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PLANAR GRAPHS WITHOUT 4-, 5- AND 8-CYCLES ARE 3-COLORABLE

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Abstract

In this paper we prove that every planar graph without 4, 5 and 8-cycles is 3-colorable.

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1. Introduction

In 1959 Grötsch [9] proved that every planar graph without 3-cycles is 3-colorable. In 1976 Steinberg [12] conjectured that every planar graph without 4- and 5-cycles is 3-colorable. In fact, there exist 4-critical planar graphs which have only 4-cycles but no 5-cycles or only 5-cycles but no 4-cycles [1]. In 1990, Erdös proposed the following relaxed conjecture: every planar graph without cycles of size $\{4, 5, \dots, k\}, k \geq 5$, is 3-colorable. Abbott and Zhou [1] proved that the above conjecture holds for $k = 11$. Borodin [3] improved the result by showing that the result holds for $k = 10$. Borodin [2] and Sanders and Zhao [11] further improved the result showing that $k = 9$. To date, the best known result is by Borodin *et al.* [4], where it is shown that any planar graph without cycles of length in $\{4, 5, 6, 7\}$ is 3-colorable. Xiaofang, Chen and Wang [14] showed that a planar graph without cycles of length $4, 6, 7$ and 8 is 3-colorable. Chen, Raspaud and Wang [8] showed that a planar graph without cycles of length $4, 6, 7$ and 9 is 3-colorable. Zhang and Wu [16] showed that every planar graph without $4, 5, 6$ and 9-cycles is 3-choosable. Wang and Chen [13] proved that every planar graph without $4, 6$, and 8-cycles is 3-colorable. In this article, we show that the result holds true for planar graphs without cycles of length in $\{4, 5, 8\}$.

Another problem somewhat related to Steinberg's conjecture is the Havel's conjecture [10]. In 1969, Havel [10] posed the following problem: Does there exist a constant d such that every planar graph with the minimum distance between triangles at least d is 3-colorable? Some of the recent results on Havel's problems are that every planar graph without 3-cycles at distance less than d and without 5-cycles is 3-colorable ($d = 4$ [6] and $d = 3$ [5, 15]). Borodin *et al.* [7] proved that a planar graph without adjacent triangles and without 5- and 7-cycles is 3-colorable. In this paper, we intend to prove the following result:

Theorem 1. *Every planar graph without* 4-*,* 5- *and* 8-*cycles is* 3-*colorable.*

We use $\mathcal{G}$ to denote the class of planar graphs without 4-, 5-, and 8-cycles. Let C_i denote an i-cycle. A 9- or a 12-cycle is bad if the subgraph inside the cycle has a partition into 6- and 3-cycles. We call a cycle of length $\{3, 6, 7, 9, 10, 11, 12\}$ that is not bad a good cycle. We would prove a stronger version of Theorem 1 as given below:

Theorem 2. *Let G be a graph in $\mathcal{G}$. Let D be an arbitrary good cycle of G. Then every proper* 3-*coloring of D can be extended to a proper* 3-*coloring of the whole graph G.*

Assuming that Theorem 2 holds, we can easily establish Theorem 1. Suppose $G \in \mathcal{G}$, namely, G contains no 4-, 5- and 8-cycles. We confirm that G contains C_i for some fixed $i \in \{6, 9\}$, or else, G is 3-colorable by the result of [8] or [13]. Suppose that G contains C_6. It is easy to see that C_6 is chordless and has a proper 3-coloring ϕ. By Theorem 2, ϕ can be extended to both inside and outside of C_6 to make a proper 3-coloring of G. If G contains C_9, it is again easy to see that C_9 is chordless and has a proper 3-coloring ϕ. By Theorem 2, ϕ can be extended to both inside and outside of C_9 to make a proper 3-coloring of G.

Only simple graphs are considered in this paper. A plane graph is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its vertices, edges, faces and maximum degree by $V(G)$,

$E(G)$, $F(G)$, and $\Delta(G)$ respectively. We use k-vertex, k^+-vertex, k^--vertex, $> k$-vertex, $< k$-vertex to denote a vertex of degree k, at least k, or at most k, greater than k, less than k respectively. Similarly, we can define k-face, k^+-face, k^--face, $> k$-face, $< k$-face. We say that two cycles or faces are adjacent if they share at least one common edge. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f. If $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in the clockwise order, we write $f = [u_1u_2 \ldots u_n]$. Given two vertices u and v in a cycle C, let $C[u, v]$ denote the path of C in the clockwise order from u to v (including u and v), and let $C(u, v) = C[u, v] \backslash \{u, v\}$. A cycle C in a plane graph G is called separating if $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$, where $int(C)$ and $ext(C)$ represent the sets of vertices located inside and outside C, respectively.

