

## SOME RESULTS ON SEMI-TOTAL SIGNED GRAPHS <sup>1</sup>

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### Abstract

A *signed graph* (or *sigraph* in short) is an ordered pair  $S = (S^u, \sigma)$ , where  $S^u$  is a graph  $G = (V, E)$ , called the *underlying graph* of  $S$  and  $\sigma : E \rightarrow \{+, -\}$  is a function from the edge set  $E$  of  $S^u$  into the set  $\{+, -\}$ , called the *signature* of  $S$ . The  $\times$ -*line sigraph* of  $S$  denoted by  $L_{\times}(S)$  is a sigraph defined on the line graph  $L(S^u)$  of the graph  $S^u$  by assigning to each edge  $ef$  of  $L(S^u)$ , the product of signs of the adjacent edges  $e$  and  $f$  in  $S$ . In this paper, first we define *semi-total line sigraph* and *semi-total point sigraph* of a given sigraph and then characterize balance and consistency of semi-total line sigraph and semi-total point sigraph.

**Keywords:** sigraph, semi-total line sigraph, semi-total point sigraph, balanced sigraph, consistent sigraph.

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### 1. INTRODUCTION

For standard terminology and notation in graph theory we refer Harary [14] and West [21] and Zaslavsky [22, 23] for sigraphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

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A *signed graph* (or *sigraph* in short; see [7, 11]) is an ordered pair  $S = (S^u, \sigma)$ , where  $S^u$  is a graph  $G = (V, E)$ , called the *underlying graph* of  $S$  and  $\sigma : E \rightarrow \{+, -\}$  is a function from the edge set  $E$  of  $S^u$  into the set  $\{+, -\}$ , called the *signature* of  $S$ . Alternatively, the sigraph can be written as  $S = (V, E, \sigma)$ , with  $V, E, \sigma$  in the above sense. Let  $E^+(S) = \{e \in E(G) : \sigma(e) = +\}$  and  $E^-(S) = \{e \in E(G) : \sigma(e) = -\}$ . The elements of  $E^+(S)$  and  $E^-(S)$  are called *positive* and *negative* edges of  $S$ , respectively. A sigraph is said to be *homogeneous* if all its edges are of the same sign and *heterogeneous* otherwise.

A sigraph  $S$  is called a *regular sigraph* if the number of positive edges,  $d^+(v)$  incident at a vertex  $v$  in  $S$ , is independent of the choice of  $v$  in  $S$  and the number of negative edges,  $d^-(v)$  incident at a vertex  $v$  in  $S$  is also independent of the choice of  $v$  in  $S$ , i.e.,  $S$  is a sigraph of order  $n$  and regular of degree  $k = i + j$ , where  $i = d^+(v)$  is the positive degree of  $v$  in  $S$  and  $j = d^-(v)$  is the negative degree of  $v$  in  $S$ .

For a sigraph  $S$ , Behzad and Chartrand [7] defined its *line sigraph*  $L(S)$  as the sigraph in which the edges of  $S$  are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in  $S$  have a vertex in common and any such edge  $ef$  is defined to be negative whenever both  $e$  and  $f$  are negative edges in  $S$ . In [12], the author introduced a variation of the above standard notion of line sigraph  $L(S)$  of a given sigraph  $S$  as follows:  $L_\times(S)$  is a sigraph defined on the line graph  $L(S^u)$  of the graph  $S^u$  by assigning to each edge  $ef$  of  $L(S^u)$ , the product of signs of the adjacent edges  $e$  and  $f$  of  $S$ .  $L_\times(S)$  is called the  $\times$ -*line sigraph* of  $S$ .

A path in a sigraph  $S$  is said to be *all-negative* if each of its edge is negative. A cycle in a sigraph  $S$  is said to be *all-positive*(*all-negative*) if each of its edge is positive (negative). A cycle in a sigraph  $S$  is said to be *positive* if it contains an even number of negative edges. A given sigraph  $S$  is said to be *balanced* if every cycle in  $S$  is positive, i.e., it contains an even number of negative edges [4, 10, 13]. A spectral characterization of balanced sigraphs was given by Acharya [2]. Harary and Kabell [15, 16] developed a simple algorithm to get balanced sigraphs and also enumerated them. The following important lemma on balanced sigraph is given by Zaslavsky.

