MONOCHROMATIC CYCLES AND MONOCHROMATIC PATHS IN ARC-COLORED DIGRAPHS

Hortensia Galeana-Sánchez
Guadalupe Gaytán-Gómez

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F. 04510, México

E-mail: hgaleana@matem.unam.mx
        gaytan@matem.unam.mx

AND

Rocío Rojas-Monroy

Facultad de Ciencias
Universidad Autónoma del Estado de México
Instituto Literario No. 100, Centro 50000, Toluca, Edo. de México, México

E-mail: mrrm@uaemex.mx

Abstract

We call the digraph $D$ an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. A path (or a cycle) is called monochromatic if all of its arcs are colored alike. A cycle is called a quasi-monochromatic cycle if with at most one exception all of its arcs are colored alike. A subdigraph $H$ in $D$ is called rainbow if all its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them and; (ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic path. The closure of $D$, denoted by $\mathcal{C}(D)$, is the $m$-colored multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, $A(\mathcal{C}(D)) = A(D) \cup \{(u, v) \mid \text{there exists a } uv\text{-monochromatic path colored } i \text{ contained in } D\}$. Notice that for
any digraph $D$, $\mathcal{C}(\mathcal{C}(D)) \simeq \mathcal{C}(D)$ and $D$ has a kernel by monochromatic paths if and only if $\mathcal{C}(D)$ has a kernel.

Let $D$ be a finite $m$-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colors in $C_i$ is monochromatic. We show that if $\mathcal{C}(D)$ does not contain neither rainbow triangles nor rainbow $P_3$ involving colors of both $C_1$ and $C_2$, then $D$ has a kernel by monochromatic paths.

This result is a wide extension of the original result by Sands, Sauer and Woodrow that asserts: Every 2-colored digraph has a kernel by monochromatic paths (since in this case there are no rainbow triangles in $\mathcal{C}(D)$).

**Keywords:** kernel, kernel by monochromatic paths, monochromatic cycles.

**2010 Mathematics Subject Classification:** 05C20.

1. Introduction

For general concepts we may refer the reader to [1]. Let $D$ be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of $D$, respectively. We recall that a subdigraph $D_1$ of $D$ is a spanning subdigraph if $V(D_1) = V(D)$. If $S$ is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by $S$ is the digraph having vertex set $S$, and whose arcs are all those arcs of $D$ joining vertices of $S$. An arc $u_1u_2$ of $D$ will be called an $S_1S_2$-arc of $D$ whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$, that is an arc from $z$ towards some vertex in $N$. A digraph $D$ is called a kernel-perfect digraph when every induced subdigraph of $D$ has a kernel. Sufficient conditions for the existence of kernels in digraphs have been investigated by several authors, Duchet and Meyniel [4]; Duchet [2, 3]; Galeana-Sánchez and Neumann-Lara [5, 6]. The concept of kernel is very useful in applications.

We call the digraph $D$ an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. Along this paper, all the paths and cycles will be directed paths and directed cycles. A path is called monochromatic if all of its arcs are colored alike. A subdigraph $H$ of $D$ is called rainbow if all its arcs have distinct colors. A set $N \subseteq V(D)$ is called a kernel by monochromatic paths
if for every pair of different vertices \( u, v \in N \) there is no monochromatic path between them and for every vertex \( v \in V(D) - N \) there is a monochromatic path from \( v \) to some vertex in \( N \).

In [12] Sands, Sauer and Woodrow have proved that any 2-colored digraph \( D \) has an independent set \( S \) of vertices of \( D \) such that, for every vertex \( x \notin S \), there is a monochromatic path from \( x \) to a vertex of \( S \) (i.e., \( D \) has a kernel by monochromatic paths, concept that was introduced later by Galeana-Sánchez [7].) In particular, they proved that any 2-colored tournament \( T \) has a kernel by monochromatic paths. They also raised the following problem: Let \( T \) be a 3-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must \( T \) have a kernel by monochromatic paths? (This question still remains open.) In [11] Shen Minggang proved that if \( T \) is an \( m \)-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle, and every transitive tournament of order 3 is quasi-monochromatic, then \( T \) has a kernel by monochromatic paths. He also proved that this result is the best possible for \( m \)-colored tournaments with \( m \geq 5 \). In fact, he proved that for each \( m \geq 5 \) there exists an \( m \)-colored tournament \( T \) such that every cycle of length 3 is quasi-monochromatic and \( T \) has no kernel by monochromatic paths. Also for every \( m \geq 3 \) there exists an \( m \)-colored tournament \( T' \) such that every transitive tournament of order 3 is quasi-monochromatic and \( T' \) has no kernel by monochromatic paths. In 2004 [10] H. Galeana-Sánchez and R. Rojas-Monroy presented a 4-colored tournament \( T \) such that every cycle of order 3 is quasi-monochromatic; but \( T \) has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in \( m \)-colored (\( m \geq 3 \)) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. More information on \( m \)-colored digraphs can be found in [7, 8, 9, 13, 14].