2. Proof of Theorem 2

Assume that G is a minimal (least number of vertices) counterexample to Theorem 2. Without loss of generality, assume that the outside face f_0 is of degree $6, 7, 9, 10, 11$ or 12 such that a proper 3-coloring ϕ of the boundary vertices of f_0 cannot be extended to the whole graph G. This implies that there exists at least one vertex in the interior of $b(f_0)$. In fact, $\Delta(G) \geq 3$ in this case. In the sequel, we write C as the boundary walk of f_0, i.e., $C = b(f_0)$. Other faces in G different from f_0 are called the internal faces. The vertices in C are called the outer vertices and other vertices the internal vertices. An internal 3-vertex incident to a 3-face is called bad.

Claim 1. G does not contain a separating good cycle.

Proof. Suppose that G has such a separating cycle C_i. Then we can extend ϕ to $G - int(C_i)$ by the minimality of G. Subsequently, we delete the (possible) chords from C_i and extend the 3-coloring of C_i induced by ϕ to $G - ext(C_i)$ (this is possible due to the minimality of G). □

Claim 2. G is 2-connected.

Proof. Assume that C contains a cut vertex u. Assume that B is an end block with a cut vertex $u \in V(G)\backslash V(C)$. Due to minimality of G, we can extend ϕ to $G - (B - u)$, then 3-color B, and thus obtain an extension of ϕ to G. □

Claim 3. Each 2-vertex in G belongs to C; no 2-vertex in C is incident to a 3-face.

Proof. Let G contains a 2-vertex $v \in V(G)\backslash V(C)$. Then we can extend coloring ϕ to $G-v$ by the minimality of G, then color v with a color different from the colors of its neighbors in G. If a 2-vertex v in C is incident to a 3-face, we can extend ϕ to $G-v$ (due to minimality of G) and then recolor v with a color different from the colors of its neighbors in G. □

Claim 4. No cycle of length at most 9 in G has a non-triangular chord. In particular, if C is a good cycle and boundary of the external face, it has no chord at all.

Proof. If G contains a cycle of length at most 9 with a non-triangular chord, then it is easy to show that G must contain a cycle of length 4, 5 or 8, contradicting the assumption.

Suppose that C has a chord e. If e cuts a 3-cycle C_3 from C, then C_3 forms a 3-face by Claim 1, which contradicts Claim 3. Otherwise, it follows that $|C| = 10$ or $|C| = 11$ or $|C| = 12$ by the previous argument.

Assume that $|C| = 10$. Since G contains no 4-, 5-, 8-cycles, e cuts C into two cycles $C^1 = C_6$ and $C^2 = C_6$. If both $int(C^1)$ and $int(C^2)$ are empty, then it is straightforward to derive that G is 3-colorable. Otherwise, at least one of C^1 and C^2 is a separating cycle, which contradicts Claim 1.

Assume that $|C| = 11$. Since G contains no 4, 5, 8-cycles, e cuts C into two cycles $C^1 = C_6$ and $C^2 = C_7$. If both $int(C^1)$ and $int(C^2)$ are empty, then it is straightforward to derive that G is 3-colorable. Otherwise, at least one of C^1 and C^2 is a separating cycle, which contradicts Claim 1.

Assume that $|C| = 12$. Then e must cut C into two cycles $C^1 = C_7$ and $C^2 = C_7$. If $int(C^1)$ and $int(C^2)$ are empty, then G is 3-colorable. Otherwise, either C^1 or C^2 is a separating cycle, again contradicting Claim 1. Thus, C has no chord. The proof of Claim 4 is complete. □

Claim 5. Let C be a good cycle. For $v_1, v_2 \in C$ and $x \notin C$, if $xv_1, xv_2 \in E(G)$, then $v_1v_2 \in E(C)$.

