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ON FULKERSON CONJECTURE

Jean-Luc Fouquet and Jean-Marie Vanherpe

L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans
45067 Orléans Cedex 2, France

Abstract

If G is a bridgeless cubic graph, Fulkerson conjectured that we can find 6 perfect matchings (a *Fulkerson covering*) with the property that every edge of G is contained in exactly two of them. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (this problem is known as the Fan Raspaud Conjecture). A *FR-triple* is a set of 3 such perfect matchings. We show here how to derive a Fulkerson covering from two FR-triples.

Moreover, we give a simple proof that the Fulkerson conjecture holds true for some classes of well known snarks.

Keywords: cubic graph, perfect matchings.

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1. Introduction

The following conjecture is due to Fulkerson, and appears first in [4].

Conjecture 1.1. If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.

We shall say that $\mathcal{F} = \{M_1, \ldots, M_6\}$, in the above conjecture, is a *Fulkerson covering*. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take any 3 of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.

Conjecture 1.2. There exists a fixed integer k such that every bridgeless cubic graph has a list of k perfect matchings with empty intersection.

For $k = 3$ this conjecture is known as the Fan Raspaud Conjecture.

Conjecture 1.3 [2]**.** Every bridgeless cubic graph contains perfect matching M_1, M_2, M_3 such that

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

Let G be a cubic graph with 3 perfect matchings M_1, M_2 and M_3 having an empty intersection. Since G satisfies the Fan Raspaud conjecture, when considering these perfect matchings, we shall say that $\mathcal{T} = (M_1, M_2, M_3)$ is a *FR-triple*. We define $T_i \subset E(G)$ $(i = 0..2)$ as the set of edges of G which are covered i times by $\mathcal{T}$. It will be convenient to use $T'_i(i = 0, \ldots, 2)$ for the FR-triple $\mathcal{T}'$.

2. FR-triples and Fulkerson Covering

In this section, we are concerned with the relationship between FR-triples and Fulkerson coverings.

2.1. On FR-triples

Proposition 2.1. *Let G be a bridgeless cubic graph with $\mathcal{T}$ a FR-triple. Then T_0 and T_2 are disjoint matchings.*

Proof. Let v be a vertex incident to an edge of T_0. Since v must be incident to each perfect matching of $\mathcal{T}$ and since the three perfect matchings have an empty intersection, one of the remaining edges incident to v must be contained in 2 perfect matchings while the other is contained in exactly one perfect matching. The result follows. ■

We introduce now concepts and definitions coming from [10]. Let ab be an edge of bridgeless cubic graph G. We shall say that we have *splitted* the edge ab when we have applied the operation depicted in Figure 1. The resulting graph is no longer cubic since we get 4 vertices with degree 2 instead of two vertices of degree 3. Let A_1 and A_2 be two disjoint matchings of G (we insist to say that these matchings are not, necessarily, perfect matchings). For $i = 1, 2$, let G_{A_i} be the graph obtained by splitting the edges of A_i and let $\overline{G_{A_i}}$ be the graph homeomorphic to G_{A_i} when the degree 2 vertices are deleted. The connected component of $\overline{G_{A_i}}$ are cubic graphs and *vertexless loop graphs* (graph with one edge and no vertex). We shall say that $\overline{G_{A_i}}$ is 3-edge colourable whenever the cubic components are 3-edge colourable (any colour can be given to the vertexless loops).

The following Lemma can be obtained from the work of Hao and al. [10] when considering FR-triples.

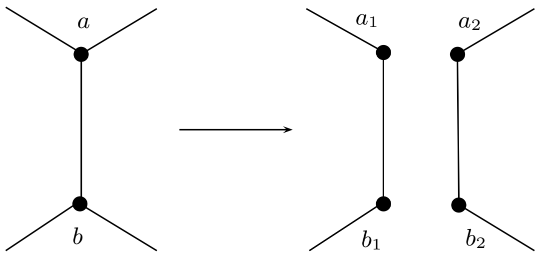

Figure 1. Splitting an edge.

Lemma 2.2. *Let G be a bridgeless cubic graph and let $\mathcal{T}$ be a FR-triple. Then $\overline{G_{T_2}}$ is 3-edge colourable.*

Proof. Assume that $T = (M_1, M_2, M_3)$ is a FR-triple. Let ab be an edge of T_2 then the two edges of T_1 incident with ab must be in the same perfect matching of $\mathcal{T}$. Hence, these two edges are identified in some sens. If we colour the edges of T_1 with 1, 2 or 3 when they are in M_1, M_2 or M_3 respectively, we get a natural 3-edge colouring of $\overline{G_{T_2}}$. ∎

Lemma 2.3. *Let G be a bridgeless cubic graph containing two disjoint matchings A_1 and A_2 such that $\overline{G_{A_1}}$ is 3-edge colourable and $A_1 \cup A_2$ forms an union of disjoint cycles. Then G has a FR-triple $\mathcal{T}$ where $T_2 = A_1$ and $T_0 = A_2$.*

Proof. Obviously, $A_1 \cup A_2$ forms an union of disjoint even cycles in G. Let $C = a_0a_1 \dots a_{2p-1}$ be an even cycle of $A_1 \cup A_2$ and assume that $a_ia_{i+1} \in A_1$ when $i \equiv 0(2)$.