If \( C = (z_0, z_1, \ldots, z_n, z_0) \) is a cycle, we will denote by \( \ell(C) \) its length and if \( z_i, z_j \in V(C) \) with \( i \leq j \) we denote by \( (z_i, C, z_j) \) the \( z_iz_j \)-path contained in \( C \), and \( \ell(z_i, C, z_j) \) will denote its length.

The following is our main result:

**Theorem 1.** Let \( D \) be a finite \( m \)-colored digraph. Suppose that there is a partition \( C = C_1 \cup C_2 \) of the set of colors of \( D \) such that every cycle in the subdigraph \( D[C_i] \) spanned by the arcs with colors in \( C_i \) is monochromatic. Suppose, moreover, that \( C(D) \) does not contain neither rainbow triangles nor rainbow \( P_3 \) involving colors of both \( C_1 \) and \( C_2 \). Then \( D \) has a kernel by monochromatic paths.
Notice that the Theorem 1 implies the Theorem of Sands, Sauer and Woodrow in the finite case by taking as a partition each of the two colors: all cycles in each color class are trivially monochromatic and $\mathcal{C}(D)$ has no rainbow subdigraphs.

We will need the following basic elementary results.

**Lemma 2.** Let $D$ be a digraph; $u, v \in V(D)$. Every $uv$-monochromatic walk in $D$ contains a $uv$-monochromatic path.

**Lemma 3.** Let $D$ be a digraph. Every closed walk in $D$ contains a cycle.

**Lemma 4.** Let $D$ be a digraph. If for every $v \in V(D)$ fulfills that $\delta^{-}(v) \geq 1$ ($\delta^{+}(v) \geq 1$) then $D$ contains a cycle.

And the following Theorem.

**Theorem 5** (Berge-Duchet [2]). If $D$ is a digraph such that every cycle of $D$ has at least one symmetrical arc, then $D$ is a kernel-perfect digraph.

### 2. Monochromatic Cycles and Monochromatic Paths in Arc-colored Digraphs

The following lemmas are about $m$-colored digraphs such that each cycle is monochromatic, and they are useful to prove our main result.

**Lemma 6.** Let $D$ be a finite or infinite $m$-colored digraph such that every cycle in $D$ is monochromatic. If $C = (u_0, u_1, \ldots, u_{n-1})$ is a sequence of $n \geq 2$ vertices, different by pairs, such that for every $i \in \{0, \ldots, n-1\}$ $T_i$ is some $u_iu_{i+1}$-monochromatic path then the set of paths $\{T_i \mid i \in \{0, \ldots, n-1\}\}$ is monochromatic, that is, the paths $T_i$ are of the same color by pairs (the indices of the vertices will be taken modulo $n$.)

**Proof.** Assume, by contradiction, that there exists a sequence of vertices $(u_0, u_1, \ldots, u_{n-1})$ such that for every $i \in \{0, \ldots, n-1\}$ there exists a $T_i = u_iu_{i+1}$-monochromatic path in $D$ and the set of paths $\{T_i \mid i \in \{0, \ldots, n-1\}\}$ is not monochromatic. Choose such a counterexample with a minimal number of arcs. Then from Lemma 3 the subdigraph induced by this walk contains a cycle which involves more than one path. Since all cycles in $D$ are monochromatic, we can not consider the arcs of the cycle and obtain a counterexample with a smaller number of arcs, a contradiction.
As a direct result from Lemma 6 we have:

**Remark 7.** If $D$ is an $m$-colored digraph such that every cycle is monochromatic then in $\mathcal{C}(D)$ every cycle is monochromatic.