Proof. Assume on the contrary that v_1v_2 does not belong to $E(C)$. Let l denote the number of edges in sector $C[v_1, v_2]$ i.e., $|C[v_1, v_2]| = l \leq |C[v_2, v_1]|$. Then $2 \leq l \leq 6$, by $|C| \leq 12$. Let $C^1 = C[v_1, v_2] \cup v_2xv_1$ and $C^2 = C[v_2, v_1] \cup v_1xv_2$. Then C^1 is an $(l+2)$-cycle and C^2 is a $(|C|-l+2)$-cycle. Since G contains no 4, 5 and 8-cycles, $l \neq 2, 3, 6$.

Assume that $l = 4$. Then C^1 is a 6-cycle and C^2 is a $(|C| - 2)$-cycle. Thus, $|C| \neq 6, 7, 10$. By Claim 1, neither C^1 nor C^2 is separating. It is easy to see that only way C^2 can have a chord is when $|C^2| = 10$, and then it is split into two 6-cycles. In this case, G consists of three 6-cycles which can be 3-colored easily. For all the other cases of C^2 , there is no chord (otherwise, it implies presence of a cycle of length 4, 5 or 8). Hence, Both C^1 and C^2 form the faces of G, which implies that x is an internal 2-vertex. This contradicts Claim 3.

Assume that $l = 5$. C^1 is a 7-cycle and C^2 is a $(|C| - 3)$-cycle. Thus, $|C| \neq 7, 8, 11$. By Claim 1, neither C^1 nor C^2 is separating. It is easy to see that C^2 does not have any chord (otherwise, it implies presence of a cycle of length 4, 5 or 8. Hence, Both C^1 and C^2 form the faces of G, which implies that x is an internal 2-vertex. This contradicts Claim 3. □

Claim 6. Let C be a good cycle. For $v_1, v_2 \in C$ and if $v_1x, xy, yv_2 \in E(G)$ and $x, y \in int(C)$, then $v_1v_2 \in E(C)$.

Proof. Assume on the contrary that v_1v_2 does not belong to $E(C)$. Let l denote the number of edges in sector $C[v_1, v_2]$ i.e., $|C[v_1, v_2]| = l \leq |C[v_2, v_1]|$. Then $2 \leq l \leq 6$, by $|C| \leq 12$. Let $C^1 = C[v_1, v_2] \cup v_2xyv_1$ and $C^2 = C[v_2, v_1] \cup v_2yxv_1$. Then C^1 is an $(l+3)$-cycle and C^2 is a $(|C|-l+3)$-cycle.

Assume that $l = 2$. Then C^1 is a 5-cycle, contradicting assumption.

Assume that $l = 3$. Then C^1 is a 6-cycle and C^2 is a $|C|$-cycle. Note that C^1 is not separating. Also C^1 cannot have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y) \geq 3$, there must be at least two chords of C^2. If $|C^2| = 6$, there cannot exist two chords without creating a 4-cycle (contradicting the assumption). If $|C^2| = 7$, there is a 5-cycle contradicting assumption again. When $|C^2| = 9$, either there is a 4- or 5- cycle or there are two 3-cycles adjacent to C^1 creating a 8-cycle, contradicting assumption. If $|C^2| = 10$ or 11, there is a 4-, 5- or 8-cycle. If $|C^2| = 12$, then either there is a 4-, 5- or 8-cycle or a bad cycle. If C^2 is bad, then we cannot have $d(x) \geq 3$ and $d(y) \geq 3$, contradicting Claim 3.

When $l = 4$, C^1 is a 7-cycle and C^2 is a $(|C| - 1)$-cycle. By Claim 6, neither C^1 nor C^2 is separating(unless bad). First assume that C^1 does not have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y) \geq 3$, there must be at least two chords of C^2. There are four possibilities of good C^2: 6-cycle, 8-cycle, 9-cycle, 10-cycle or 11-cycle. In all these cases, we can establish that there is a cycle of length in $\{4, 5, 8\}$ or a bad C^2. When C^2 is bad, there is a contradiction as either

$d(x) = 2$ or $d(y) = 2$ or there is 8-cycle. Next we assume that C^1 has an internal chord. The only possible chord divides it into 6- and 3-faces. By Claim 1, C^2 (when good) cannot be separating. Let us assume that C^2 is good. Hence, as $d(x), d(y) \geq 3$, there must be at least one chord of C^2 with one end at x or y. If $|C^2| = 6$, there is a 4- or 5-cycle (contradicting the assumption). If $|C^2| = 8$, there is a 4- or 5-cycle or a 8-cycle (a 6-cycle adjacent to two 3-cycles or two adjacent 3-cycles), contradicting the assumption. When $|C^2| = 9$, either there is a 4-, 5- or 8-cycle. If $|C^2| = 10$, there is a 4- or 5-cycle or a 8-cycle (a 6-cycle adjacent to two 3-cycles) or C is bad, contradicting the assumption. If C^2 is bad, then we have 6-cycle adjacent to two 3-cycles (hence a 8-cycle) or two adjacent 3-cycles (hence a 4-cycle) contradicting assumption.