**Lemma 1** [24]. *A signed graph in which every chordless cycle is positive, is balanced.*

A *marked signed graph* is an ordered pair  $S_\mu = (S, \mu)$ , where  $S = (S^u, \sigma)$  is a sigraph and  $\mu : V(S^u) \rightarrow \{+, -\}$  is a function from the vertex set  $V(S^u)$

of  $S^u$  into the set  $\{+, -\}$ , called a *marking* of  $S$ . A cycle  $Z$  in  $S_\mu$  is said to be *consistent* if it contains an even number of negative vertices. A given sigraph  $S$  is said be *consistent* if every cycle in it is consistent [8, 9]. To this end, we define the following *canonical marking* on  $S$ : for each vertex  $v \in V(S)$ ,

$$\mu(v) = \prod_{e \in E_v} \sigma(e)$$

where  $E_v$  is set of edges  $e$  incident at  $v$  in  $S$ .

The *semi-total line graph*  $T_1(G)$  of a graph  $G$  [19] is the graph whose vertex set is  $V(G) \cup E(G)$  where  $V(G)$  and  $E(G)$  are vertex set and edge set of  $G$ , respectively and in  $T_1(G)$  two vertices are adjacent if and only if (i) they are adjacent edges in  $G$  (ii) one is a vertex and the other is an edge in  $G$  incident to it.

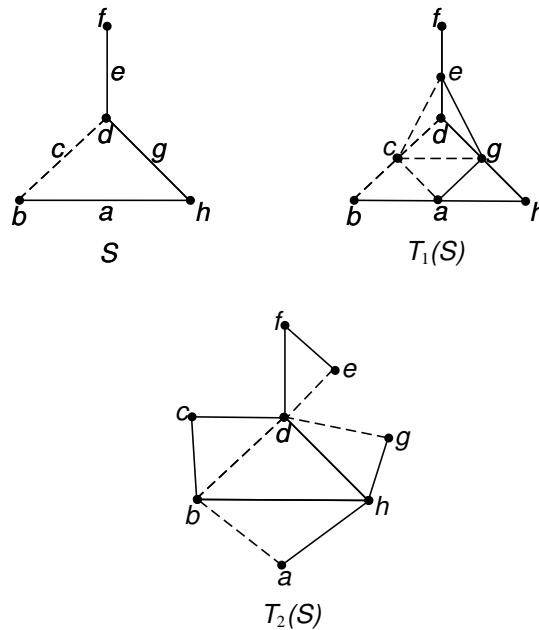


Figure 1. Showing  $T_1(S)$  and  $T_2(S)$  of a sigraph  $S$ .

The *semi-total point graph*  $T_2(G)$  of a graph  $G$  [19] is the graph whose vertex set is  $V(G) \cup E(G)$  where  $V(G)$  and  $E(G)$  are vertex set and edge set of  $G$ , respectively and in  $T_2(G)$  two vertices are adjacent if and only if (i) they

are adjacent vertices in  $G$ , (ii) one is a vertex and the other is an edge in  $G$  incident to it.

Let  $S = (V, E, \sigma)$  be any sigraph. Its *semi-total line sigraph*  $T_1(S)$  [as shown in Figure 1] has  $T_1(S^u)$  as its underlying graph and for any edge  $uv$  of  $T_1(S^u)$

$$\sigma_{T_1}(uv) = \begin{cases} \sigma(u)\sigma(v) & \text{if } u, v \in E, \\ \sigma(v) & \text{if } u \in V \text{ and } v \in E. \end{cases}$$

Let  $S = (V, E, \sigma)$  be any sigraph. Its *semi-total point sigraph*  $T_2(S)$  [as shown in Figure 1] has  $T_2(S^u)$  as its underlying graph and for any edge  $uv$  of  $T_2(S^u)$

$$\sigma_{T_2}(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V, \\ \sigma(u) \prod_{e \in E_v} \sigma(e) & \text{if } u \in E \text{ and } v \in V. \end{cases}$$

We observe that the  $\times$ -line sigraph,  $L_\times(S)$  is an induced subsigraph of  $T_1(S)$  and  $S$  is an induced subsigraph of  $T_2(S)$ .

