A CHARACTERIZATION OF COMPLETE TRIPARTITE DEGREE-MAGIC GRAPHS

LUDMILA BEZEGOVÁ AND JAROSLAV IVANČO

Institute of Mathematics,
P. J. Šafárik University, Jesenná 5,
040 01 Košice, Slovakia

e-mail: ludmila.bezegova@student.upjs.sk
jaroslav.ivanco@upjs.sk

Abstract

A graph is called degree-magic if it admits a labelling of the edges by integers 1, 2, . . . , |E(G)| such that the sum of the labels of the edges incident with any vertex v is equal to \(1 + |E(G)| \deg(v)\). Degree-magic graphs extend supermagic regular graphs. In this paper we characterize complete tripartite degree-magic graphs.

Keywords: supermagic graphs, degree-magic graphs, complete tripartite graphs.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If \(G\) is a graph, then \(V(G)\) and \(E(G)\) stand for the vertex set and the edge set of \(G\), respectively. Cardinalities of these sets are called the order and size of \(G\).

Let a graph \(G\) and a mapping \(f\) from \(E(G)\) into positive integers be given. The index mapping of \(f\) is the mapping \(f^*\) from \(V(G)\) into positive integers defined by

\[
f^*(v) = \sum_{e \in E(G)} \eta(v, e)f(e) \quad \text{for every } v \in V(G),
\]

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0023-10, by the Slovak VEGA Grant 1/0428/10 and VVGS Grant No. 617-B(I-10-032-00).
where $\eta(v,e)$ is equal to 1 when $e$ is an edge incident with a vertex $v$, and 0 otherwise. An injective mapping $f$ from $E(G)$ into positive integers is called a magic labelling of $G$ for an index $\lambda$ if its index mapping $f^*$ satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling $f$ of a graph $G$ is called a supermagic labelling if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph $G$ is supermagic (magic) whenever there exists a supermagic (magic) labelling of $G$.

A bijection $f$ from $E(G)$ into $\{1, 2, \ldots, |E(G)|\}$ is called a degree-magic labelling (or only d-magic labelling) of a graph $G$ if its index mapping $f^*$ satisfies

$$f^*(v) = 1 + \frac{|E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G).$$

A d-magic labelling $f$ of a graph $G$ is called balanced if for all $v \in V(G)$ it holds

$$|\{e \in E(G) : \eta(v,e) = 1, f(e) \leq \frac{|E(G)|}{2}\}| = |\{e \in E(G) : \eta(v,e) = 1, f(e) > \frac{|E(G)|}{2}\}|.$$

We say that a graph $G$ is degree-magic (balanced degree-magic) (or only d-magic) when there exists a d-magic (balanced d-magic) labelling of $G$.

The concept of magic graphs was introduced by Sedláček [7]. Supermagic graphs were introduced by M.B. Stewart [8]. There is by now a considerable number of papers published on magic and supermagic graphs; we refer the reader to [4] for comprehensive references. The concept of degree-magic graphs was introduced in [1] as some extension of supermagic regular graphs. Basic properties of degree-magic graphs were also established in [1]. Let us recall those, which we shall use hereinafter.

**Theorem 1.** Let $G$ be a regular graph. Then $G$ is supermagic if and only if it is degree-magic.

**Theorem 2.** Let $G$ be a d-magic graph of even size. Then every vertex of $G$ has an even degree and every component of $G$ has an even size.

**Theorem 3.** Let $H_1$ and $H_2$ be edge-disjoint subgraphs of a graph $G$ which form its decomposition. If $H_1$ is d-magic and $H_2$ is balanced d-magic then $G$ is a d-magic graph. Moreover, if $H_1$ and $H_2$ are both balanced d-magic then $G$ is a balanced d-magic graph.

A complete $k$-partite graph is a graph whose vertices can be partitioned into $k \geq 2$ disjoint classes $V_1, \ldots, V_k$ such that two vertices are adjacent whenever they belong to distinct classes. If $|V_i| = n_i$, $i = 1, \ldots, k$, then the complete $k$-partite graph is denoted by $K_{n_1, \ldots, n_k}$.