Let M_1, M_2 and M_3 be the three matchings associated to a 3 edge-colouring of $\overline{G_{A_1}}$. Thanks to the construction of $\overline{G_{A_1}}$ for some $i \equiv 0[2]$, the third edge incident to a_i, say e, and the third one incident to a_{i+1}, say e' lead to a unique edge of $\overline{G_{A_1}}$. Assume that this edge of $\overline{G_{A_1}}$ is in M_1, then M_1 can be extended naturally to a matching of G containing $\{e, e'\}$. Moreover we add $a_i a_{i+1}$ to M_2 and $a_i a_{i+1}$ to M_3. When applying this process to all edges of A_1 on all cycles of $A_1 \cup A_2$ we extend the colours of $\overline{G_{A_1}}$ into perfect matchings of G. Since every edge of G belongs to at most 2 matchings in $\{M_1, M_2, M_3\}$ we have a FR-triple with $\mathcal{T} = \{M_1, M_2, M_3\}$. By construction, we have $T_2 = A_1$ and $T_0 = A_2$, as claimed. ■

Proposition 2.4. *Let G be a bridgeless cubic graph then G has a FR-triple if and only if G has two disjoint matchings A_1 and A_2 such that $A_1 \cup A_2$ forms an union of disjoint cycles, moreover $\overline{G_{A_1}}$ or $\overline{G_{A_2}}$ is 3-edge colourable.*

Proof. Assume that G has two disjoint matchings A_1 and A_2 such that, without loss of generality, $\overline{G_{A_1}}$ is 3-edge colourable. From Lemma 2.3, G has a FR-triple $\mathcal{T}$ where $T_2 = A_1$ and $T_0 = A_2$.

Conversely, assume that $\mathcal{T}$ is a FR-triple. From Lemma 2.2 $\overline{G_{T_2}}$ is 3-edge colourable. Let $A_1 = T_0$ and $A_2 = T_2$. Then A_1 and A_2 are two disjoint matchings and $\overline{G_{A_2}}$ is 3-edge colourable. ■

2.2. On compatible FR-triples

As pointed out in the introduction, any three perfect matchings in a Fulkerson covering lead to a FR-triple. Is it possible to get a Fulkerson covering when we know one or more FR-triples? In fact, we can characterize a Fulkerson covering in terms of FR-triples in the following way.

Let G be a bridgeless cubic graph with $\mathcal{T} = (M_1, M_2, M_3)$ and $\mathcal{T}' = (M'_1, M'_2, M'_3)$ two FR-triples. We shall say that $\mathcal{T}$ and $\mathcal{T}'$ are *compatible* whenever $T_0 = T'_2$ and $T_2 = T'_0$ (and hence $T_1 = T'_1$).

Theorem 2.5. *Let G be a bridgeless cubic graph then G can be provided with a Fulkerson covering if and only if G has two compatible FR-triples.*

Proof. Let $\mathcal{F} = \{M_1, \ldots, M_6\}$ be a Fulkerson covering of G and let $\mathcal{T} = (M_1, M_2, M_3)$ and $\mathcal{T}' = (M_4, M_5, M_6)$. $\mathcal{T}$ and $\mathcal{T}'$ are two FR-triples and we claim that they are compatible. Since each edge of G is covered exactly twice by $\mathcal{F}$, T_1 the set of edges covered only once by $\mathcal{T}$ must be covered also only once by $\mathcal{T}'$, T_0 the set of edges not covered by $\mathcal{T}$ must be covered exactly twice by $\mathcal{T}'$ and T_2 the set of edges covered exactly twice by $\mathcal{T}$ is not covered by $\mathcal{T}'$. Which means that $T_1 = T_1'$, $T_0 = T_2'$ and $T_2 = T_0'$, that is $\mathcal{T}$ and $\mathcal{T}'$ are compatible.

Conversely, assume that $\mathcal{T}$ and $\mathcal{T}'$ are two FR-triples compatible. Then it is an easy task to check that each edge of G is contained in exactly 2 perfect matchings of the 6 perfect matchings involved in $\mathcal{T}$ or $\mathcal{T}'$. ■

Proposition 2.6. *Let G be a bridgeless cubic graph then G has two compatible FR-triples if and only if G has two disjoint matchings A_1 and A_2 such that $A_1 \cup A_2$ forms an union of disjoint cycles and $\overline{G_{A_1}}$ and $\overline{G_{A_2}}$ are 3-edge colourable.*

Proof. Let $\mathcal{T}$ and $\mathcal{T}'$ be 2 compatible FR-triples. From Lemma 2.2 we know that $\overline{G_{T_2}}$ and $\overline{G_{T_2'}}$ are 3-edge colourable. Since $T_0 = T_2'$ and $T_0' = T_2$ by the compatibility of $\mathcal{T}$ and $\mathcal{T}'$, the result holds when we set $A_1 = T_0$ and $A_2 = T_2$.

Conversely, assume that G has two disjoint matchings A_1 and A_2 such that $\overline{G_{A_1}}$ and $\overline{G_{A_2}}$ are 3-edge colourable. From Lemma 2.3, G has a FR-triple $\mathcal{T}$ where $T_2 = A_1$ and $T_0 = A_2$ as well as a FR-triple $\mathcal{T}'$ where $T_2' = A_2$ and $T_0' = A_1$. These two FR-triples are obviously compatible. ■

Proposition 2.7 [10]. *Let G be a bridgeless cubic graph then G can be provided with a Fulkerson covering if and only if G has two disjoint matchings A_1 and A_2 such that $A_1 \cup A_2$ forms an union of disjoint cycles and $\overline{G_{A_1}}$ and $\overline{G_{A_2}}$ are 3-edge colourable.*

Proof. Obvious in view of Theorem 2.5 and Proposition 2.6. ■

3. Fulkerson Covering for Some Classical Snarks

A non 3-edge colourable, bridgeless, cyclically 4-edge-connected cubic graph is called a *snark*.