When $l = 5$, C^1 is a 8-cycle, a contradiction.

When $l = 6$, C^1 is a 9-cycle and C^2 is a $|C| - 3$ cycle. Hence, $l \neq 7, 8, 11$. By Claim 6, neither C^1 nor C^2 is separating (unless bad). Note that C^1 cannot have any chord without creating a cycle of length in $\{4, 5, 8\}$. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y) \geq 3$, there must be at least two chords of C^2. There are four possibilities of good C^2: 3-cycle, 6-cycle, 7-cycle, or 9-cycle. In the first case, there cannot be any chord. For all the other cases, we can establish that there is a cycle of length in $\{4, 5, 8\}$. When C^2 is bad, there is a contradiction as either $d(x) = 2$ or $d(y) = 2$ or there is 8-cycle. Hence, Claim 6 is proved. □

Claim 7. Let C be a good cycle. For $v_1, v_2 \in C$ and if $v_1x, xy, yz, zv_2 \in E(G)$ and $x, y, z \in int(C)$, then $v_1v_2 \in E(C)$.

Proof. Assume on the contrary that v_1v_2 does not belong to $E(C)$. Let l denote the number of edges in sector $C[v_1, v_2]$ i.e., $|C[v_1, v_2]| = l \leq |C[v_2, v_1]|$. Then $2 \leq l \leq 6$, by $|C| \leq 12$. Let $C^1 = C[v_1, v_2] \cup v_2xyv_1$ and $C^2 = C[v_2, v_1] \cup v_2yxv_1$. Then C^1 is an $(l+4)$-cycle and C^2 is a $(|C| - l + 4)$-cycle.

Assume that $l = 2$. Then C^1 is a 6-cycle and C^2 is a $(|C| + 2)$-cycle. Note that C^1 is not separating. Also C^1 cannot have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y), d(z) \geq 3$, there must be at least two chords of C^2. If $|C^2| = 9$, C is a bad cycle contradicting assumption. When $|C^2| = 10, 11, 12, 13$ or 14, either there is a 4-, 5- or 8-cycle, contradicting assumption.

When $l = 3$, C^1 is a 7-cycle and C^2 is a $(|C| + 1)$-cycle. Note that C^1 is not separating. First assume that C^1 does not have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y), d(z) \geq 3$, there must be at least three chords of C^2. There are four possibilities of good C^2: 7-cycle, 10-cycle, 11-cycle, 12-cycle or 13-cycle. In all these cases, we can establish that there is a cycle of length in $\{4, 5, 8\}$ or a bad C^2. When C^2 is bad, there is a contradiction as there is an internal 2-vertex or there is 8-cycle. Next we assume that C^1 has an internal chord. The only possible chord divides it into 6- and 3-faces. By Claim 1, C^2 (when good) cannot be separating. Let us assume that C^2 is good. Hence, as $d(x), d(y), d(z) \geq 3$, there must be at least two chords of C^2 with one end at x, y or z. If $|C^2| = 7$, there is a 4-, 5- or 8-cycle (contradicting the assumption). If $|C^2| = 10, 11, 12$ or 13, we can again show that there is a 4- or 5-cycle or a 8-cycle, contradicting the assumption. If C^2 is bad, then we have 6-cycle adjacent to two 3-cycles (hence a 8-cycle) contradicting assumption.

When $l = 4$, C^1 is a 8-cycle, a contradiction.

When $l = 5$, C^1 is a 9-cycle and C^2 is a $(|C| - 1)$-cycle. Hence, $l \neq 5, 6, 9$. Neither C^1 nor C^2 is separating (unless bad). Note that C^1 cannot have any chord without creating a cycle of length in $\{4, 5, 8\}$. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y), d(z) \geq 3$, there must be at least three chords of C^2. There are four possibilities of good C^2: 6-cycle, 9-cycle, 10-cycle or 11-cycle. In the first case, there cannot be any chord. For all the other cases, we can establish that there is a cycle of length in $\{4, 5, 8\}$. When C^2 is bad, there is a contradiction as either there is an internal 2-vertex or there is 8-cycle.