**Remark 8.** If $D$ is an $m$-colored digraph such that every cycle is monochromatic then in $\mathcal{C}(D)$ every cycle is symmetrical.

**Proof.** It follows from Remark 7 and the fact that $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$.

**Lemma 9.** Let $D$ be a finite $m$-colored digraph such that every cycle in $D$ is monochromatic. Then there exists $x_0 \in V(D)$ such that for every $z \in V(D) - \{x_0\}$ if there exists a $x_0z$-monochromatic path contained in $D$ then there exists a $zx_0$-monochromatic path contained in $D$.

**Proof.** Assume, for a contradiction, that $D$ is a digraph as in the hypothesis and that there is no vertex $x_0$ satisfying the affirmation from Lemma 9. It follows that $\text{Asym}\mathcal{C}(D)$ has a cycle. On the other hand, from Remark 8 we have that every cycle in $\mathcal{C}(D)$ is symmetric, a contradiction.

Let $D$ be an $m$-colored digraph and let $H$ be a subdigraph of $D$. We will say that $S \subseteq V(D)$ is a semikernel by monochromatic paths modulo $H$ of $D$ if $S$ is independent by monochromatic paths in $D$ and for every $z \in V(D) - S$, if there is a $Sz$-monochromatic path contained in $D - H$ then there is a $zS$-monochromatic path contained in $D$.

**Lemma 10.** Let $D$ be a finite $m$-colored digraph. Let $H$ be a subdigraph of $D$ such that every directed cycle in $D - H$ is monochromatic. Then there exists $x_0 \in V(D)$ which satisfies that $\{x_0\}$ is a semikernel by monochromatic paths modulo $H$ of $D$.

**Proof.** It follows by applying Lemma 9 to $D - H$.

Let

$$S = \{\emptyset \neq S \mid S \text{ is a semikernel by monochromatic paths mod } D_2 \text{ of } D\}.$$

Whenever $S \neq \emptyset$, we will denote by $D_S$ the digraph defined as follows: $V(D_S) = S$ (i.e., for every element of $S$ we put a vertex in $D_S$) and $(S_1, S_2) \in A(D_S)$ if and only if for every $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 = s_2$, or there exists a $s_1s_2$-monochromatic path contained in $D_2$ and there is no $s_2s_1$-monochromatic path contained in $D$. 
Lemma 11. Let $D$ be a finite $m$-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colors in $C_i$ is monochromatic. Then $D_S$ is an acyclic digraph.

Proof. Observe that by Lemma 10, there exists a semikernel by monochromatic paths mod $D_2$ of $D$. Thus $S = \emptyset$ and we can consider the digraph $D_S$. Suppose for a contradiction, that $D_S$ contains some cycle, say $C = (S_0, S_1, \ldots, S_{n-1}, S_0)$ of length $n \geq 2$. Since $C$ is a cycle in $D_S$, we have that $S_i \neq S_j$ whenever $i \neq j$.

Claim 1. There exists $i_0 \in \{0, 1, 2, \ldots, n - 1\}$ such that for some $z \in S_{i_0}$, $z \notin S_{i_0+1} \ (\text{mod} \ n)$. 
Otherwise, for every $i \in \{0, 1, \ldots, n - 1\}$ and every $z \in S_i$ we have that $z \in S_{i+1}$ and then $S_i = S_j$ for all $i, j \in \{0, 1, \ldots, n - 1\}$. So, $C = (S_0)$, which is a contradiction since a cycle contains at least two vertices.

Claim 2. If there exists $i_0 \in \{0, 1, \ldots, n - 1\}$ such that for some $z \in S_{i_0}$ and some $w \in S_{i_0+1} \ (\text{mod} \ n)$ there exists a $zw$-monochromatic path; then there exists $j_0 \neq i_0, j_0 \in \{0, 1, \ldots, n - 1\}$ such that $w \in S_{j_0}$ and $w \notin S_{j_0+1} \ (\text{mod} \ n)$.
Suppose without loss of generality that $i_0 = 0$. First, observe that $w \notin S_n = S_0$ since otherwise we have a $zw$-monochromatic path with $\{z, w\} \subseteq S_0$, contradicting that $S_0$ is independent by monochromatic paths. Since $w \in S_1$, let $j_0 = \max \{i \in \{0, 1, \ldots, n - 1\} \mid w \in S_i\}$ (notice that for both previous observations $j_0$ is well defined.) So, $w \in S_{j_0}$ and $w \notin S_{j_0+1}$.