Stewart [9] characterized supermagic complete graphs. Supermagic regular complete multipartite graphs were characterized in [6]. Thus, according to Theorem
1, degree-magic regular complete multipartite graphs are characterized as well. All balanced d-magic complete multipartite graphs are characterized in [2]. In particular for the complete bipartite graphs we have

**Theorem 4** [1]. The complete bipartite graph $K_{m,n}$ is balanced d-magic if and only if the following statements hold:

(i) $m \equiv n \equiv 0 \pmod{2}$,
(ii) if $m \equiv n \equiv 2 \pmod{4}$, then $\min\{m, n\} \geq 6$.

The complete bipartite graph $K_{m,n}$ is d-magic if and only if there exists a magic $(m, n)$-rectangle (see [1] for details). Thus, the known result on magic rectangles (e.g., Theorem 1 in [5] or Theorem 2 in [3]) can be rewritten as follows.

**Theorem 5.** The complete bipartite graph $K_{m,n}$, for $m \geq n$, is d-magic if and only if the following statements hold:

(i) $m \equiv n \pmod{2}$,
(ii) if $n = 2$ then $m > 2$,
(iii) if $n = 1$ then $m = 1$.

The problem of characterizing d-magic complete multipartite graphs seems to be difficult. It is solved in this paper for complete tripartite graphs.

## 2. Complete Tripartite Graphs

First we present some sufficient conditions for complete tripartite graphs to possess the d-magic property.

**Lemma 1.** Let $m$, $n$ and $o$ be even positive integers. Then the complete tripartite graph $K_{m,n,o}$ is balanced d-magic.

**Proof.** Suppose that $m \geq n \geq o$ and consider the following cases.

**Case A.** Let $o > 2$, or $n > o = 2$ and $m + n \equiv 0 \pmod{4}$. Evidently, the graph $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m,n}$ and $K_{m+n,o}$. According to Theorem 4, both of these subgraphs are balanced d-magic. Thus, by Theorem 3, $K_{m,n,o}$ is balanced d-magic, too.

**Case B.** Let $n > o = 2$ and $m + n \not\equiv 0 \pmod{4}$. In this case we have either $m \equiv 0 \pmod{4}$, or $n \equiv 0 \pmod{4}$. Without loss of generality, assume that $m \equiv 0 \pmod{4}$. The graph $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m,o}$ and $K_{n,m+n,o}$. By Theorem 4, both of these subgraphs are balanced d-magic. Therefore, $K_{m,n,o}$ is balanced d-magic because of Theorem 3.
Case C. Let \( n = o = 2 \). A balanced d-magic labelling of \( K_{2,2,2} \) is given in Figure 1. Thus, \( K_{2,2,2} \) is balanced d-magic. If \( m > 2 \), then the graph \( K_{m,n,o} \) is decomposable into edge-disjoint subgraphs isomorphic to \( K_{2,n,o} \) and \( K_{m-2,n+o} \). As \( K_{2,2,2} \) and \( K_{m-2,4} \) are balanced d-magic, \( K_{m,n,o} \) is balanced d-magic by Theorem 3.

Lemma 2. Let \( m \geq n \geq o \) be odd positive integers such that \( m \equiv 3 \pmod{4} \) whenever \( n = 1 \). Then the complete tripartite graph \( K_{m,n,o} \) is d-magic.

Proof. Let us assume to the contrary that \( K_{m,n,o} \) (where \( m \geq n \geq o \) are odd positive integers such that \( m \equiv 3 \pmod{4} \) whenever \( n = 1 \)) is a complete tripartite graph with a minimum number of vertices which is not d-magic. Consider the following cases.

Case A. \( n = 1 \). Then \( o = 1 \) and \( m \equiv 3 \pmod{4} \) in this case. If \( m > 3 \) then \( K_{m,n,o} \) is decomposable into edge-disjoint subgraphs isomorphic to \( K_{m-4,n,o} \) and \( K_{4,n+o} \). By the minimality of \( K_{m,n,o} \), the graph \( K_{m-4,n,o} \) is d-magic and according to Theorem 4, \( K_{4,2} \) is balanced d-magic. Thus, by Theorem 3, \( K_{m,n,o} \) is d-magic, contrary to the choice of \( K_{m,n,o} \). Therefore, \( m = 3 \). However, \( K_{3,1,1} \) admits a d-magic labelling (see Figure 2) and so it is d-magic, a contradiction.