When $l = 6$, then C^1 is a 10-cycle and C^2 is a $(|C| - 2)$-cycle. Hence, $l \neq 6, 7, 10$. Neither C^1 nor C^2 is separating (unless bad). C^1 can have a chord only in two possible ways (the chord divides C^1 as $3 + 9$ or $6 + 6$ cycles). If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y), d(z) \geq 3$, there must be at least two chords of C^2. There are four possibilities of good C^2: 6-cycle, 7-cycle, 9-cycle or 10-cycle. In the first case, there cannot be any chord. For all the other cases, we can establish that there is a cycle of length in $\{4, 5, 8\}$ or a bad cycle. When C^2 is bad, there is a contradiction as either there is an internal 2-vertex or there is 8-cycle. Hence, Claim 7 is proved. □

Now, we shall make G into smaller graphs by identifying vertices. In doing so, we should be sure that we do not

(i) identify two vertices of C (because then C is not a cycle anymore),

(ii) create an edge between two vertices of C colored the same (for otherwise our precoloring ϕ of C would be destroyed),

(iii) create loops,

(iv) create multiple edges,

(v) create cycles of length 4, 5 or 8, and

(vi) make C a bad cycle.

Claim 8. G has no 6-face other than C.

Proof. Suppose $f = wxyzpq$ is a face inside C. If f has an adjacent 3-cycle, we remove the common edge between f and the 3-cycle. The resulting graph is smaller than G, and does not have any 4-, 5- or 8-cycle. So we assume that G does not have any adjacent 3-cycle. By Claim 4, f has at least one internal vertex. Let y be an internal vertex. Identifying x with p within f cannot violate (i). Suppose $x, p \in C$. Let $z \in C$. Then z and x cannot be consecutive along C as otherwise, it violates the assumption of no 5-cycle. This implies by Claim 5 that y cannot be internal (a contradiction) or z is internal. If z is internal, then by Claim 6, x and p are consecutive on C, but then there is a 4-cycle in G.

Next suppose (ii) is an obstacle for identifying x with p. With out loss of generality, $x \in C$, p does not belong to C, and there is an edge pv_i such that $v_i \in C$, where v_i is not adjacent to x along C. If q is on C, by Claim 5, q is adjacent to v_i. In this case, there is a 3-cycle adjacent to f, contradicting assumption. If w is on C, by Claim 6, it must be adjacent to v_i. This creates a 4-cycle, contradicting assumption. Similarly y and z cannot be on C. Hence, all of y, z, p, q and w must be internal. Hence, by Claim 7, v_i is adjacent to x along C, contradicting the assumption.

The property (iii) follows from the absence of 4-cycles in G. The property (iv) is true else there is a 5-cycles in G.

Suppose we have created a 4-, 5- or 8-cycle $C' = xv_1 \dots v_k$, where $y \in int(C')$ and $k \in \{3, 4, 7\}$. If $k = 3$ then there is a separating 7-cycle if y does not belong $b(C')$. However, y cannot actually coincide with one of v_i's as then there is a 4-cycle in G. If $k = 4$ then there is a separating 8-cycle in G contradicting assumption. If $k = 7$ and y does not coincide with any of the v_i's, there is a separating cycle $(zwxv_1 \dots v_k)$ of length 10, contradicting Claim 1. If y coincides with one of the v_i's, then the only possible case without creating a 4-, 5- or 8-cycle is when y coincides with v_2 or v_6. In both the cases there is a 3-cycle incident at y. This contradicts the assumption that there is no 3-cycle adjacent to f.

Finally, collapsing the 6-face f by identifying x with p cannot make C bad. Hence Claim 8 is proved. □

We use the definition of good path as in [13]. A path $P = v_1v_2v_3v_4$ in the interior of C is called good if the following properties hold:

(a) $d(v_i) = 3$ for each $i = 1, 2, 3, 4$;

(b) $\ldots xPx' \ldots$ is on the boundary of a face;

(c) there is a triangle $[uv_1v_2]$ with $u \neq x$;

(d) $tv_3, t'v_4 \in E(G)$, where $t \neq x'$ and $t' \neq x'$.

Obviously, when $t = t'$, a good path is just a tetrad defined in [4].

Claim 9. G does not contain a good path P.