It follows from Claim 1 that there exists $i_0 \in \{0, \ldots, n - 1\}$ and $t_0 \in S_{i_0}$ such that $t_0 \notin S_{i_0+1}$. It follows from the fact that $(S_{i_0}, S_{i_0+1}) \in F(D_S)$ that there exists $t_1 \in S_{i_0+1}$ such that there exists a $t_0t_1$-monochromatic path contained in $D_2$ and there is no $t_1S_{i_0}$-monochromatic path contained in $D$. From Claim 2, it follows that there exists an index $i_1 \in \{0, \ldots, n - 1\}$ such that $t_1 \in S_{i_1}$ and $t_1 \notin S_{i_1+1}$. Since $(S_{i_1}, S_{i_1+1}) \in F(D_S)$ it follows that there exists $t_2 \in S_{i_1+1}$ such that there is a $t_1t_2$-monochromatic path contained in $D_2$ and there is no $t_2S_{i_1}$-monochromatic path contained in $D$. Since $D$ is finite, we obtain a sequence of vertices $(t_0, t_1, t_2, \ldots, t_{m-1})$ such that there exists a $t_i t_{i+1}$-monochromatic path contained in $D_2$ and there is no $t_{i+1}t_i$-monochromatic path contained in $D$ for every $i \in \{0, 1, 2, \ldots, m - 1\} \ (\text{mod} \ m)$. But this contradicts Lemma 6. Therefore $D_S$ is an acyclic digraph. ■
3. The Main Result

The following theorem is a particular case from our Main Result.

**Theorem 12.** Let $D$ be an $m$-colored digraph such that every cycle in $D$ is monochromatic, then $D$ has a kernel by monochromatic paths.

**Proof.** It follows from Remark 8 and Theorem 5 that $\mathcal{C}(D)$ has a kernel and so $D$ has a kernel by monochromatic paths.

The main idea of the proof of our main theorem is to select $S \in V(D_S)$ such that $\delta^+_D(S) = 0$ (such $S$ exists since $D_S$ is acyclic) and prove that $S$ is a kernel by monochromatic paths of $D$.

We next proceed to prove our main result, Theorem 1.

**Proof of Theorem 1.** Consider the digraph $D_S$ of the digraph $D$. Since $D_S$ is a finite digraph and from Lemma 11 it does not contain cycles, it follows that $D_S$ contains at least a vertex of zero outdegree. Let $S \in V(D_S)$ be such that $\delta^+_D(S) = 0$.

We will prove that $S$ is a kernel by monochromatic paths of $D$.

Suppose for a contradiction, that $S$ is not a kernel by monochromatic paths of $D$. Since $S \in V(D_S)$, we have that $S$ is independent by monochromatic paths.

Let

$$X = \{z \in V(D) \mid \text{there is no } zS\text{-monochromatic path in } D\}.$$ 

It follows from our assumption that $X \neq \emptyset$. Since $D[X]$ is an induced subdigraph of $D$, we have that $D[X]$ satisfies the hypotheses from Lemma 11. So, it follows that there exists $x_0 \in X$ such that $\{x_0\}$ is a semikernel by monochromatic paths $mod$ $D_2$ of $D$.

Let

$$T = \{z \in S \mid \text{there is no } zx_0\text{-monochromatic path in } D_2\}.$$ 

From the definition of $T$, we have that for every $z \in (S - T)$ there exists a $zx_0$-monochromatic path contained in $D_2$.

**Claim 13.** $T \cup \{x_0\}$ is independent by monochromatic paths.

It follows directly from the facts that $T \subseteq S$, $S \in S$ and $x_0 \in X$. 


Claim 14. For each \( z \in V(D) - T \cup \{x_0\} \), if there exists a \((T \cup \{x_0\})z\)-monochromatic path contained in \( D_1 \), then there exists a \( z(T \cup \{x_0\})\)-monochromatic path contained in \( D \).