## 2. BALANCED SEMI-TOTAL LINE SIGRAPH

In this section, we obtain a characterization of balanced semi-total line sigraph.

**Theorem 2** [6]. *The  $\times$ -line sigraph  $L_\times(S)$  of a sigraph  $S$  is a balanced sigraph.*

**Theorem 3** [18]. *A sigraph  $S = (S^u, \sigma)$  is balanced if and only if there exists a marking  $\mu$  of its vertices such that for each edge  $uv$  in  $S$  one has  $\sigma(uv) = \mu(u)\mu(v)$ .*

**Theorem 4.** *The semi-total line sigraph  $T_1(S)$  of a sigraph  $S$  is a balanced sigraph.*

**Proof.** By the definition of  $T_1(S)$ , it contains  $L_\times(S)$  as induced subsigraph, triangles formed by the adjacent edges  $e$  and  $f$  in  $S$  and the vertex  $v$  such that  $e \cap f = \{v\}$  and cycles formed by the symmetric difference of these triangles and cycles in  $L_\times(S)$ . Since  $L_\times(S)$  is a balanced sigraph due to Theorem 2, we have to only show that triangles and cycles formed as above in  $T_1(S)$  are positive.

*Case (i).* Suppose  $e$  and  $f$  both are positive adjacent edges with  $v$  as their common vertex in  $S$ , then  $ef$  will be a positive edge in  $T_1(S)$ . Now by the definition of  $T_1(S)$ , the triangle formed by the vertices  $e$ ,  $f$  and  $v$  does not contain any negative edge. Thus, such triangles are positive.

*Case (ii).* Suppose  $e$  and  $f$  both are negative adjacent edges with  $v$  as their common vertex in  $S$ , then  $ef$  will be a positive edge in  $T_1(S)$ . Now by the definition of  $T_1(S)$ , the triangle formed by the vertices  $e$ ,  $f$  and  $v$  contain two negative edges. Thus, such triangles are positive too.

*Case (iii).* Suppose  $e$  and  $f$  are edges of opposite parity and they are adjacent with  $v$  as their common vertex in  $S$ , then  $ef$  will be a negative edge in  $T_1(S)$ . Now by the definition of  $T_1(S)$ , the triangle formed by the vertices  $e$ ,  $f$  and  $v$  contain two negative edges. Thus, such triangles are also positive.

Now, due to Lemma 1, it follows that  $T_1(S)$  is a balanced sigraph. Hence the theorem. ■

### 3. CONSISTENT SEMI-TOTAL LINE SIGRAPH

Beineke and Harary [8, 9] were the first to pose the problem of characterizing consistent marked graphs, which was subsequently settled by Acharya [1, 2] and Hoede [17]. Acharya and Sinha obtained consistency of sigraphs that satisfy certain sigraph equations in [20, 5]. In this section, first we define a  $\mu_1$ -marking and then obtain a characterization of  $\mu_1$ -consistent semi-total line sigraph.

For any sigraph  $S = (S^u, \sigma)$ , we define  $\mu_1$ -marking in semi-total line sigraph  $T_1(S)$  as  $\mu_1 : V(T_1(S)) \rightarrow \{+, -\}$  such that

$$\mu_1(v_i) = \prod_{e_j \in E_{v_i}} \sigma(e_j)$$

and

$$\mu_1(e_i) = \sigma(e_i).$$

**Theorem 5** [17]. *A marked graph  $G_\mu$  is consistent if and only if for any spanning tree  $T$  of  $G$  all fundamental cycles with respect to  $T$  are consistent and all common paths of pairs of those fundamental cycles have end vertices carrying the same marks.*

**Theorem 6.** *The semi-total line sigraph  $T_1(S)$  of a sigraph  $S$  is  $\mu_1$ -consistent if and only if the following conditions hold in  $S$ :*

- (i) *each cycle  $Z$  in  $S$  is homogeneous and positive,*
- (ii) *if  $d(v) \geq 3$ , then  $d^-(v) = 0$  for every vertex  $v$  in  $S$ .*