For an odd $k \geq 3$, let J_k be the cubic graph on $4k$ vertices $x_0, x_1, \ldots, x_{k-1}$, $y_0, y_1, \ldots, y_{k-1}$, $z_0, z_1, \ldots, z_{k-1}$, $t_0, t_1, \ldots, t_{k-1}$ such that $x_0x_1, \ldots, x_{k-1}$ is an induced cycle of length k, $y_0y_1, \ldots, y_{k-1}$ $z_0z_1, \ldots, z_{k-1}$ is an induced cycle of length $2k$ and for $i = 0, \ldots, k-1$ the vertex t_i is adjacent to x_i, y_i and z_i. The set $\{t_i, x_i, y_i, z_i\}$ induces the claw C_i. In Figure 2 we have a representation of J_3, the half edges (to the left and to the right in the figure) with same labels are identified. For $k \geq 5$ those graphs were introduced by

Isaacs in [6] under the name of flower snarks in order to provide an infinite family of snarks.

Proposition 2.7 is essentially used in [10] in order to show that the so called flower snarks and Goldberg snarks can be provided with a Fulkerson covering. We shall see, in this section, that this result can be directly obtained.

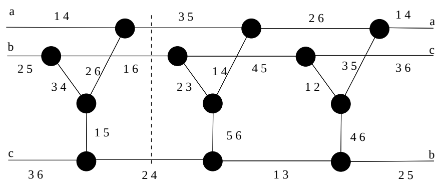

Figure 2. J_3

Theorem 3.1. *For any odd $k \geq 3$, J_k can be provided with a Fulkerson covering.*

Proof. For $k = 3$ the Fulkerson covering is given in Figure 2. We obtain a Fulkerson covering of J_k by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 3 when we cut J_3 along the dashed line of Figure 2. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.

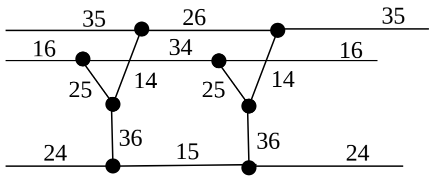

Figure 3. A block for the flower snark. ■

Let H be the graph depicted in Figure 4.

Let G_k (k odd) be a cubic graph obtained from k copies of H ($H_0, \ldots, H_{k-1}$ where the name of vertices are indexed by i) by adding edges $a_i a_{i+1}$, $c_i c_{i+1}$, $e_i e_{i+1}$, $f_i f_{i+1}$ and $h_i h_{i+1}$ (subscripts are taken modulo k).

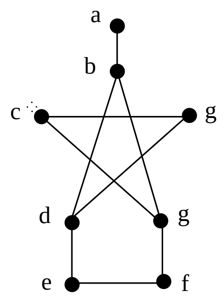

Figure 4. H

If $k = 5$, then G_k is known as the Goldberg snark (see [5]). Accordingly, we refer to all graphs G_k as Goldberg graphs. The graph G_5 is shown in Figure 5. The half edges (to the left and to the right in the figure) with same labels are identified.

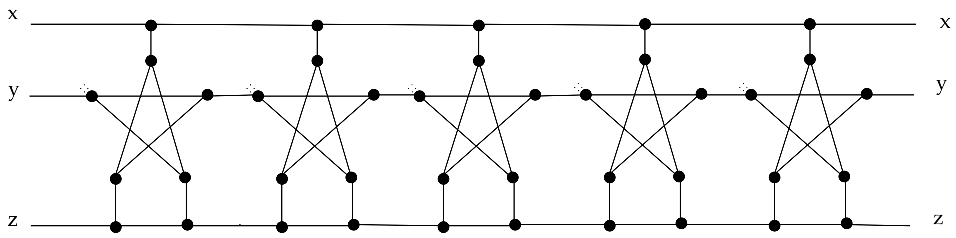

Figure 5. Goldberg snark G_5.

Theorem 3.2. *For any odd $k \geq 5$, G_k can be provided with a Fulkerson covering.*

Proof. We give first a Fulkerson covering of G_3 in Figure 6(a). The reader will complete easily the matchings along the 5-cycles by remarking that these cycles are incident to 5 edges with a common label from 1 to 6 and to exactly one edge of each remaining label. We obtain a Fulkerson covering of G_k with odd $k \geq 5$ by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 6(b) when we cut G_3 along the dashed line. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.

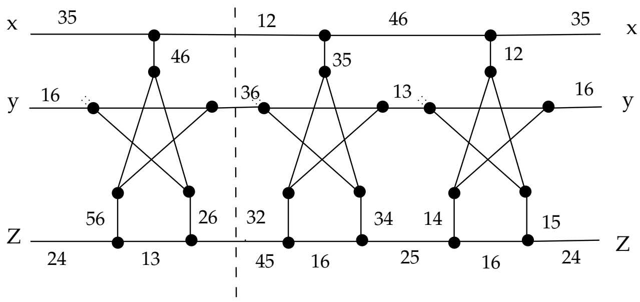

Figure (a). A Fulkerson covering for G_3.

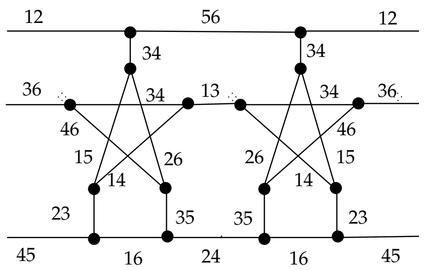

Figure (b). A block for the Goldberg snark.

Figure 6. Fulkerson covering for the Golberg Snarks. ■

4. A TECHNICAL TOOL

Let M be a perfect matching, a set $A \subseteq E(G)$ is an *M-balanced matching* when we can find a perfect matching M' such that $A = M \cap M'$. Assume that $\mathcal{M} = \{A, B, C, D\}$ are 4 pairwise disjoint M-balanced matchings, we shall say that $\mathcal{M}$ is an *F-family for M* whenever the three following conditions are fulfilled:

(i) Every odd cycle of $G \backslash M$ has exactly one vertex incident with one edge of each matching in $\mathcal{M}$.