Figure 2. Degree-magic labelling of \( K_{3,1,1} \)

Case B. \( o = 1 \) and \( n = 3 \). As \( m \geq n \), the graph \( K_{m,n,o} \) is decomposable into subgraphs isomorphic to \( K_{m-2,n,o} \) and \( K_{2,n+o} \). By the minimality of \( K_{m,n,o} \), the
A Characterization of Complete Tripartite ...  

graph $K_{m-2,n,o}$ is $d$-magic and according to Theorem 4, $K_{2,4}$ is balanced $d$-magic. Thus, by Theorem 3, $K_{m,n,o}$ is $d$-magic, a contradiction.

Thus, by Theorem 3, $K_{m,n,o}$ is $d$-magic, a contradiction.

Figure 3. Degree-magic labelling of $G_1$

Case C. $o = 1$ and $n > 3$. If $m > 5$ then $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n,o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is $d$-magic and by Theorem 4, $K_{4,n,o}$ is balanced $d$-magic. According to Theorem 3, $K_{m,n,o}$ is $d$-magic, a contradiction. Therefore, $m = n = 5$. The graph $K_{5,5,1}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{4,4}$ and $G_1$ which is depicted in Figure 3. The graph $K_{4,4}$ is balanced $d$-magic by Theorem 4 and $G_1$ is $d$-magic (see Figure 3). Thus, using Theorem 3, $K_{5,5,1}$ is $d$-magic, a contradiction.

Figure 4. Degree-magic labelling of $G_2$

Case D. $o > 1$. If $m > 3$ then $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n,o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is $d$-magic and by Theorem 4, $K_{4,n,o}$ is balanced $d$-magic. According to Theorem 3, $K_{m,n,o}$ is $d$-magic, a contradiction. Therefore, $m = n = o = 3$. The graph $K_{3,3,3}$ is decomposable into subgraphs isomorphic to $K_{2,2,2}$ and $G_2$ which is depicted in Figure 4. The graph $K_{2,2,2}$ is balanced $d$-magic by Lemma 1 and $G_2$ is $d$-magic (see Figure 4). Thus by Theorem 3, $K_{3,3,3}$ is $d$-magic, a contradiction.
Lemma 3. Let $n \geq o$ be odd positive integers and let $m$ be an even positive integer such that $m \equiv 0 \pmod{4}$ whenever $n = 1$. Then the complete tripartite graph $K_{m,n,o}$ is d-magic.

Proof. Let us assume to the contrary that $K_{m,n,o}$ (where $n \geq o$ are odd positive integers and $m$ is an even positive integer such that $m \equiv 0 \pmod{4}$ whenever $n = 1$) is a complete tripartite graph with a minimum number of vertices which is not d-magic. Consider the following cases.

Case A. $m > 4$. The graph $K_{m,n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{m-4,n,o}$ and $K_{4,n,o}$. By the minimality of $K_{m,n,o}$, the graph $K_{m-4,n,o}$ is d-magic and by Theorem 4, $K_{4,n,o}$ is balanced d-magic. According to Theorem 3, $K_{m,n,o}$ is d-magic, contrary to the choice of $m,n,o$.

Case B. $m = 4$. The graph $K_{m,n,o}$ is decomposable into subgraphs isomorphic to $K_{m,n+o}$ and $K_{n,o}$. Thus, if $n = 1$ or $o > 1$, then by Theorems 4, 5 and 3, $K_{m,n,o}$ is d-magic, a contradiction. Therefore, $o = 1$ and $n > 1$. $K_{m,n,o}$ can be decomposed into subgraphs isomorphic to $K_{m-2,n,o}$ and $K_{2,n+o}$. If $n \equiv 3 \pmod{4}$, then, according to the minimality of $K_{m,n,o}$ and Theorems 4, 3, the graph $K_{m,n,o}$ is d-magic, a contradiction. So, $1 < n \equiv 1 \pmod{4}$, i.e., there is a positive integer $k$ such that $n = 4k + 1$. Denote the vertices of $K_{4,n,1}$ by $u_1, \ldots, u_4, v_1, \ldots, v_n, w$ in such a way that $\{u_1, \ldots, u_4\}, \{v_1, \ldots, v_n\}$ and $\{w\}$ are its maximal independent sets. Consider the mapping $f : E(K_{4,n,1}) \rightarrow \{1,2,\ldots,5n+4\}$ given by