Proof. Suppose on the contrary that such a good path P exists in G. Let G' denote the graph obtained from G by deleting vertices v_1, v_2, v_3 and v_4 and identifying x and t. It is easy to see that G' contains no 4, 5 and 8-faces. In order to show that $G' \in \mathcal{G}$, we have the following argument. We first notice that G' has neither loops nor multiple edges. Indeed, if G' has a loop, then x is adjacent to t in G which leads to a 5-cycle $xtv_3v_2v_1x$. If G' has multiple edges, then both x and t are adjacent to a common vertex y so that a 6-cycle $xytv_3v_2v_1x$ is established. This implies presence of a 8-cycle $xytv_4v_3v_2uv_1x$.

Next, we claim that G' does not contain a separating cycle of length 4, 5 or 8. In fact, if $C^* = xy_1y_2 \ldots y_kt$ is a separating cycle in G', where $k \in \{3, 4, 7\}$, then $C' = xy_1y_2 \ldots y_ktv_3v_2v_1x$ is a cycle of length 8, 9 or 12 in G. Clearly, u does not belong C'. Thus, C' separates v_4 from u in G, which contradicts Claim 1 unless C' is bad. If C' is bad, there is a 6-cycle adjacent to two 3-cycles. This implies presence of 8-cycle, contradicting assumption.

We need to prove that identifying x and t cannot damage the coloring of C. If this is not true, then we either identify two vertices of C colored differently, or insert an edge between two vertices of C colored by the same color. This means that the total distance from x and t to C is at most 1, that is, at least one of x and t lies on C. Without loss of generality, assume that $t \in C$ and let $C = u_1u_2 \ldots u_{|C|}u_1$, where the subscripts increase in the clockwise order. Suppose that $u_{|C|}$ is a vertex of C nearest to x. Since $|C| \in \{6, 7, 9, 10, 11, 12\}$, C is split by $u_{|C|}$ and t into two paths, P_1 and P_2, one of which, say $P_1 = u_{|C|}u_1 \ldots u_jt$, consists of at most six edges. Thus, P_1 and the path $tv_3v_2v_1xu_{|C|}$ yield a cycle of length at most 11. Since $xv_1v_2v_3v_4x'$ is on the boundary of a face, $C' = u_{|C|}u_1u_2 \ldots u_jtv_3v_2v_1xu_{|C|}$ separates u from v_4, contradicting Claim 1.

Finally, we prove that any 3-coloring ϕ of G' can be extended to a 3-coloring of G in the following two ways:

(i) Assume that $t = t'$. We first color v_4 and v_3 in succession, and then properly color v_1 and v_2. Since x and t have the same color, x and v_3 must have different colors, therefore the required coloring exists.

(ii) Assume that $t \neq t'$, i.e., v_4 is not adjacent to t. If $\phi(t)$ does not belong to $\{\phi(t'), \phi(x')\}$, we color v_4 with $\phi(t)$ and then color v_3, since $\phi(x) = \phi(t)$, $\phi(x) \neq \phi(v_3)$. Thus, v_1 and v_2 can be properly colored in this case. Suppose that $\phi(t) \in \{\phi(t'), \phi(x')\}$. We can properly color v_4 with a color different from $\phi(t)$. Afterwards we color v_3, v_2 and v_1 in succession. □

Claim 10. No 3-face is adjacent to a k-face for $k = 3, 7$.

Proof. Suppose that G contains a 3-face f adjacent to a k-face $f' = [v_1v_2 \ldots v_k]$ for some $k \in \{3, 7\}$. If $k = 3$, it is easy to derive that $b(f) \cup b(f')$ contains a 4-cycle, a contradiction.

Assume that $k = 7$. If f' and f have two common boundary edges, then G has an internal 2-vertex, contradicting Claim 3. So we may suppose that $f = [v_1uv_2]$. If u does not belong to $b(f')$, then a 8-cycle $uv_2 \ldots v_7v_1u$ is constructed in G, which is impossible. So, $u \in b(f')$.

Clearly, $u \neq v_3$. If $u = v_4$, a 5-cycle $v_1v_4v_5v_6v_7v_1$ is established. If $u = v_5$, a 4-cycle $v_2v_3v_4v_5v_2$ is established. We always get a contradiction. We can give a similar proof for $u = v_6$ or $u = v_7$. This proves Claim 10. □

Claim 11. No two 6-faces can have more than one common edge.

