

where $n = q + s$.

By computer, we verified Conjecture 9 (and, therefore, the sharpness of the bound in Theorem 8) for C_q^s if $q \in \{5, 7, 9, 21\}$ and $s \in \{1, 2, \ldots, 35 - q\}$.
To give one of the examples in detail, consider C_7^4. By computer, we obtained $cn(C_7^4) = 5$ and from $q = 7$, $s = 4$, $n = 11$ it follows $\lceil 1 + \log_2(n - 2) \rceil = \lceil 1 + \log_2(11 - 2) \rceil = 5$ as well as $\lceil 1 + \log_2(n + 1) \rceil = \lceil 1 + \log_2(11 + 1) \rceil = 5$.

We close this subsection with the remark that, for infinitely many graphs, our results are better than the bound (EH) of Exoo and Harary [5] given at the beginning of Section 3. As a first example, consider K_3^{10} (cf. Figure 2). Then Theorem 5 yields $cn(K_3^{10}) = 5$, but from (EH) we would obtain $cn(K_3^{10}) \le \lceil \log_2 10 + 3 \rceil = 7$. As a second example, for C_{21}^4 Theorem 8 provides the bound $cn(C_{21}^4) \le \lceil 2 + \log_2 13 \rceil = 6$, and from (EH) it follows $cn(C_{21}^4) \le \lceil \log_2 4 + 21 \rceil = 23$.

In general, with increasing length of the (odd) cycle considered in the graph, the bound (EH) becomes more blurred.

3.3. Neighborhood completeness number and diameter

We can observe that the diameter $diam(G)$ (the maximum distance between two vertices in the graph G) is closely related to the neighborhood completeness number $cn(G)$. But at least in the class of graphs consisting of a clique K_l ($l \geq 3$) and some vertex disjoint tails, the length s ($s \geq 1$) of a longest tail is a more elegant measure to determine $cn(G)$. For illustration, consider the graph $K_l^{s,s}$ consisting of an l-clique K_l with two (vertex disjoint) tails of length s. Because of $diam(K_l^s) = s+1$ and $diam(K_l^{s,s}) = 2s+1$ Corollary 7 implies

Remark 10. $cn(K_l^s) = \lceil 1+\log_2(diam(K_l^s)) \rceil$ and $cn(K_l^{s,s}) = \lceil \log_2(diam(K_l^{s,s})+1) \rceil$.

Hence, using the diameter, we obtain two different formulas for the neighborhood completeness numbers $cn(K_l^s)$ and $cn(K_l^{s,s})$. By contrast, using the length s of a longest tail as a parameter, we obtain one and the same formula for both types of graphs: Corollary 7 leads to $cn(K_l^s) = \lceil 1 + \log_2(s+1) \rceil = cn(K_l^{s,s})$, since the length of a longest tail is the same (namely s) in both K_l^s and $K_l^{s,s}$.

A recent result of Schweitzer [17] immediately implies

Theorem 11 [17]. *If G is connected, non-bipartite and not an odd cycle, then* $\log_2(diam(G)) \leq cn(G) \leq \lceil 2 + \log_2(diam(G)) \rceil$.

Note that $2+\log_2(diam(G))$ is not an upper bound for $cn(G)$: taking the above example C_7^4 we obtain $diam(C_7^4) = 7$ and $cn(C_7^4) = 5 > 2 + \log_2(7)$.

For special classes of graphs the upper bound in Theorem 11 follows from our results. Additionally to K_l^s and $K_l^{s,s}$ (cf. Remark 10) we mention the following two classes:

(A) Consider the graphs $\widehat{G}$ being investigated in Corollary 7, which have a K_l-path-covering with a longest tail w_1 of length $s = l(w_1)$, such that only the end vertex $v_1 \in V(K_l)$ of w_1 has neighbors in $V(\widehat{G}) \setminus V(w_1)$. The diameter of such a graph is at least $s+1$, consequently $cn(\widehat{G}) = \lceil 1 + \log_2(s+1) \rceil < \lceil 2 + \log_2(diam(\widehat{G})) \rceil$.

(B) Similarly, using Theorem 8 we obtain a corresponding result for certain triangle-free, connected, non-bipartite graphs being no odd cycles.

Let G be a unicyclic graph consisting of a cycle C of odd length $q > 3$ and several trees (one with at least two vertices), where each of the trees has exactly one end vertex in common with C.

Moreover, let $\mathcal{W}_C = \{w_1, \ldots, w_p\}$ be a system of paths of length at most $s := s_{max}(C)$ in G such that $V \setminus V(C) \subseteq V(\mathcal{W}_C) := \bigcup_{i=1}^{p} V(w_i)$ and every path $w_i \in \mathcal{W}_C$ has exactly one end vertex v_i in common with C, for $i \in \{1, \ldots, p\}$. Since at least one of the trees in G is nontrivial, $s \geq 2$ is valid.

Then $diam(G) \geq \frac{q-1}{2} + s > \frac{q-1}{2} + \lceil \frac{s}{2} \rceil = \frac{l(C)-1}{2} + \left\lceil \frac{s_{max}(C)}{2} \right\rceil = s'$. Theorem 8 implies $cn(G) \leq \lceil 2 + \log_2(s'+1) \rceil \leq \lceil 2 + \log_2(diam(G)) \rceil$.

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