(ii) Every even cycle of $G \backslash M$ incident with some matching in $\mathcal{M}$ contains 4 vertices such that two of them are incident to one matching in $\mathcal{M}$

while the other are incident to another matching in $\mathcal{M}$ or the 4 vertices are incident to the same matching in $\mathcal{M}$.

(iii) The subgraph induced by 4 vertices so determined in the previous items has a matching.

It will be convenient to denote the set of edges described in the third item by N.

Theorem 4.1. *Let G be a bridgeless cubic graph together with a perfect matching M and an F-family $\mathcal{M}$ for M. Then G can be provided with a Fulkerson covering.*

Proof. Since A, B, C and D are M-balanced matchings, we can find 4 perfect matchings M_A, M_B, M_C and M_D such that

$$M \cap M_A = A \quad M \cap M_B = B \quad M \cap M_C = C \quad M \cap M_D = D.$$

Let $M' = M \setminus \{A, B, C, D\} \cup N$, we will prove that $\mathcal{F} = \{M, M_A, M_B, M_C, M_D, M'\}$ is a Fulkerson covering of G.

Claim 4.1.1. M' is a perfect matching.

Proof. The vertices of G which are not incident with some edge in $M \setminus \{A, B, C, D\}$ are precisely those which are end vertices of edges in $M_A \cup M_B \cup M_C \cup M_D$. From the definition of an F-family, the 4 vertices defined on each cycle of $\{C_i | i = 1 \dots k\}$ incident to edges of $\mathcal{M}$ form a matching with two edges, which insures that M' is a perfect matching. □

Let $\mathcal{C} = \{\Gamma_i | i = 1 \dots k\}$ be the set of cycles of $G \backslash M$ and let X and Y be two distinct members of $\mathcal{M}$.

Claim 4.1.2. Let $\Gamma \in \mathcal{C}$ be an odd cycle. Assume that X and Y have ends x and y on Γ. Then xy is the only edge of C not covered by $M_X \cup M_Y$.

Proof. Since M_X (M_Y respectively) is a perfect matching, the edges of M_X (M_Y respectively) contained in Γ saturate every vertex of Γ with the exception of x (y respectively). The result follows. □

Claim 4.1.3. Let $\Gamma \in \mathcal{C}$ be an even cycle. Assume that X and Y have ends x_1, x_2 and y_1, y_2 on C with $x_1 y_1 \in N$ and $x_2 y_2 \in N$. Then $x_1 y_1$ and $x_2 y_2$ are the only edges of Γ not covered by $M_X \cup M_Y$.

Proof. The perfect matching M_X must saturate every vertex of Γ with the exception of x_1 and x_2. The same holds with M_Y and y_1 and y_2. Since x_1y_1 and x_2y_2 are edges of Γ, these two edges are not covered by $M_X \cup M_Y$ and we can easily check that the other edges are covered. $\square$

Claim 4.1.4. *Let $\Gamma \in \mathcal{C}$ be an even cycle. Assume that X and Y have ends x_1, x_2 and y_1, y_2 on C with $x_1x_2 \in N$ and $y_1y_2 \in N$. Then either x_1x_2 and y_1y_2 are the only edges of Γ not covered by $M_X \cup M_Y$ or $M_X \cup M_Y$ induces a perfect matching on Γ such that every edge in that perfect matching is covered by M_X and M_Y with the exception of x_1x_2 which belongs to M_Y and y_1y_2 which belongs to M_X.*

Proof. The perfect matching M_X must saturate every vertex of Γ with the exception of the two consecutive vertices x_1 and x_2. The same holds with M_Y and y_1 and y_2.

Let us recall here that, since X (Y respectively) is a balanced matching, the paths determined by x_1 and x_2 on Γ have odd lengths (the paths determined by y_1 and y_2 respectively). Two cases may occur.

Case 1. The two paths obtained on Γ by deleting the edges x_1x_2 and y_1y_2 have odd lengths.

We can check that $M_X \cup M_Y$ determines a perfect matching on Γ such that every edge in that perfect matching is covered by M_X and M_Y with the exception of x_1x_2 which belongs to M_Y and y_1y_2 which belongs to M_X.

Case 2. The two paths obtained on Γ by deleting x_1x_2 and y_1y_2 have even lengths.

We can check that $M_X \cup M_Y$ covers every edge of Γ with the exception of x_1x_2 and y_1y_2. $\square$

Claim 4.1.5. Let $\Gamma \in \mathcal{C}$ be an even cycle. Assume that X have ends x_1, x_2, x_3 and x_4 on Γ with $x_1x_2 \in N$ and $x_3x_4 \in N$. Then we can choose a perfect matching M_Y in such a way that x_1x_2 and x_3x_4 are the only edges of Γ not covered by $M_X \cup M_Y$.

Proof. Since M_X is a perfect matching, the edges of M_X contained in Γ saturate every vertex of Γ with the exception of x_1, x_2, x_3 and x_4. Since Y is not incident to Γ the perfect matching M_Y can be chosen in two ways (taking one of the two perfect matchings contained in this cycle). We can see easily that we can choose M_Y in such a way that every edge distinct from x_1x_2 and x_3x_4 is covered by M_X or M_Y. □

Since $\{A, B, C, D, M' \cap M\}$ is a partition of M, each edge of M is covered twice by some perfect matchings of $\mathcal{F}$.