$$f(u_1v_j) = \begin{cases} 
1 + 2k - \frac{j+1}{2} & \text{if } j < n, \ j \equiv 1 \pmod{2}, \\
10 + 20k - \frac{j}{2} & \text{if } j \equiv 0 \pmod{2}, \\
1 + 3k & \text{if } j = n,
\end{cases}$$

$$f(u_2v_j) = \begin{cases} 
8 + 16k - \frac{j+1}{2} & \text{if } j < n, \ j \equiv 1 \pmod{2}, \\
2 + 4k + \frac{j}{2} & \text{if } j \equiv 0 \pmod{2}, \\
7 + 13k & \text{if } j = n,
\end{cases}$$

$$f(u_3v_j) = \begin{cases} 
8 + 16k & \text{if } j = 1, \\
10 + 18k - \frac{j-1}{2} & \text{if } 1 < j \leq 1 + 2k, \ j \equiv 1 \pmod{2}, \\
9 + 18k - \frac{j-1}{2} & \text{if } j > 1 + 2k, \ j \equiv 1 \pmod{2}, \\
2 + 4k - \frac{j}{2} & \text{if } j < 2k, \ j \equiv 0 \pmod{2}, \\
1 + 4k - \frac{j}{2} & \text{if } j > 2k, \ j \equiv 0 \pmod{2}, \\
4 + 8k & \text{if } j = 1, \\
2 + 6k + \frac{j-1}{2} & \text{if } 1 < j \leq 1 + 2k, \ j \equiv 1 \pmod{2}, \\
3 + 6k + \frac{j-1}{2} & \text{if } j > 1 + 2k, \ j \equiv 1 \pmod{2}, \\
8 + 14k - \frac{j}{2} & \text{if } j < 2k, \ j \equiv 0 \pmod{2}, \\
7 + 14k - \frac{j}{2} & \text{if } j > 2k, \ j \equiv 0 \pmod{2}, 
\end{cases}$$

$$f(u_4v_j) = \begin{cases} 
3 + 6k + \frac{j-1}{2} & \text{if } j > 1 + 2k, \ j \equiv 1 \pmod{2},
\end{cases}$$
If it is easy to see that $h: G \rightarrow \text{edge-disjoint subgraphs}$ such that $i_j \in \{1, 2, \ldots, n\}$, then denote the vertices of $G$ by $i_j = 1$ is isomorphic to $K_2$. Thus, $K_{n,o}$ is decomposable into edge-disjoint subgraphs isomorphic to $K_{2,n,o}$ and $K_{n,o}$ and so, using Theorem 3, it is $d$-magic, a contradiction. Therefore, $n+o \equiv 2 \pmod 4$. As $K_{n,o}$ is $d$-magic, there is its $d$-magic labelling $g: E(K_{n,o}) \rightarrow \{1, 2, \ldots, \varepsilon\}$, where $\varepsilon = no$ is its number of edges. Suppose that $e', e^*$ are edges of $K_{n,o}$ such that $g(e') = 1$ and $g(e^*) = \varepsilon$. Consider the following subcases.