**Proof.** *Necessity:* Suppose  $T_1(S)$  of a sigraph  $S$  is  $\mu_1$ -consistent. Since  $L_\times(S)$  is an induced subsigraph of  $T_1(S)$  and  $T_1(S)$  is  $\mu_1$ -consistent, it follows that  $L_\times(S)$  is  $\mu_1$ -consistent. Let  $Z$  be a cycle in  $S$  and  $v \in V(Z)$ . Let  $d(v) = 2$ . If possible  $d^-(v) = 1$ , then let a positive edge  $e$  and a negative edge  $f$  be incident at  $v$ . Due to  $\mu_1$ -marking in  $T_1(S)$ , there is a  $\mu_1$ -consistent cycle  $Z_1$  having one positively marked vertex  $e$  and two negatively marked vertices  $v$  and  $f$  in  $T_1(S)$ . Let  $Z_2$  be a  $\mu_1$ -consistent cycle in  $L_\times(S)$  having the edge  $ef$ . Now, taking the symmetric difference of the edge sets of  $Z_1$  and  $Z_2$ , we get a  $\mu_1$ -inconsistent cycle in  $T_1(S)$ , since the end vertices of the common edge  $ef$  are oppositely marked. Thus, a contradiction to the assumption that  $T_1(S)$  is  $\mu_1$ -consistent. That means, each cycle  $Z$  in  $S$  is homogeneous. Again, the edges of each cycle  $Z$  in  $S$  create a cycle in  $L_\times(S)$  and each cycle in  $L_\times(S)$  has an even number of negatively marked vertices. So, each cycle  $Z$  in  $S$  has an even number of negative edges. That means, each cycle  $Z$  in  $S$  is positive. Thus, (i) follows.

If for a vertex  $v$  in  $S$ ,  $d^-(v) \geq 3$ , then any of the three negative edges incident to  $v$  will form a  $\mu_1$ -inconsistent triangle in  $L_\times(S)$ , a contradiction that  $L_\times(S)$  is  $\mu_1$ -consistent. So,  $d^-(v) < 3$ . Now, if  $d(v) > 3$ , then  $d^-(v)$  being equal to one or two would contradict the fact that  $L_\times(S)$  is  $\mu_1$ -consistent. If  $d(v) = 3$ , then  $d^-(v)$  being equal to one again contradicts the fact that  $L_\times(S)$  is  $\mu_1$ -consistent. If  $d(v) = 3$  and  $d^-(v) = 2$ , then let a positive edge  $e$  and two negative edges  $f$  and  $g$  be incident on  $v$ . Now due to  $\mu_1$ -marking in  $T_1(S)$ , there is a  $\mu_1$ -inconsistent cycle  $Z$  having two positively marked vertices  $v$  and  $e$  and one negatively marked vertex  $f$  in  $T_1(S)$ , a contradiction to the assumption that  $T_1(S)$  is  $\mu_1$ -consistent. Thus, (ii) follows.

*Sufficiency:* Suppose both the conditions (i) and (ii) hold for a given sigraph  $S$ . We have to show that  $T_1(S)$  is  $\mu_1$ -consistent. By the definition of  $T_1(S)$ , it contains  $L_\times(S)$  as an induced subsigraph, triangles due to the adjacent edges  $e$  and  $f$  in  $S$  and the vertex  $v$  such that  $e \cap f = \{v\}$  and cycles formed by the symmetric difference of these triangles and cycles in  $L_\times(S)$ . By these conditions,  $S$  is either an all-negative cycle of even length

or the sigraph containing all-positive cycles and the end vertices of induced all-negative path do not lie on any cycle.

*Case (i).* Suppose  $S$  is an all-negative cycle of even length. That means,  $L_{\times}(S)$  has an even number of negatively marked vertices. That means,  $L_{\times}(S)$  is  $\mu_1$ -consistent. Now we have to see the  $\mu_1$ -consistency of the triangles formed by the edges of  $S$  and  $L_{\times}(S)$ . Suppose both  $e$  and  $f$  are negative adjacent edges with  $v$  as their common vertex in  $S$ . Then due to the  $\mu_1$ -marking of  $T_1(S)$ ,  $\mu_1(e) = \mu_1(f) = -$  and  $\mu_1(v) = +$ . Hence, the triangle formed by the vertices  $e$ ,  $f$  and  $v$  in  $T_1(S)$  contains two negatively marked vertices  $e$  and  $f$ . That means, such triangles are  $\mu_1$ -consistent. Now, since the vertices  $e$  and  $f$  in  $T_1(S)$  have the same marks, so due to Theorem 5, cycles formed by the symmetric difference of these triangles and cycles in  $L_{\times}(S)$  will be  $\mu_1$ -consistent. Hence  $T_1(S)$  is  $\mu_1$ -consistent.