Case 1. There exists a \( Tz\)-monochromatic path contained in \( D_1 \). Since \( T \subseteq S \) and \( S \subseteq S \), it follows that there exists a \( zS\)-monochromatic path contained in \( D \). We may suppose that there exists a \( z(S-T)\)-monochromatic path contained in \( D \). Let \( \alpha_1 \) be a \( uw\)-monochromatic path contained in \( D_1 \) with \( u \in T \), and let \( \alpha_2 \) be a \( zw\)-monochromatic path with \( w \in (S-T) \) contained in \( D \). Since \( w \in (S-T) \) it follows from the definition of \( T \) that there exists \( \alpha_3 \) a \( wx_0\)-monochromatic path contained in \( D_2 \).

Moreover, \( \text{color}(\alpha_1) \neq \text{color}(\alpha_2) \) (\( \text{color}(\alpha) \) denotes the color used in the arcs of \( \alpha \)) otherwise there exists a \( uw\)-monochromatic path contained in \( \alpha_1 \cup \alpha_2 \), with \( \{u,w\} \subseteq S \), in contradiction with the fact that \( S \) is independent by monochromatic paths. In addition, we will suppose that color \( \alpha_2 \neq \text{color}(\alpha_3) \) since if \( \text{color}(\alpha_2) = \text{color}(\alpha_3) \) then \( \alpha_2 \cup \alpha_3 \) contains a \( zw_0\)-monochromatic path and Claim 2 is proved. Also \( \text{color}(\alpha_1) \neq \text{color}(\alpha_3) \) as \( \text{color}(\alpha_1) \in C_1 \) and \( \text{color}(\alpha_3) \in C_2 \).

So, we obtain that \((u,z,w,x_0)\) is a rainbow \( P_3 \) in \( \mathcal{C}(D) \) involving colors of both \( C_1 \) and \( C_2 \), a contradiction.

Case 2. There exists a \( x_0z\)-monochromatic path contained in \( D_1 \).

Let \( \alpha_1 \) be such a path, we may suppose that \( z \not\in X \). It follows from the definition of \( X \) that there exists some \( zS\)-monochromatic path contained in \( D \), let \( \alpha_2 \) be such path, say that \( \alpha_2 \) ends in \( w \). We will suppose that \( w \in (S-T) \). Since \( w \in (S-T) \), by the definition of \( T \), we have that there exists a \( wx_0\)-monochromatic path contained in \( D_2 \), let \( \alpha_3 \) be such a path.

Again, we have that \( \text{color}(\alpha_1) \neq \text{color}(\alpha_2) \) otherwise there exists a \( x_0w\)-monochromatic path contained in \( \alpha_1 \cup \alpha_2 \), contradicting that \( x_0 \in X \) and \( w \in S \). In addition, we will suppose that \( \text{color}(\alpha_2) \neq \text{color}(\alpha_3) \) since if \( \text{color}(\alpha_2) = \text{color}(\alpha_3) \) then \( \alpha_2 \cup \alpha_3 \) contains a \( zw_0\)-monochromatic path and Claim 2 is proved. Also \( \text{color}(\alpha_1) \neq \text{color}(\alpha_3) \) since \( \alpha_1 \subseteq D_1 \) and \( \alpha_3 \subseteq D_2 \).

Then \((x_0,z,w,x_0)\) is a rainbow \( C_3 \) in \( \mathcal{C}(D) \) which involves colors of both \( C_1 \) and \( C_2 \), a contradiction.

We conclude from Claims 1 and 2 that \( T \cup \{x_0\} \in \mathcal{S} \) and therefore \( T \cup \{x_0\} \in V(D_S) \). We have that \((S,T \cup \{x_0\}) \in F(D_S) \) since \( T \subseteq T \cup \{x_0\} \), and for each \( s \in S - T \) there exists a \( sx_0\)-monochromatic path contained in \( D_2 \) and there is no \( x_0S\)-monochromatic path contained in \( D \). But this contradicts
the fact that $\delta^+_{DS}(S) = 0$. Therefore $S$ is a kernel by monochromatic paths in $D$ and the Theorem is proved. ■

**Remark 15.** Notice that while in Theorem 12 it is asked for every cycle to be monochromatic, in the Theorem 1 there could exist non monochromatic cycles since the monochromatic cycles only are asked for each $D_i$, $i \in \{1, 2\}$.

**Acknowledgement**

The authors thank the anonymous referee for many suggestions which improve substantially the rewriting of this paper.

**References**


Received 26 November 2009
Revised 18 December 2010
Accepted 19 December 2010