Subcase C1. If $e'$ and $e^*$ are adjacent edges (note that $n = o = 3$ belongs to this subcase), then denote the vertices of $K_{2,n,o}$ by $u_1, u_2, v_1, v_2, \ldots, v_{n+o}$ in such a way that $\{u_1, u_2\}$ is its maximal independent set, the subgraph $K_{n,o}$ is induced by $\{v_1, \ldots, v_{n+o}\}$ and $e' = v_1v_3$, $e^* = v_2v_3$. The graph $K_{2,n,o}$ is decomposable into edge-disjoint subgraphs $G_3$ (induced by $\{u_iv_j : i \in \{1, 2\}, j \in \{7, \ldots, n+o\}\}$, if $n+o > 6$) and $G_4$ (induced by remaining edges). Evidently, if $n+o > 6$ then $G_3$ is isomorphic to $K_{2,n+o-6}$, and by Theorem 4, it is balanced $d$-magic. Consider the mapping $h_1: E(G_4) \rightarrow \{1, 2, \ldots, \varepsilon + 12\}$ given by

$$h_1(e) = \begin{cases} 6 + g(e) & \text{if } e \in E(K_{n,o}) - \{e', e^*\}, \\ 6 & \text{if } e = e', \\ 7 + \varepsilon & \text{if } e = e^*, \end{cases}$$

and the values of edges $u_iv_j$ are described in the following matrix

$$
\begin{array}{cccccccc}
    & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
\hline
h_1(u_1) & \varepsilon + 9 & \varepsilon + 8 & 7 & 1 & \varepsilon + 11 & 3 \\
h_1(u_2) & 5 & 4 & \varepsilon + 6 & \varepsilon + 12 & 2 & \varepsilon + 10 \\
\end{array}
$$

It is easy to see that $h_1$ is a bijection. Since $\deg_{G_4}(v_j) = \deg_{K_{n,o}}(v_j)$, for each $j \in \{7, \ldots, n+o\}$, we have

$$
\begin{align*}
    h_1^*(v_j) &= g^*(v_j) + 6 \deg_{G_4}(v_j) = \frac{14\varepsilon}{2} \deg_{G_4}(v_j) + 6 \deg_{G_4}(v_j) \\
    &= \frac{13\varepsilon}{2} \deg_{G_4}(v_j).
\end{align*}
$$

For $3 \leq j \leq 6$, $\deg_{G_4}(v_j) = 2 + \deg_{K_{n,o}}(v_j)$ and so
Subcase C2. If \( e' \) and \( e^* \) are not adjacent edges \((n + o \geq 10 \text{ in this subcase})\), then denote the vertices of \( K_{2,n,o} \) by \( u_1, u_2, v_1, v_2, \ldots, v_{n+o} \) in such a way that \( \{u_1, u_2\} \) is its maximal independent set, the subgraph \( K_{n,o} \) is induced by \( \{v_1, \ldots, v_{n+o}\} \) and \( e' = v_1v_2, e^* = v_3v_4 \). The graph \( K_{2,n,o} \) is decomposable into edge-disjoint subgraphs \( G_5 \) (induced by \( \{u_iv_j : i \in \{1, 2\}, j \in \{11, \ldots, n+o\}\} \)), if \( n + o > 10 \) and \( G_6 \) (induced by remaining edges). Evidently, if \( n + o > 10 \) then \( G_5 \) is isomorphic to \( K_{2,n+o-10} \), and by Theorem 4, it is balanced d-magic. Consider the mapping \( h_2 : E(G_6) \rightarrow \{1, 2, \ldots, \varepsilon + 20\} \) given by

\[
h_2(e) = \begin{cases} 
10 + g(e) & \text{if } e \in E(K_{n,o}) - \{e', e^*\}, \\
10 & \text{if } e = e', \\
11 + \varepsilon & \text{if } e = e^*, 
\end{cases}
\]

and the values of edges \( u_iv_j \) are described in the following matrix:

<table>
<thead>
<tr>
<th>( h_1(u_iv_j) )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>( \varepsilon + 19 )</td>
<td>3</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>5</td>
<td>( \varepsilon + 17 )</td>
</tr>
<tr>
<td>( v_3 )</td>
<td>( \varepsilon + 18 )</td>
<td>2</td>
</tr>
<tr>
<td>( v_4 )</td>
<td>4</td>
<td>( \varepsilon + 16 )</td>
</tr>
<tr>
<td>( v_5 )</td>
<td>1</td>
<td>( \varepsilon + 20 )</td>
</tr>
<tr>
<td>( v_6 )</td>
<td>( \varepsilon + 15 )</td>
<td>6</td>
</tr>
<tr>
<td>( v_7 )</td>
<td>7</td>
<td>( \varepsilon + 14 )</td>
</tr>
<tr>
<td>( v_8 )</td>
<td>( \varepsilon + 13 )</td>
<td>8</td>
</tr>
<tr>
<td>( v_9 )</td>
<td>( \varepsilon + 12 )</td>
<td>9</td>
</tr>
<tr>
<td>( v_{10} )</td>
<td>11</td>
<td>( \varepsilon + 10 )</td>
</tr>
</tbody>
</table>