Let $\Gamma \in \mathcal{C}$ be an odd cycle, each edge of Γ distinct from the two edges of N (Claim 4.1.2) is covered twice by some perfect matchings of $\mathcal{F}$. The two edges of N are covered by exactly one perfect matching belonging to $\{M_A, M_B, M_C, M_D\}$ and by the perfect matching M'. Hence every edge of Γ is covered twice by $\mathcal{F}$.

Let $\Gamma \in \mathcal{C}$ be an even cycle. Assume first that 4 vertices of Γ are ends of some edges in A while no other set of $\mathcal{M}$ is incident with Γ. From Claim 4.1.5 we can choose M_B in such a way that every edge distinct from the two edges of N is covered by M_A or M_B. We can then choose M_C in such a way that one of the two edges of N belongs to M_C. Finally, we can choose M_D in order to cover the other edge of N. Each edge of Γ distinct from the two edges of N (Claim 4.1.5) is covered twice by some perfect matchings of $\mathcal{F}$. The two edges of N are covered by exactly one perfect matching belonging to $\{M_A, M_B, M_C, M_D\}$ and by the perfect matching M'. Hence every edge of Γ is covered twice by $\mathcal{F}$.

Assume now that 2 vertices of Γ are ends of some edges in A (say a_1 and a_2) and 2 other vertices are ends of some edges in B (say b_1 and b_2).

Case 1. $a_1b_1 \in N$ and $a_2b_2 \in N$. We can choose M_C and M_D in order to cover every edge of Γ. From Claim 4.1.3 every edge of Γ is covered by $M_A \cup M_B$ with the exception of a_1b_1 and a_2b_2. Hence every edge of Γ is covered twice by $M_A \cup M_B \cup M_C \cup M_D$ while a_1b_1 and a_2b_2 are covered twice by $M_C \cup M_D \cup M'$ Hence every edge of Γ is covered twice by $\mathcal{F}$.

Case 2. $a_1a_2 \in N$ and $b_1b_2 \in N$. Assume that a_1a_2 and b_1b_2 are the only edges of Γ not covered by $M_A \cup M_B$ (Claim 4.1.4). Then we can choose M_C and M_D in such a way that every edge of Γ is covered by $M_C \cup M_D$. In that case every edge of Γ is covered twice by $M_A \cup M_B \cup M_C \cup M_D$ with the exception of a_1a_2 and b_1b_2 which are covered twice by $M_C \cup M_D \cup M'$.

Assume now that $M_A \cup M_B$ induces a perfect matching on Γ where $a_1a_2 \in M_B$ and $b_1b_2 \in M_A$ while the other edges of this perfect matchings are in $M_A \cap M_B$ (Claim 4.1.4). Then we can choose M_C and M_D such that every edge of Γ not contained in $M_A \cup M_B$ is covered twice by $M_C \cup M_D$ ($M_C \cup M_D$ induces a perfect matching on Γ). Hence every edge of Γ is covered twice by $M_C \cup M_D$ or by $M_A \cup M_B$ with the exception of a_1a_2 which is covered twice by $M_B \cup M^{'}$ and b_1b_2 which is covered twice by $M_A \cup M^{'}$.

Finally, assume that Γ has no vertex as end of some edge in $\mathcal{M}$. Then we can choose easily M_A, M_B, M_C and M_D such that every edge of Γ is covered twice by $M_A \cup M_B \cup M_C \cup M_D$

Hence $\mathcal{F}$ is a Fulkerson covering of G. ■

Remark 4.2. Observe that the matchings of the Fulkerson covering described in the above proof are all distinct.

4.1. Dot products which preserve an F-family

In [6] Isaacs defined the *dot product* operation in order to describe infinites families of non trivial snarks.

Let G_1, G_2 be two bridgeless cubic graphs and $e_1 = u_1v_1$, $e_2 = u_2v_2 \in E(G_1)$ and $e_3 = x_1x_2 \in E(G_2)$ with $N_{G_2}(x_1) = \{y_1, y_2, x_2\}$ and $N_{G_2}(x_2) = \{z_1, z_2, x_1\}$.

The *dot product* of G_1 and G_2, denoted by $G_1 \cdot G_2$ is the bridgeless cubic graph G defined as (see Figure 7):

$$G = [G_1 \backslash \{e_1, e_2\}] \cup [G_2 \backslash \{x_1, x_2\}] \cup \{u_1y_1, v_1y_2, u_2z_1, v_2z_2\}.$$

It is well known that the dot product of two snarks remains to be a snark. It must be pointed out that in general the dot product operation does not permit to extend a Fulkerson covering, in other words, whenever G_1 and G_2 are snarks together with a Fulkerson covering, we do not know how to get a Fulkerson covering for $G_1 \cdot G_2$.

However, in some cases, the dot product operation can preserve, in some sense, an F-family, leading thus to a Fulkerson covering of the resulting graph.

Proposition 4.3. *Let M_1 be a perfect matching of a snark G_1 such that $G_1 \backslash M_1$ contains only two (odd) cycles, namely C and C'. Let ab be an edge of C and $a'b'$ be an edge of C'.*

Let M_2 be a perfect matching of a snark G_2 where $\{A, B, C, D\}$ is an F-family for M_2. Let xy be an edge of $M_2 \backslash \{A \cup B \cup C \cup D\}$, with x and y vertices of two distinct odd cycles of $G_2 \backslash M_2$.

Then $\{A, B, C, D\}$ is an F-family for the perfect matching M of $G = G_1 \cdot G_2$ with $M = M_1 \cup M_2 \backslash \{xy\}$.