Proof. Suppose that there are two adjacent 6-faces $f = [v_1v_2v_3v_4v_5v_6]$ and $f' = [v_1v_2u_1u_2u_3u_4]$ with v_1v_2 as a common edge. If f and f' have any other common edge then it is easy to establish presence of a separating cycle of length at most 12 (contradicting Claim 1) or of a 4-, 5-, or 8-cycle contradicting that $G \in \mathcal{G}$. □

Claim 12. No two adjacent 6-faces can have 3 or more common vertices.

Proof. If the adjacent 6-faces have 3 or more common vertices then it is easy to establish presence of a separating cycle of length at most 12 (contradicting Claim 1) or of a 4-,5-, or 8-cycle contradicting that $G \in \mathcal{G}$. □

Claim 13. The following properties hold true.

1. No 7-face shares more than one edge with a $\leq$ 9-face.
2. No 6-face shares more than one edge with a $\leq$ 10-face.
3. No 3-face shares more than one edge with a $\leq$ 13-face.

Proof. In all these cases, it is easy to establish presence of a separating cycle of length at most 12 (contradicting Claim 1) otherwise. □

Claim 14. The following properties hold true.

1. No 6-face is adjacent to two or more 3-faces.
2. No 7-face is adjacent to a 3-faces.

Proof. In both the cases, it is easy to establish presence of a 8-cycle (contradicting that $G \in \mathcal{G}$) otherwise. □

Claim 15. There cannot be three 6-faces incident with a 3-vertex.

Proof. In this case, it is easy to establish presence of a separating 12-cycle (contradicting contradicting Claim 1) otherwise. □

Claim 16. Let us consider a 3-face. The following properties hold with respect to the adjacent faces:

1. There cannot be any adjacent face of degree 7.
2. There cannot be two faces of degree 6 adjacent to the 3-face.
3. There cannot be three faces of length 6, 9, and 3 mutually adjacent to each other.

Proof. (1) is basically restatement of 14(2). This is true as there is no 8-cycle by assumption. If (2) is false, there is separating 9-cycle contradicting claim 1. If (3) is false, there is separating 12-cycle contradicting Claim 1. □

3. Discharging

Since by Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$,

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8. \tag{1}$$

We define a charge function w by $w(v) = d(v) - 4$ for each vertex $v \in V(G)$, $w(f) = d(f) - 4$ for each internal face $f \in \{F(G) \backslash f_0\}$, and $w(f_0) = d(f_0) + 4$. It follows from identity (1) that the total sum of charge is equal to 0. We intend to design appropriate discharging rules and redistribute charges so that once the discharging is finished, a new charge function w' is produced. The discharging rules maintain that the total charge is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new charge function $w'(x)$ satisfies the following properties:

1. $w'(x) \geq 0 \; \forall x \in V(G) \cup F(G)$;
2. there exists some $x^* \in V(G) \cup F(G)$ such that $w'(x^*) > 0$.

This leads to the following obvious contradiction,

$$0 < \sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = 0. \tag{2}$$

Our discharging rules are as follows:

R0. Each 3-face $f = xyz$ receives $\frac{1}{3}$ from each adjacent face, unless $d(x) = 3$, $d(y) \geq 4$, and $d(z) \geq 4$, in which case f receives $\frac{1}{6}$ each from faces adjacent to xy, and xz, and receives $\frac{2}{3}$ from the face adjacent to yz.

R1. Every 3-vertex $v \notin C$ receives $\frac{1}{3}$ from each incident face, unless v is incident with one 3-face, in which case v receives $\frac{1}{2}$ from each of the two > 3 faces.

R2. Every 2-vertex receives $\frac{5}{3}$ from the external face, and $\frac{1}{3}$ from the other adjacent (i.e., internal) face.

R3. The external face f_0 gives 1 to each incident vertex of degree at least 3.

R4. Let v_1, v_2, v_3 be consecutive vertices of external face f_0 with $d(v_2) \geq 4$. Then v_2 gives 1 to each incident face not incident with edges v_1v_2 and v_2v_3. Furthermore, if the internal face receiving 1 is a 3-face (v_2xy) where x and y do not belong to f_0, then it passes the 1 to the neighboring internal face (one with the common edge xy).

R5. Each 9^+-face $f \neq f_0$ gives $\frac{d(f)-4}{2}$ to f_0.

Claim 17. For all $v \in V(G)$, $w'(v) \geq 0$.