*Case (ii).* Suppose  $S$  is the graph containing all-positive cycles and by the condition (ii), such cycles will be adjacent with positive edges only. That means the end vertices of induced all-negative path do not lie on any cycle and due to condition (ii), the end vertices of these induced all-negative path are of degree two. Let  $e$  and  $f$  be positive and negative adjacent edges, respectively with  $v$  as their common vertex in  $S$ , then by the  $\mu_1$ -marking of  $T_1(S)$ , there is a  $\mu_1$ -consistent cycle  $Z$  having one positively marked vertex  $e$  and two negatively marked vertices  $v$  and  $f$  in  $T_1(S)$ . Again, let both  $e$  and  $f$  be negative adjacent edges with  $v$  as their common vertex in  $S$ . Then, by the  $\mu_1$ -marking of  $T_1(S)$ , there is a consistent cycle  $Z$  having one positively marked vertex  $v$  and two negatively marked vertices  $e$  and  $f$  in  $T_1(S)$ . Hence  $T_1(S)$  is  $\mu_1$ -consistent. ■

#### 4. BALANCED SEMI-TOTAL POINT SIGGRAPH

In this section, we define a  $\mu_1$ -marking in semi-total point sigraph and obtain a characterization of balanced semi-total point sigraph.

For any sigraph  $S = (S^u, \sigma)$ , we define  $\mu_1$ -marking in semi-total point sigraph  $T_2(S)$  as  $\mu_1 : V(T_2(S)) \rightarrow \{+, -\}$  such that  $\mu_1(v_i) = \prod_{e_j \in E_{v_i}} \sigma(e_j)$  and  $\mu_1(e_i) = \sigma(e_i)$ .

**Theorem 7.** *The semi-total point sigraph  $T_2(S)$  of a sigraph  $S$  is balanced if and only if the following conditions hold in  $S$ :*

- (i) if  $e$  is a positive edge in  $S$  and  $u, v$  are the end vertices of  $e$ , then the number of negative edges incident at  $u$  and  $v$  are of the same parity,
- (ii) if  $e$  is a negative edge in  $S$  and  $u, v$  are the end vertices of  $e$ , then the number of negative edges incident at  $u$  and  $v$  are of the opposite parity.

**Proof.** *Necessity:* Suppose  $T_2(S)$  is a balanced sigraph, then every cycle in  $T_2(S)$  must have an even number of negative edges. The vertex  $e$  being adjacent to the vertices  $u$  and  $v$  in  $T_2(S)$ , where  $uv$  is an edge  $e$  in  $S$ , we get a triangle  $Z$  in  $T_2(S)$  due to the vertices  $u, v$  and  $e$  which is balanced due to hypothesis. Now,

*Case (i).* If  $e$  is a positive edge in  $Z$ , then the edges  $ue$  and  $ve$  must be of the same parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) = \prod_{e_j \in E_v} \sigma(e_j),$$

whence, the number of negative edges incident at  $u$  and  $v$  are of the same parity. Thus, (i) follows.

*Case (ii).* If  $e$  is a negative edge in  $Z$ , then the edges  $ue$  and  $ve$  must be of the opposite parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) \neq \prod_{e_j \in E_v} \sigma(e_j),$$

whence, the number of negative edges incident at  $u$  and  $v$  are of the opposite parity. Thus, (ii) follows.

*Sufficiency:* Suppose conditions (i) and (ii) hold for a given sigraph  $S$ . We have to show that  $T_2(S)$  is a balanced sigraph. Let  $e$  be an edge in  $S$  whose end vertices are  $u$  and  $v$ . By the definition of  $T_2(S)$ ,  $T_2(S)$  contains  $S$  as an induced subgraph, triangles due to the vertices  $u, v$  and  $e$  and the cycles due to the symmetric difference of these triangles and cycles in  $S$ .