Proof. Obvious by the definition of the F-family and the construction of the graph resulting of the dot product. ■

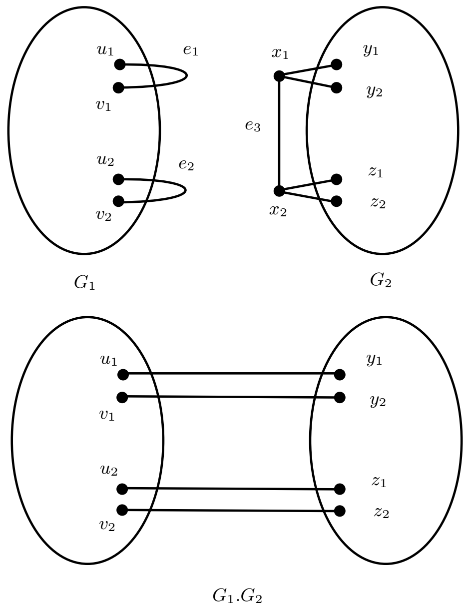

Figure 7. The dot product operation.

Proposition 4.4. *Let M_1 be a perfect matching of a snark G_1 where $\{A, B, C, D\}$ is an F-family for M_1. Let xy and zt be two edges of $E(G_1)\backslash M_1$ not contained in N.*

Let M_2 be a perfect matching of a snark G_2 such that $G_2\backslash M_2$ contains only two (odd) cycles, namely C and C'. Let $xy \in M_2$, with $x \in V(C)$ and $y \in V(C')$.

Then $\{A, B, C, D\}$ is an F-family for the perfect matching M of $G = G_1 \cdot G_2$ with $M = M_1 \cup M_2\backslash\{xy\}$.

Proof. Obvious by the definition of the F-family and the construction of the graph resulting of the dot product. ■

We remark that the graphs obtained via Propositions 4.3 and 4.4 can be provided with a Fulkerson covering by Theorem 4.1.

The dot product operations described in Propositions 4.3 and 4.4 will be said to *preserve* the F-family.

5. Applications

5.1. Fulkerson coverings, more examples

Figures 8 and 9(a) show that the Petersen Graph as well as the flower snark J_5 have oddness 2 and have an F-family (the dashed edges denote the related perfect matching).

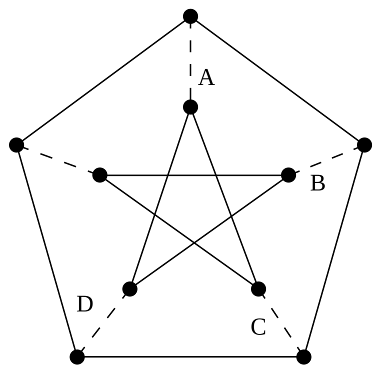

Figure 8. An F-family $\{A, B, C, D\}$ for the Petersen graph.

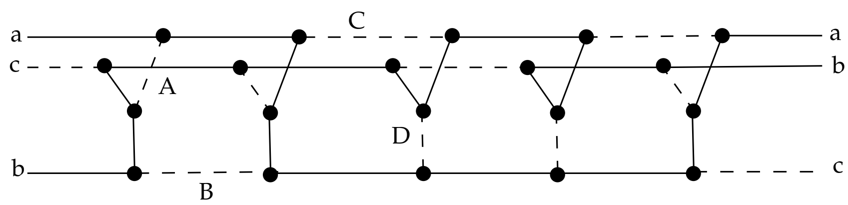

Figure (a). An F-family $\{A, B, C, D\}$ for the flower snark J_5.

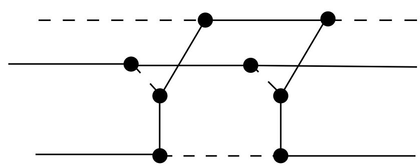

Figure (b). Two more Claws.

Figure 9. An F-family $\{A, B, C, D\}$ for the flower snark J_k.

Moreover, as shown in Figure 9(b) the F-family of J_5 can be extended by induction to all the J_k's (k odd).

Thus, following the above Propositions we can define a sequence $(G_n)_{n\in\mathbb{N}}$ of cubic graphs as follows:

- Let G_0 be the Petersen graph or the flower snark J_k ($k > 3$, k odd).
- For $n \in \mathbb{N}^*$, $G_n = G_{n-1}.G$ where G is either the Petersen graph or the flower snark J_k ($k > 3$, k odd) and the dot product operation preserves the F-family.

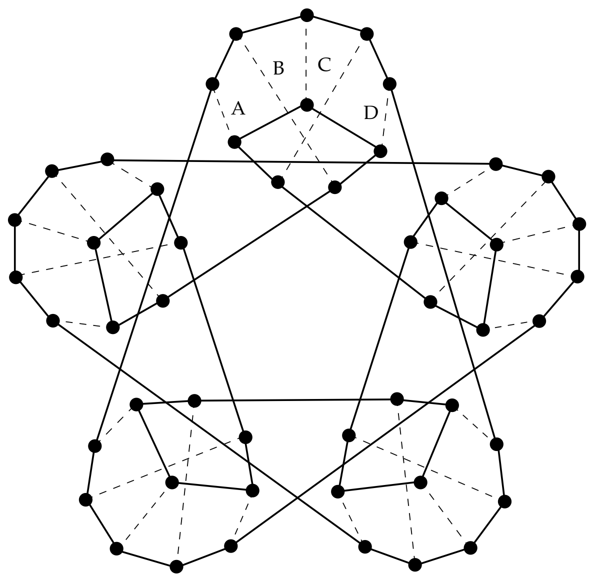

Figure 10. An F-family $\{A, B, C, D\}$ for the Szekeres Snark.