Proof. Let us assume that v does not belong to C. If $d(v) = 3$ and v is not incident with a 3-face, $w'(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$. If $d(v) = 3$ and v is incident with a 3-face, $w'(v) = 3 - 4 + 2 \times \frac{1}{2} = 0$. If $d(v) \geq 4$,

$w'(v) = w(v) \geq 0$. Now suppose $v \in C$. If $d(v) = 2$ then by (R2), $w'(v) = 2 - 4 + \frac{5}{3} + \frac{1}{3} = 0$. If $d(v) = 3$, by (R3) $w'(v) = 3 - 4 + 1 = 0$. If $d(v) \geq 4$, $w'(v) = d(v) - 4 + 1 - (d(v) - 3) \times 1 = 0$, by (R3) and (R4). □

Claim 18. For all $f \in F(G) \backslash C$, $w'(f) \geq 0$.

Proof. If $d(f) = 3$, then $w'(f) \geq 3-4+3\times\frac{1}{3} = 0$ or $3-4+1\times\frac{2}{3}+2\times\frac{1}{6} = 0$, by (R0). If f appears in R4, then it may have additional charge, hence $w'(f) \geq 0$.

Let us assume $d(f) = 7$. Let $f = v_1v_2v_3v_4v_5v_6v_7$. We have seen that f cannot be adjacent to any 3-face. Hence by (R1) and (R2), $w'(f) \geq 7 - 4 - 7 \times \frac{1}{3} > 0$.

Let us consider the case of $d(f) \geq 9$. Note that Claim 9 holds. We can partition the donation of f to the vertices by (R1), (R2) and to the edges by (R0) into $d(f)$ groups so that the total donation per group is at most $\frac{1}{2}$. For example, if f gives the edge v_iv_{i+1} (notation is in *mod* $d(f)$) a charge of $\frac{2}{3}$ (by (R0), then we can split the charge as $\frac{1}{3}$ each to the two vertices as v_i and v_{i+1}. If f gives the edge v_iv_{i+1} a charge of $\frac{1}{3}$ or $\frac{1}{6}$ (by (R0), then we can split the charge as at most $\frac{1}{6}$ each to the two vertices as v_{i-1} and v_{i+2}, which can receive $\frac{1}{3}$ each at most by (R1) and (R2). So each of v_{i-1} and v_{i+2} gets at most $\frac{1}{3} + \frac{1}{6}$ or $\frac{1}{2}$. Hence, $w'(f) \geq d(f) - 4 - d(f) \times \frac{1}{2} - (d(f) - 8) \times \frac{1}{2} \geq 0$. □

Claim 19. $w'(f_0) > 0$.

Proof. If f is the outer face f_0, then $d(f_0) \in \{6, 7, 9, 10, 11, 12\}$. Since G is different from C, and G is 2-connected, it follows that C has at least two ≥ 3-vertices. Thus $w'(f_0) \geq d(f_0)+4-\frac{2}{3}-\frac{5}{3}\times(d(f_0)-2)-2\times1$. Since there is no 4-, 5-, 8-cycle, there is an internal non-triangular face with at least 4 internal vertices. This implies an internal face of dimension at least $d(f_0)-2+4$, i.e., $d(f_0) + 2$. This face gives at least $(d(f_0) + 2 - 8) \times \frac{1}{2}$, i.e., $(d(f_0) - 6) \times \frac{1}{2}$ to f_0. Hence, $w'(f_0) = d(f_0) + 4 - \frac{2}{3} - \frac{5}{3} \times (d(f_0) - 2) - 2 \times 1 + (d(f_0) - 6) \times \frac{1}{2} = \frac{1}{6} \times (22 - d(f_0))$. For the case, there is no 2-vertex, by rules (R0), (R3) and (R5), $w'(f_0) > d(f_0) + 4 - 1 \times d(f_0) - \frac{2}{3} \times (\frac{1}{2} \times d(f_0)) = \frac{1}{3} \times (12 - d(f_0))$. This implies $w'(f_0) > 0$. □

4. CONCLUSION

To date, the best known result towards Steinberg's conjecture is by [4] that states that any planar graph without cycles of length in $\{4, 5, 6, 7\}$ is 3-colorable. In this article, we show that the result holds true for any planar graph without cycles of length in $\{4, 5, 8\}$. This takes us one step closer to resolving the Steinberg's conjecture. The result is promising with respect to Havel's problem too.

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