By condition (i) and (ii), the sign of each edge in  $S$  is the product of  $\mu_1$ -marking of corresponding end vertices in  $S$ . So, using Theorem 3,  $S$  is balanced. Now, we have to only show that the triangles and cycles formed as above in  $T_2(S)$  are also positive.

By condition (i),  $e$  is a positive edge in  $S$  whose end vertices are  $u$  and  $v$  and the number of negative edges incident at  $u$  and  $v$  are of the same parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) = \prod_{e_j \in E_v} \sigma(e_j)$$



and

$$\sigma(e) = +.$$

Hence, by the definition of  $T_2(S)$ , the triangle due to the vertices  $u, v$  and  $e$ , contains either no negative edge or two negative edges. Thus, such triangles are positive.

By condition (ii),  $e$  is a negative edge in  $S$ , whose end vertices are  $u$  and  $v$  and the number of negative edges incident at  $u$  and  $v$  are of the opposite parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) \neq \prod_{e_j \in E_v} \sigma(e_j)$$

and

$$\sigma(e) = -.$$

Hence, by the definition of  $T_2(S)$ , the triangle due to the vertices  $u, v$  and  $e$ , contains two negative edges. Thus, such triangles are positive.

Thus, due to Lemma 1, it follows that  $T_1(S)$  is a balanced sigraph. Hence the theorem. ■

**Corollary 8.** *The semi-total point sigraph  $T_2(S)$  of a regular heterogeneous sigraph  $S$  is not balanced.*

### 5. CONSISTENT SEMI-TOTAL POINT SIGRAPH

In this section, we obtain a characterization of  $\mu_1$ -consistent semi-total point sigraph.

**Theorem 9.** *The semi-total point sigraph  $T_2(S)$  of a sigraph  $S = (S^u, \sigma)$  is  $\mu_1$ -consistent if and only if the following conditions hold in  $S$ :*

- (i) *if  $e$  is a positive edge in  $S$  and  $u, v$  are the end vertices of  $e$ , then the vertices  $u$  and  $v$  are of the same parity,*
- (ii) *if  $e$  is a negative edge in  $S$  and  $u, v$  are the end vertices of  $e$ , then the vertices  $u$  and  $v$  are of the opposite parity,*
- (iii) *each cycle  $Z$  in  $S$  is all-positive and if for any  $v \in V(Z)$*

$$\prod_{e \in E_v} \sigma(e) = -,$$

*then  $Z$  is of even length.*

**Proof. Necessity:** Suppose  $T_2(S)$  is  $\mu_1$ -consistent, then every cycle in  $T_2(S)$  must have an even number of negative vertices. Since  $T_2(S)$  has  $S$  as an induced subgraph and  $T_2(S)$  is  $\mu_1$ -consistent, it follows that the induced  $S$  in  $T_2(S)$  is  $\mu_1$ -consistent. Now, the vertex  $e$  being adjacent to the vertices  $u$  and  $v$  in  $T_2(S)$ , where  $uv$  is an edge  $e$  in  $S$ , we get a triangle  $Z$  in  $T_2(S)$  due to the vertices  $u, v$  and  $e$  which is  $\mu_1$ -consistent by hypothesis.

*Case (i).* Let  $e$  be a positive edge in  $Z$  and  $\mu_1(e) = \sigma(e)$ , then the number of negative edges incident at  $u$  and  $v$  are of the same parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) = \prod_{e_j \in E_v} \sigma(e_j).$$

This implies, the vertices  $u$  and  $v$  are of the same parity. Thus, (i) follows.

*Case (ii).* Let  $e$  be a negative edge in  $Z$  and  $\mu_1(e) = \sigma(e)$ , then the number of negative edges incident at  $u$  and  $v$  are of the opposite parity. That means,

$$\prod_{e_i \in E_u} \sigma(e_i) \neq \prod_{e_j \in E_v} \sigma(e_j).$$

This implies, the vertices  $u$  and  $v$  are of the opposite parity. Thus, (ii) follows.