Analogously as in the Case C1 it is easy to verify that \( h_2 \) is a d-magic labelling. Thus, \( G_6 \) is a d-magic graph and consequently, the graph \( K_{2,n,o} \) is d-magic, a contradiction.

Case D. \( m = 2 \) and \( o = 1 \). In this case there is a positive integer \( k \) such that \( n = 2k + 1 \). Denote the vertices of \( K_{2,n,1} \) by \( u_0, u_1, u_2, v_{-k}, \ldots, v_k \) in such a way that \( \{u_1, u_2\}, \{v_{-k}, \ldots, v_k\} \) and \( \{u_0\} \) are its maximal independent sets.
Put \( r = \left\lceil \frac{2k}{3} \right\rceil \) (note that \( 3r - 2k \in \{0, 1, 2\} \)) and define

\[
R = \begin{cases}
\{0, 1\} & \text{if } k = 1, \\
\{0, k\} & \text{if } k \text{ is even,} \\
\{0, r\} & \text{if } k > 1 \text{ is odd and } 3r - 2k \neq 1, \\
\{0, r, k\} & \text{if } k > 1 \text{ is odd and } 3r - 2k = 1.
\end{cases}
\]

Let \( P \) and \( Q \) be disjoint subsets of the set \( \{0, 1, \ldots, k\} - R \) such that

\[
P \cup Q \cup R = \{0, 1, \ldots, k\} \quad \text{and} \quad 0 \leq |P| - |Q| \leq 1.
\]

Consider the mapping \( \xi : E(K_{2,n,1}) \to \{1, 2, \ldots, 6k + 5\} \) given by

\[
\xi(u_0v_1) = 6k + 5, \quad \xi(u_0u_2) = 1,
\]

\[
\xi(u_jv_i) = \begin{cases}
3k + 3 + i & \text{if } j = 0, i \in P \cup Q, \\
i + 2 & \text{if } j = 1, i \in P \text{ or } j = 2, i \in Q, \\
6k + 4 - 2i & \text{if } j = 2, i \in P \text{ or } j = 1, i \in Q,
\end{cases}
\]

\[
\xi(u_jv_{-i}) = \begin{cases}
3k + 3 - i & \text{if } j = 0, i \in P \cup Q, \\
2k + 3 - i & \text{if } j = 1, i \in P \text{ or } j = 2, i \in Q, \\
4k + 3 + 2i & \text{if } j = 2, i \in P \text{ or } j = 1, i \in Q,
\end{cases}
\]

and the values of edges \( u_jv_i, |i| \in R, \) are described in the following matrices:

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_1 & v_{-1} & \\
u_0 & 10 & 3 & 5 \end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_k & v_{-k} & \\
v_0 & 6k + 4 & k + 2 & 2k + 3
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_1 & v_{-1} & \\
u_1 & 2 & 7 & 4
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_k & v_{-k} & \\
v_0 & 6k + 4 & k + 2 & 2k + 3
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_k & v_{-k} & \\
u_0 & 6k + 4 & 6k + 4 - 2r & 2k + 3 - r
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_{-1} & \\
u_0 & 3k + 3 & 3k + 3 + r & 3k + 3 - r
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_1 & v_0 & v_r & v_{-r} & \\
u_1 & 2 & r + 2 & 4k + 3 + 2r
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_1 & v_0 & v_{-1} & \\
u_1 & 6k + 4 & 6k + 4 - 2r & 2k + 3 - r
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_r & v_{-r} & \\
u_0 & 6k + 4 & r + 2 & 3k + 3 - r
\end{array}
\]

\[
\xi(u_jv_i) = \begin{array}{cccc}
u_0 & v_0 & v_{-1} & \\
u_0 & 6k + 4 & r + 2 & 3k + 3 - r
\end{array}
\]