As a matter of fact this sequence of iterated dot products of the Petersen graph and/or the flower snark J_k forms a family of exponentially many snarks including the Szekeres Snark (see Figure 10) as well as the two types of generalized Blanuša snarks proposed by Watkins in [9] (see Figure 11).

The family obtained when reducing the possible values of k to $k = 5$ has already been defined by Skupień in [8], in order to provide a family of hypohamiltonian snarks in using the so-called *Flip-flop construction* introduced by Chvátal in [1].

As far as we know there is no Fulkerson family for the Golberg snark.

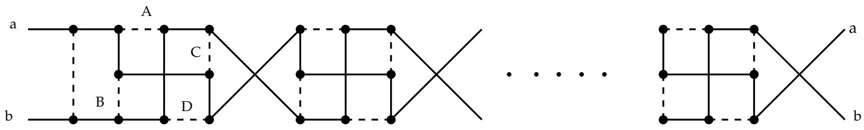

Figure (a). Blanuša snark of type 1.

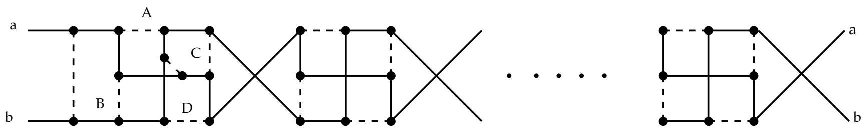

Figure (b). Blanuša snark of type 2.

Figure 11. An F-family $\{A, B, C, D\}$ for the Generalized Blanuša snarks.

5.2. Graphs with a 2-factor of C_5's.

Let G be a bridgeless cubic graph having a 2-factor where each cycle is isomorphic to a chordless C_5. We denote by G^* the multigraph obtained from G by shrinking each C_5 of this 2-factor in a single vertex. The graph G^* is 5-regular and we can easily check that it is bridgeless.

Theorem 5.1. *Let G be a bridgeless cubic graph having a 2-factor of chordless C_5. Assume that G^* has chromatic index 5. Then G can be provided with a Fulkerson covering.*

Proof. Let M be the perfect matching complementary of the 2-factor of C_5. Let $\{A, B, C, D, E\}$ be a 5-edge colouring of G^*. Each colour corresponds to a matching of G (let us denote these matchings by A, B, C, D and E). Then it is an easy task to see that $\mathcal{M} = \{A, B, C, D\}$ is an F-family for M and the result follows from Theorem 4.1. ■

Theorem 5.2. *Let G be a bridgeless cubic graph having a 2-factor of chordless C_5. Assume that G^* is bipartite. Then G can be provided with a Fulkerson covering.*

Proof. It is well known, in that case, the chromatic index of G^* is 5. The result follows from Theorem 5.1. ■

Remark that, when considering the Petersen graph P, the graph associated P^* is reduced to two vertices and is thus bipartite.

We can construct cubic graphs with chromatic index 4 which are cyclically 4-edge connected (*snarks* in the literature) and having a 2-factor of C_5's. Indeed, let G be cyclically 4-edge connected snark of size n and M be a perfect matching of G, $M = \{x_i y_i | i = 1, \ldots, \frac{n}{2}\}$. Let $G_1, \ldots, G_{\frac{n}{2}}$ be $\frac{n}{2}$ cyclically 4-edge connected snarks (each of them having a 2-factor of C_5). For each G_i $(i = 1, \ldots, \frac{n}{2})$ we consider two edges e_i^1 and e_i^2 of the perfect matching which is the complement of the 2-factor.

We construct then a new cyclically 4-edge connected snark H by applying the dot-product operation on $\{e_i^1, e_i^2\}$ and the edge $x_i y_i$ $(i = 1, \ldots, \frac{n}{2})$. We remark that the vertices of G vanish in the operation and the resulting graph H has a 2 factor of C_5, which is the union of the 2-factors of C_5 of the G_i. Unfortunately, when considering the graph H^*, derived from H, we cannot insure, in general, that H^* is 5-edge colourable in order to apply Theorem 5.1 and obtain hence a Fulkerson covering of H.

An interesting case is obtained when, in the above construction of H, each graph G_i is isomorphic to the Petersen graph. Indeed, the 2-factor of C_5's obtained then is such that we can find a partition of the vertex set of H in sets of 2 C_5 joined by 3 edges. These sets lead to pairs of vertices of H^* joined by three parallel edges. We can thus see H^* as a cubic graph where a perfect matching is taken 3 times. Let us denote by $\tilde{H}$ this cubic graph (by the way $\tilde{H}$ is 3-connected). It is an easy task to see that, when $\tilde{H}$ is 3-edge colourable, H^* is 5-edge colourable and hence, Theorem 5.1 can be applied.

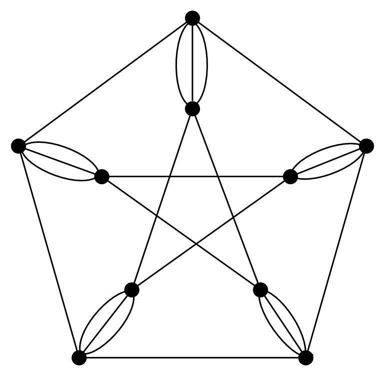

Figure 12. H^* isomorphic to $\mathcal{P}(3)$.

Let us consider by example the graph H obtained with 5 copies of the Petersen graph following the above construction (let us remark that the graph G involved in our construction must be isomorphic also to the Petersen graph). This graph is a snark on 50 vertices. Since $\tilde{H}$ is a bridgeless cubic graph, the only case for which we cannot say whether H has a Fulkerson covering occurs when $\tilde{H}$ is isomorphic to the Petersen graph and, hence H^* is isomorphic to the *unslicable* graph $\mathcal{P}(3)$ described by Rizzi [7] (see Figure 12). As a matter of fact we do not know if it is possible to construct a graph H as described above such that H^* is isomorphic to the graph $\mathcal{P}(3)$.