Now, let  $e$  be a negative edge contained in a cycle  $Z$  in  $S$  and  $u, v$  are the end vertices of  $e$ , then due to condition (ii),  $u$  and  $v$  are of the opposite parity. That means,

$$\mu_1(u) \neq \mu_1(v)$$

and

$$\mu_1(e) = \sigma(e)$$

whence, we get a triangle  $Z_1$  in  $T_2(S)$  due to the vertices  $u, v$  and  $e$  which is  $\mu_1$ -consistent by hypothesis. Let  $Z_2$  be the consistent cycle in  $T_2(S)$  containing the edge  $e = uv$ . Now, if we take the symmetric difference of  $Z_1$  and  $Z_2$ , then by Theorem 5,  $T_2(S)$  will not be  $\mu_1$ -consistent, a contradiction of our hypothesis. Thus,  $e$  can not be contained in any cycle in  $S$ . This implies, each cycle  $Z$  in  $S$  is all-positive. Now, let for any  $v \in V(Z)$

$$\prod_{e \in E_v} \sigma(e) = -.$$

Since  $Z$  is all-positive, then by condition (i), for each  $v \in V(Z)$

$$\prod_{e \in E_v} \sigma(e) = -,$$

whence, for each  $v \in V(Z)$

$$\mu_1(v) = -.$$

Since  $S$  is an induced subgraph of  $T_2(S)$ , the cycle  $Z$  will be the cycle in  $T_2(S)$  and  $Z$  will be  $\mu_1$ -consistent due to hypothesis. It follows that,  $Z$  is of even length. Thus (iii) follows.

*Sufficiency:* Suppose conditions (i), (ii) and (iii) hold for a given sigraph  $S$ . We have to show that  $T_2(S)$  is  $\mu_1$ -consistent. Let  $e$  be an edge in  $S$  whose end vertices are  $u$  and  $v$ . By the definition of  $T_2(S)$ ,  $T_2(S)$  contains  $S$  as an induced subgraph, triangles due to the vertices  $u, v$  and  $e$  and cycles due to the symmetric difference of these triangles and cycles in  $S$ .

By the condition (i),  $e$  is a positive edge in  $S$  whose end vertices are  $u$  and  $v$  and the vertices  $u$  and  $v$  are of the same parity. That means,

$$\mu_1(u) = \mu_1(v)$$

and

$$\mu_1(e) = \sigma(e) = +.$$

By the definition of  $T_2(S)$ , the triangle due to the vertices  $u, v$  and  $e$ , contains either no negatively marked vertex or two negatively marked vertices. Therefore, such triangles are  $\mu_1$ -consistent.

By the condition (ii),  $e$  is a negative edge in  $S$  whose end vertices are  $u$  and  $v$  and the vertices  $u$  and  $v$  are of the opposite parity. That means,

$$\mu_1(u) \neq \mu_1(v)$$

and

$$\mu_1(e) = \sigma(e) = -.$$

By the definition of  $T_2(S)$ , the triangle due to the vertices  $u, v$  and  $e$ , contains two negatively marked vertices. Therefore, such triangles are  $\mu_1$ -consistent.

By the condition (iii), each cycle  $Z$  in  $S$  is all-positive. Let  $e$  be a positive edge of such a cycle and  $u, v$  be the end vertices of  $e$ . Then, by condition (i) and the definition of  $T_2(S)$ , we get a  $\mu_1$ -consistent triangle  $Z_1$

due to the vertices  $u$ ,  $v$  and  $e$ , containing the edge  $e$ . Let  $Z_2$  be the  $\mu_1$ -consistent cycle in  $S$  containing the edge  $e$ . Now, we take the symmetric difference of  $Z_1$  and  $Z_2$ . Then, by Theorem 5, we get a  $\mu_1$ -consistent cycle. Since cycles in  $S$  are cycles in  $T_2(S)$ ,  $Z$  will be the cycle in  $T_2(S)$  and if for any  $v \in V(Z)$

$$\prod_{e \in E_v} \sigma(e) = -$$

then  $Z$  is of even length. Thus,  $Z$  is a  $\mu_1$ -consistent cycle. Hence the theorem. ■

**Corollary 10.** *The semi-total point sigraph  $T_2(S)$  of a regular heterogeneous sigraph  $S$  is not  $\mu_1$ -consistent.*

**Corollary 11.** *The semi-total point sigraph  $T_2(S)$  of a heterogeneous cycle  $S$  is not  $\mu_1$ -consistent.*

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