As \( \bigcup_{j=0}^n \{\xi(u_jv_i)\} = \{i + 2, 3k + 3 + i, 6k + 4 - 2i\}, \) for \( 0 \leq i \leq k, \) and

\[
\bigcup_{j=0}^2 \{\xi(u_jv_{-i})\} = \{2k + 3 - i, 3k + 3 - i, 4k + 3 + 2i\}, \text{ for } 1 \leq i \leq k, \text{ it is not}.
\]
Therefore, \( \xi_j \in \{\xi_0, \ldots, \xi_k\} \) and consequently, \( \sum_{i} \alpha_i = 0 \) for each \( i \). Moreover, \( \sum_{i} \beta_i = 0 \) for each \( i \). Thus, \( \sum_{i} (\alpha_i + \beta_i) = 0 \) for each \( i \).

Moreover, \( K_{m,n,o} \) denote by \( f \). Clearly, \( f \) is a \( d \)-magic labelling, a contradiction.

Proof. Denote the vertices of \( K_{m,n,o} \) by \( u_1, \ldots, u_m, v, w \) in such a way that \( \{u_1, \ldots, u_m\}, \{v\} \) and \( \{w\} \) are its maximal independent sets. The size of \( K_{m,n,o} \) denote by \( q \). Evidently, \( q = 2m + 1 \). Suppose that \( f \) is a \( d \)-magic labelling of \( K_{m,n,o} \). Then,

\[
(1 + q)(1 + m) = f^*(v) + f^*(w) = (1 + 2 + \cdots + q) + f vw,
\]

and consequently, \( f vw = \frac{1 + q}{2} = 1 + m \). Put \( A := \{i : f(u_i) \leq m\} \) and \( B := \{i : f(wu_i) \leq m\} \). Clearly, \( A \cap B = \emptyset \) and \( A \cup B = \{1, 2, \ldots, m\} \), because \( f(v, u_i) + f(w, u_i) = f^*(u_i) = 1 + q \) for each \( i \in \{1, \ldots, m\} \). Thus,

\[
\sum_{i \in A} f vu_i + \sum_{i \in B} f vu_i = f^*(v) - f(vw) = \frac{1 + q}{2} (1 + m) - \frac{1 + q}{2} = (1 + m)m.
\]

Consequently,

\[
(1 + m)m = \sum_{i \in A} f vu_i + \sum_{i \in B} f vu_i = \sum_{i \in A} f(vu_i) + \sum_{i \in B} f(vu_i) + \sum_{i \in B} (1 + q - f(wu_i))
\]

\[
= \sum_{i \in A} f vu_i - \sum_{i \in B} f vu_i + |B|(1 + q).
\]

Thus, \( \sum_{i \in A} f vu_i \equiv \sum_{i \in B} f(wu_i) \) (mod 2), because \( (1 + m)m \) and \( 1 + q \) are even integers. This implies that \( \sum_{i \in A} f vu_i + \sum_{i \in B} f(wu_i) \) is an even integer. However, \( \sum_{i \in A} f vu_i + \sum_{i \in B} f(wu_i) = 1 + 2 + \cdots + m = \frac{m^2}{2} (1 + m) \), and it is even only for \( m \equiv 0 \) (mod 4) or \( m \equiv 3 \) (mod 4).
Suppose that two integers of \( \{m, n, o\} \) are even and the third is odd. In this case the graph \( K_{m,n,o} \) has an even number of edges and it contains some vertices of odd degree. According to Theorem 2, \( K_{m,n,o} \) is not a \( d \)-magic graph. This proves that condition (ii) holds.

On the other hand, if conditions (i) and (ii) are satisfied then the complete tripartite graph \( K_{m,n,o} \) is \( d \)-magic by Lemmas 1, 2 and 3.

References


Received 14 December 2010
Revised 7 April 2011
Accepted 28 April 2011