By the way, we do not know example of cyclically 5-edge connected snarks (excepted the Petersen graph) with a 2-factor of induced cycles of length 5. We have proposed in [3] the following problem.

Problem 5.3. *Is there any* 5*-edge connected snark distinct from the Petersen graph with a* 2*-factor of* C_5*'s* ?

6. On Proper Fulkerson Covering

As noticed in the introduction, when a cubic graph is 3-edge colourable, we can find a Fulkerson covering by using a 3-edge colouring and considering each colour twice.

Proposition 6.1. *Let* G *be a bridgeless cubic graph with chromatic index* 4*. Assume that* G *has a Fulkerson covering* $\mathcal{F} = \{M_1, M_2, M_3, M_4, M_5, M_6\}$ *of its edge set. Then the* 6 *perfect matchings are distinct.*

Proof. Assume, without loss of generality that $M_1 = M_2$. Since each edge is contained in exactly 2 perfect matchings of $\mathcal{F}$, we must have $M_3 \cap M_1 = \emptyset$. Hence G is 3-edge colourable, a contradiction. ■

Let us say that a Fulkerson covering is *proper* whenever the 6 perfect matchings involved in this covering are distinct. An interesting question is thus to determine which cubic bridgeless graph have a proper Fulkerson covering.

A 3-edge colourable graph is said to be *bi-hamiltonian* whenever in any 3-edge colouring, there are at least two colours whose removing leads to an hamiltonian 2-factor.

Proposition 6.2. *Let* G *be a bridgeless* 3*-edge colourable cubic graph which is not bi-hamiltonian. Then* G *has a proper Fulkerson covering.*

Proof. Let $\Phi \quad E(G) \to \{\alpha, \beta, \gamma\}$ be a 3-edge colouring of G. When x and y are colours in $\{\alpha, \beta, \gamma\}$, $\Phi(x, y)$ denotes the set of disjoint even cycles induced by the two colours x and y.

Since the graph G is not bi-hamiltonian we may assume that the 2-factors $\Phi(\alpha, \beta)$ and $\Phi(\beta, \gamma)$ are not hamiltonian cycles. Let C be a cycle in $\Phi(\alpha, \beta)$, we get a new 3-edge colouring $\Phi^{'}$ by exchanging the two colours α and β along C. We get hence a partition of $E(G)$ into 3 perfect matching $\alpha^{'}, \beta^{'}$ and γ. In the same way, when considering a cycle D in $\Phi(\beta, \gamma)$, we get a new 3-edge colouring $\Phi^{''}$ of G by exchanging β and γ along D. Let α,$\beta^{''}$ and $\gamma^{''}$ be the 3 perfect matchings so obtained.

Since we have two distinct 3-edge colourings of G, $\Phi^{'}$ and $\Phi^{''}$, the set of 6 perfect matchings so involved $\{\alpha, \alpha^{'}, \beta^{'}, \beta^{''}, \gamma, \gamma^{''}\}$ is a Fulkerson covering. It remains to show that this set is actually a proper Fulkerson covering.

The exchange operated in order to get $\Phi^{'}$ involve some edges in α and some edges in β (those which are on C_1) while the other edges keep their colour. In the same way, the exchange operated in order to get $\Phi^{''}$ involve some edges in β and some edges in γ (those which are on D_1) while the other edges keep their colour.

The 3 perfect matchings of $\Phi^{'}$ ($\alpha^{'}, \beta^{'}$ and γ) are pairwise disjoint as well as those of $\Phi^{''}$ ($\alpha, \beta^{''}$ and $\gamma^{''}$). We have $\alpha \neq \alpha^{'}$ since $\alpha^{'}$ contains some edges of β. We have $\alpha \cap \beta^{''} = \emptyset$ and $\alpha \cap \gamma^{''} = \emptyset$ since we have exchanged β and γ in order to obtain $\beta^{''}$ and $\gamma^{''}$. We have $\beta^{'} \neq \beta^{''}$ since $\beta^{'}$ contains some edges of α while $\beta^{''}$ contains some edges of γ. We have $\beta^{'} \neq \gamma^{''}$ since $\beta^{'}$ contains some edges of α and $\gamma^{''}$ contains only edges in β or in γ. We have $\gamma \neq \gamma^{''}$ since $\gamma^{''}$ contains some edges of β.

Hence $\{\alpha, \alpha^{'}, \beta^{'}, \beta^{''}, \gamma, \gamma^{''}\}$ is a proper Fulkerson covering. ■

The *theta graph* (2 vertices joined by 3 edges), K_4, $K_{3,3}$ are examples of small bridgeless cubic graph without proper Fulkerson covering. The infinite family of bridgeless cubic bi-hamiltonian graphs obtained by doubling the edges of a perfect matching of an even cycle has no proper Fulkerson covering. On the other hand, we can provide a bi-hamiltonian graph together with a proper Fulkerson covering. Consider for example the graph G on 10 vertices which have a 2 factor of C_5's, namely *abcde* and 12345 with the additional edges edges $a2$, $b4$, $c3$, $d5$ and $e1$, it is not difficult to check that this graph is bi-hamiltonian. Moreover since the following four balanced matchings $\{a2\}$, $\{b4\}$, $\{c3\}$ and $\{d5\}$ form an F-family for the perfect matching $\{a2, b4, c3, d5, e1\}$, due to Theorem 4.1 and Remark 4.2, the graph G has a proper Fulkerson covering.

A challenging problem is thus to characterize those bridgeless cubic graphs having a proper Fulkerson covering.

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